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Ternary (σ, τ, ξ)-derivations on Banach ternary algebras

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Dedicated to the Memory of Charalambos J. Papaioannou

(Communicated by Th. M. Rassias)

Abstract

Let A be a Banach ternary algebra over a scalar field $\mathbb{R}$ or $\mathbb{C}$ and X be a Banach ternary A-module. Let σ, τ and ξ be linear mappings on A, a linear mapping $D : (A, [\ ]_A) \to (X, [\ ]_X)$ is called a ternary (σ, τ, ξ)-derivation, if

$$D([xyz]_A) = [D(x)\tau(y)\xi(z)]_X + [\sigma(x)D(y)\xi(z)]_X + [\sigma(x)\tau(y)D(z)]_X$$

for all $x, y, z \in A$.

In this paper, we investigate ternary (σ, τ, ξ)-derivation on Banach ternary algebras, associated with the following functional equation

$$f(\frac{x+y+z}{4}) + f(\frac{3x-y-4z}{4}) + f(\frac{4x+3z}{4}) = 2f(x)\ .$$

Moreover, we prove the generalized Ulam–Hyers stability of ternary (σ, τ, ξ)-derivations on Banach ternary algebras.

Keywords: Banach ternary algebra; Banach ternary A-module; Ternary (σ, τ, ξ)-derivation.
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1. Introduction

Ternary algebraic operations were considered in the 19 th century by several mathematicians such as A. Cayley [12] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 ([30]). The simplest example of such non-trivial ternary operation is given by the following composition rule:

$$\{a, b, c\}_{ijk} = \sum_{l,m,n} a_{nil} b_{ljm} c_{mkn}, \qquad i, j, k... = 1, 2, ..., N.$$

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their possible applications in physics. Some significant physical applications are as follows (see [3], [4], [8], [15]- [22], [24], [31], [32], [37], [54] and [56]). A ternary (associative) algebra $(A, [\ ])$ is a linear space A over a scalar field $\mathbb{F} = (\mathbb{R}$ or $\mathbb{C})$ equipped with a linear mapping, the so-called ternary product, $[\ ]$: $A \times A \times A \to A$ such that $[[abc]de] = [a[bcd]e]$ for all $a, b, c, d, e \in A$. This notion is a natural generalization of the binary case. Indeed if $(A, \odot)$ is a usual (binary) algebra then $[abc] := (a \odot b) \odot c$ induced a ternary product making A into a ternary algebra which will be called trivial. It is known that unital ternary algebras are trivial and finitely generated ternary algebras are ternary subalgebras of trivial ternary algebras [9]. There are other types of ternary algebras in which one may consider other versions of associativity. Some examples of ternary algebras are (i) "cubic matrices" introduced by Cayley [12] which were in turn generalized by Kapranov, Gelfand and Zelevinskii [30]; (ii) the ternary algebra of polynomials of odd degrees in one variable equipped with the ternary operation $[p_1p_2p_3] = p_1 \odot p_2 \odot p_3$, where $\odot$ denotes the usual multiplication of polynomials.

By a Banach ternary algebra we mean a ternary algebra equipped with a complete norm $\|.\|$ such that $\|[abc]\| \leq \|a\|\|b\|\|c\|$. If a ternary algebra $(A,[\ ])$ has an identity, i.e. an element e such that $a = [aee] = [eae] = [eea]$ for all $a \in A$, then $a \odot b := [aeb]$ is a binary product for which we have

$$(a \odot b) \odot c = [[aeb]ec] = [ae[bec]] = a \odot (b \odot c)$$

and

$$a \odot e = [aee] = a = [eea] = e \odot a,$$

for all $a, b, c \in A$ and so $(A,[\ ])$ may be considered as a (binary) algebra. Conversely, if $(A, \odot)$ is any (binary) algebra, then $[abc] := a \odot b \odot c$ makes A into a ternary algebra with the unit e such that $a \odot b = [aeb]$.

Let A be a Banach ternary algebra and X be a Banach space. Then X is called a ternary Banach A-module, if module operations $A \times A \times X \to X$, $A \times X \times A \to X$, and $X \times A \times A \to X$ are $\mathbb{C}$-linear in every variable. Moreover satisfy:

$$[[abc]_A\ dx]_X = [a[bcd]_A\ x]_X = [ab[cdx]_X]_X,$$

$$[abc]_A\ xd]_X = [a[bcx]_X\ d]_X = [ab[cxd]_X]_X,$$

$$[[xab]_X\ cd]_X = [x[abc]_A\ d]_X = [xa[bcd]_A]_X,$$

$$[[axb]_X\ cd]_X = [a[xbc]_X\ d]_X = [ax[bcd]_A]_X,$$

$$[[abx]_X \ cd]_X = [a[bxc]_X \ d]_X = [ab[xcd]_X]_X,$$

for all $x \in X$ and all $a, b, c, d \in A$, and

$$\max\{\|[xab]_X\|, \|[axb]_X\|, \|[abx]_X\|\} \leq \|a\|\|b\|\|x\|$$

for all $x \in X$ and all $a, b \in A$.

Let $(A, [\ ]_A)$ be a Banach ternary algebra over a scalar field $\mathbb{R}$ or $\mathbb{C}$ and $(X, [\ ]_X)$ be a ternary Banach A-module. A linear mapping $D : (A, [\ ]_A) \to (X, [\ ]_X)$ is called a ternary derivation, if

$$D([abc]_A) = [D(a)bc]_X + [aD(b)c]_X + [abD(c)]_X \tag{1.1}$$

for all $a, b, c \in A$.

Let σ, τ and ξ be linear mappings on A. A linear mapping $D : (A, [\ ]_A) \to (X, [\ ]_X)$ is called a ternary (σ, τ, ξ)-derivation, if

$$D([abc]_A) = [D(a)\tau(b)\xi(c)]_X + [\sigma(a)D(b)\xi(c)]_X + [\sigma(a)\tau(b)D(c)]_X \tag{1.2}$$

for all $a, b, c \in A$.

The stability of functional equations was first introduced by S. M. Ulam [55] in 1940. More precisely, let G_1, be a group, (G_2, d) be a metric group and ϵ be a positive number, S. M. Ulam asked, does there exist a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$?. When this problem has a solution, we say that the homomorphism from G_1 to G_2 is stable.

In 1941, D. H. Hyers [27] gave a partial solution of *Ulam's* problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1950, T. Aoki [1] was the second author to treat this problem for additive mappings (see also [11]). In 1978, Th. M. Rassias [48] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Th. M. Rassias [48] is called the Hyers-Ulam-Rassias stability, according to J. M. Rassias Theorem, as follows:

Theorem 1.1. *Let $f : V \longrightarrow W$ be a mapping from a norm vector space V into a Banach space W subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.3}$$

for all $x, y \in V$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : V \longrightarrow W$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.4}$$

for all $x \in V$. If $p < 0$ then inequality (1.3) holds for all $x, y \neq 0$, and (1.4) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into W is continuous for each fixed $x \in V$, then T is linear.

On the other hand J. M. Rassias ([44]- [46]) generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. According to J. M. Rassias Theorem [51]:

Theorem 1.2. *If it is assumed that there exist constants $\Theta \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : V \to W$ is a mapping from a norm space V into a Banach space W such that the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \Theta \|x\|^{p_1} \|y\|^{p_2}$$

for all $x, y \in V$ holds, then there exists a unique additive mapping $T : V \to W$ such that

$$\|f(x) - T(x)\| \leq \frac{\Theta}{2 - 2^p} \|x\|^p,$$

for all $x \in V$. If in addition for every $x \in V$, $f(tx)$ is continuous in real t for each fixed x, then T is linear (see [38]-[46]).

During the last decades several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [5], [6], [11], [13]-[24], [25], [28], [33], [47] and [49]-[53]. Recently, R. Badora , [7] and T. Miura et al. [34] proved the Ulam–Hyers stability, the Isac and Rassias–type stability [29], the Hyers–Ulam–Rassias stability and the Bourgin–type superstability of ring derivations on Banach algebras. On the other hand, C. Park [36] has contributed works to the stability problem of ternary homomorphisms and ternary derivations(see also [26]).

The main purpose of the present paper is to offer the Ulam–Hyers stability of ternary (σ, τ, ξ)-derivations on Banach ternary algebras subjected with the following functional equation

$$f\Big(\frac{x+y+z}{4}\Big) + f\Big(\frac{3x-y-4z}{4}\Big) + f\Big(\frac{4x+3z}{4}\Big) = 2f(x)\ . \tag{1.5}$$

2. Ternary (σ, τ, ξ)-derivations on Banach ternary algebras

In this section, we investigate ternary (σ, τ, ξ)-derivations on Banach ternary algebras. Throughout this section, assume that $(A, [\ ]_A)$ is a Banach ternary algebra and $(X, [\ ]_X)$ is a ternary Banach A-module.

Lemma 2.1. *Let V and W be linear spaces and let $f : V \to W$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in V$ and all $\mu \in \mathbb{T}^1 (:= \{\lambda \in \mathbb{C}\ ; |\lambda| = 1\})$. Then the mapping f is $\mathbb{C}$-linear. [35]*

Lemma 2.2. *Let $f : A \to X$ be a mapping such that*

$$f\Big(\frac{\mu x+y+z}{4}\Big) + f\Big(\frac{3\mu x-y-4z}{4}\Big) + f\Big(\frac{4\mu x+3z}{4}\Big) = 2\mu f(x), \tag{2.1}$$

for all $x, y, z \in A$ and $\mu \in \mathbb{T}^1$. Then f is $\mathbb{C}$-linear. [10]

The first result is as follows:

Theorem 2.3. *Let $p \neq 1$ and θ be nonnegative real numbers, and let $f : A \to X$ be a mapping and σ, τ, and ξ be linear mappings on A such that*

$$f\Big(\frac{\mu x+y+z}{4}\Big) + f\Big(\frac{3\mu x-y-4z}{4}\Big) + f\Big(\frac{4\mu x+3z}{4}\Big) = 2\mu f(x), \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$,

$$\|f([xyz]_A) - [f(x)\tau(y)\xi(z)]_X - [\sigma(x)f(y)\xi(z)]_X - [\sigma(x)\tau(y)f(z)]_X\| \leq \theta \|x\|^p \|y\|^p \|z\|^p \tag{2.3}$$

for all $x, y, z \in A$. Then the mapping $f : A \to X$ is a ternary (σ, τ, ξ)-derivation.

Proof . Assume $p < 1$. By Lemma 2.2, the mapping $f : A \to X$ is $\mathbb{C}$-linear. It follows from (2.3) that

$$\begin{aligned}
&\|f([xyz]_A) - [f(x)\tau(y)\xi(z)]_X - [\sigma(x)f(y)\xi(z)]_X - [\sigma(x)\tau(y)f(z)]_X\| \\
&= \frac{1}{n^3}\|f([(nx)(ny)(nz)]_A) - [f(nx)\tau(ny)\xi(nz)]_X - [\sigma(nx)f(ny)\xi(nz)]_X - [\sigma(nx)\tau(ny)f(nz)]_X\| \\
&\leq \frac{\theta}{n^3}n^{3p}\|x\|^p\|y\|^p\|z\|^p
\end{aligned}$$

for all $x, y, z \in A$. Thus, since $p < 1$, by letting n tend to ∞ in last inequality, we obtain

$$f([xyz]_A) = [f(x)\tau(y)\xi(z)]_X + [\sigma(x)f(y)\xi(z)]_X + [\sigma(x)\tau(y)f(z)]_X$$

for all $x, y, z \in A$. Hence, the mapping $f : A \to X$ is a ternary (σ,τ,ξ)-derivation. Similarly, one obtains the result for the case $p > 1$. $\square$

We prove the following Ulam stability problem for functional equation (1.5) controlled by the mixed type product-sum function

$$(x, y) \to \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3} + \|x\|^p + \|y\|^p + \|z\|^p)$$

introduced by J. M. Rassias (see [47],[52]).

Theorem 2.4. *Let p, p_1, p_2, p_3 be real numbers such that $p < 1$, $p_1 + p_2 + p_3 < 1$, and $\theta > 0$. Suppose $f : A \to X$ is a mapping for which there exist mappings $g, h, k : A \to A$ whit $g(0) = h(0) = k(0) = 0$ such that*

$$\begin{aligned}
&\|f(\frac{\mu x + y + z}{4}) + f(\frac{3\mu x - y - 4z}{4}) + f(\frac{4\mu x + 3z}{4}) - 2\mu f(x)\| \\
&\leq \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3} + \|x\|^p + \|y\|^p + \|z\|^p), \qquad (2.4)
\end{aligned}$$

$$\|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| \leq \theta(\|x\|^p + \|y\|^p) \qquad (2.5)$$

$$\|h(\lambda x + \lambda y) - \lambda h(x) - \lambda h(y)\| \leq \theta(\|x\|^p + \|y\|^p) \qquad (2.6)$$

$$\|k(\lambda x + \lambda y) - \lambda k(x) - \lambda k(y)\| \leq \theta(\|x\|^p + \|y\|^p) \qquad (2.7)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$,

$$\|f([xyz]_A) - [f(x)h(y)k(z)]_X - [g(x)f(y)k(z)]_X - [g(x)h(y)f(z)]_X\| \leq \theta\|x\|^p\|y\|^p\|z\|^p \qquad (2.8)$$

for all $x, y, z \in A$. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique ternary (σ,τ,ξ)-derivation $D : A \to X$ satisfying

$$\|g(x) - \sigma(x)\| \leq \theta\frac{2}{2 - 2^p}\|x\|^p \qquad (2.9)$$

$$\|h(x) - \tau(x)\| \leq \theta\frac{2}{2 - 2^p}\|x\|^p \qquad (2.10)$$

$$\|k(x) - \xi(x)\| \leq \theta\frac{2}{2 - 2^p}\|x\|^p \qquad (2.11)$$

$$\|f(x) - D(x)\| \leq 2\theta\frac{2^p}{2 - 2^p}\|x\|^p \qquad (2.12)$$

for all $x \in A$.

Proof . Setting $\mu = 1$ and $x = y = z = 0$ in (2.4), yields $f(0) = 0$. Let us take $\mu = 1$, $z = 0$ and $y = x$ in (2.4). Then we obtain

$$\|2f(\frac{x}{2}) - f(x)\| \le 2\theta \|x\|^p \ , \tag{2.13}$$

for all $x \in A$. In (2.13), replacing $\frac{x}{2}$ by x and then dividing by 2, we get

$$\|f(x) - \frac{1}{2}f(2x)\| \le 2^p \theta \|x\|^p \ ,$$

for all $x \in A$. We easily prove that by induction that

$$\|f(x) - \frac{1}{2^n}f(2^n x)\| \le 2\theta \|x\|^p \sum_{i=1}^{n} 2^{i(p-1)}. \tag{2.14}$$

In order to show that the functions $D_n(x) = \frac{1}{2^n}f(2^n x)$ form a convergent sequence, we use the Cauchy convergence criterion. Indeed, replace x by $2^m x$ and divide by 2^m in (2.14), where m is an arbitrary positive integer. We find that

$$\|\frac{1}{2^m}f(2^m x) - \frac{1}{2^{m+n}}f(2^{m+n}x)\| \le 2\theta \|x\|^p \sum_{i=m+1}^{m+n} 2^{i(p-1)}$$

for all positive integers. Hence, by the Cauchy criterion the limit $D(x) = \lim_{n\to\infty} D_n(x)$ exists for each $x \in A$. By taking the limit as $n \to \infty$ in (2.12),

$$\|f(x) - D(x)\| \le 2\theta \|x\|^p \sum_{i=1}^{\infty} 2^{i(p-1)}$$

and (2.12) holds for all $x \in A$. Now, we have

$$\|D(\frac{\mu x + y + z}{4}) + D(\frac{3\mu x - y - 4z}{4}) + D(\frac{4\mu x + 3z}{4}) - 2\mu D(x)\|$$

$$= \lim_{n\to\infty} \frac{1}{2^n}\|f(\frac{2^n \mu x + 2^n y + 2^n z}{4}) + f(\frac{3.2^n \mu x - 2^n y - 4.2^n z}{4})$$

$$+f(\frac{4.2^n \mu x + 3.2^n z}{4}) - 2\mu f(2^n x)\|_A \le \lim_{n\to\infty} \frac{1}{2^n}\theta(\|2^n x\|^{p_1}\|2^n y\|^{p_2}\|2^n z\|^{p_3}$$

$$+\|2^n x\|^p + \|2^n y\|^p + \|2^n z\|^p) = \lim_{n\to\infty} 2^{n(p_1+p_2+p_3-1)}\theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{P_3})$$

$$+ \lim_{n\to\infty} 2^{n(p-1)}\theta(\|x\|^p + \|y\|^p + \|z\|^P) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Hence

$$D(\frac{\mu x + y + z}{4}) + D(\frac{3\mu x - y - 4z}{4}) + D(\frac{4\mu x + 3z}{4}) = 2\mu D(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. So by Lemma 2.2, D is $\mathbb{C}$-linear.

Also put $\lambda = 1$ in (2.5) to obtain

$$\|g(x+y) - g(x) - g(y)\| \leq \theta(\|x\|^p + \|y\|^P) \tag{2.15}$$

fix $x \in A$.Replacing y by x and then dividing by 2 in (2.15), we get

$$\|\frac{1}{2}g(2x) - g(x)\| \leq \theta\|x\|^p$$

one can use the induction to show that

$$\|\frac{1}{2^{m+n}}g(2^{m+n}x) - \frac{1}{2^m}g(2^m x)\| \leq \theta\|x\|^p \sum_{i=m}^{m+n-1} 2^{(i-1)(p-1)} \tag{2.16}$$

for all $x \in A$. It follows from the convergence of series (2.16) that the sequence $\{\frac{g(2^n x)}{2^n}\}$ is cauchy. Hence, the limit $\sigma(x) = \lim_{n\to\infty} \frac{g(2^n x)}{2^n}$ exists for all $x \in A$. we easily prove that by (2.5) that $\sigma(\lambda x + \lambda y) = \lambda\sigma(x) + \lambda\sigma(y)$ and by (2.16) that

$$\|g(x) - \sigma(x)\| \leq \theta\|x\|^p \sum_{i=1}^{\infty} 2^{(i-1)(p-1)}$$

and (2.9) holds for all $x \in A$. Similarly there exist linear mappings τ and ξ on A such that (2.10) and (2.11) hold for all $x \in A$. On the other hand

$$\begin{aligned}
&\|D([xyz]_A) - [D(x)\tau(y)\xi(z)]_X - [\sigma(x)D(y)\xi(z)]_X - [\sigma(x)\tau(y)D(x)]_X\| \\
&= \lim_{n\to\infty} \frac{1}{8^n}\|f([(2^n x)(2^n y)(2^n z)]_A) - [f(2^n x)h(2^n y)k(2^n z)]_X \\
&-[g(2^n x)f(2^n y)k(2^n z)]_X - [g(2^n x)h(2^n y)f(2^n z)]_X\| \\
&\leq \lim_{n\to\infty} \frac{\theta}{8^n}\|2^n x\|^p\|2^n y\|^p\|2^n z\|^p \\
&= \lim_{n\to\infty} \theta 8^{n(p-1)}\|x\|^p\|y\|^p\|z\|^p \\
&= 0
\end{aligned}$$

for all $x, y, z \in A$, which means that

$$D([xyz]_A) = [D(x)\tau(t)\xi(z)]_X + [\sigma(x)D(y)\xi(z)]_X + [\sigma(x)\tau(y)D(z)]_X.$$

Therefore, we conclude that D is a ternary (σ, τ, ξ)-derivation. Suppose that there exists another ternary (σ, τ, ξ)-derivation $D' : A \to X$ satisfying (2.12). Since $D'(x) = \frac{1}{2^n}D'(2^n x)$, we see that

$$\begin{aligned}
\|D(x) - D'(x)\| &= \tfrac{1}{2^n}\|D(2^n x) - D'(2^n x)\| \\
&\leq \tfrac{1}{2^n}(\|f(2^n x) - D(2^n x)\| + \|f(2^n x) - D'(2^n x)\|) \\
&\leq 4\theta\tfrac{2^p}{2-2^p}2^{n(p-1)}\|x\|^p ,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. Therefore $D' = D$ as claimed and similarly we can prove that σ, τ and ξ are unique on A and the proof of the theorem is complete. □

Theorem 2.5. *Let p, p_1, p_2, p_3 be real numbers such that $p > 1$, $p_1 + p_2 + p_3 > 1$, and $\theta > 0$. Suppose $f : A \to X$ is a mapping for which there exist mappings g, h, k on A whit $g(0) = h(0) = k(0) = 0$ such that*

$$\|f(\frac{\mu x + y + z}{4}) + f(\frac{3\mu x - y - 4z}{4}) + f(\frac{4\mu x + 3z}{4}) - 2\mu f(x)\|$$
$$\leq \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3} + \|x\|^p + \|y\|^p + \|z\|^p), \tag{2.17}$$

$$\|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.18}$$

$$\|h(\lambda x + \lambda y) - \lambda h(x) - \lambda h(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.19}$$

$$\|k(\lambda x + \lambda y) - \lambda k(x) - \lambda k(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.20}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$,

$$\|f([xyz]_A) - [f(x)h(y)k(z)]_X - [g(x)f(y)k(z)]_X - [g(x)h(y)f(z)]_X\| \leq \theta\|x\|^p\|y\|^p\|z\|^p \tag{2.21}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exist unique linear mappings σ, τ and k on A and a unique ternary (σ, τ, ξ)-derivation $D : A \to X$ satisfying

$$\|g(x) - \sigma(x)\| \leq \theta\frac{2}{2^p - 2}\|x\|^p \tag{2.22}$$

$$\|h(x) - \tau(x)\| \leq \theta\frac{2}{2^p - 2}\|x\|^p \tag{2.23}$$

$$\|k(x) - \xi(x)\| \leq \theta\frac{2}{2^p - 2}\|x\|^p \tag{2.24}$$

$$\|D(x) - f(x)\| \leq 2\theta\frac{2^p}{2^p - 2}\|x\|^p \tag{2.25}$$

for all $x \in A$.

Proof . Setting $\mu = 1$ and $x = y = z = 0$ in (2.17), yields $f(0) = 0$. Let us take $\mu = 1$, $z = 0$ and $y = x$ in (2.17). Then we obtain

$$\|2f(\frac{x}{2}) - f(x)\| \leq 2\theta\|x\|^p\ , \tag{2.26}$$

for all $x \in A$. By induction, we get

$$\|2^n f(\frac{x}{2^n}) - f(x)\| \leq 2\theta\|x\|^p \sum_{i=0}^{n-1} 2^{i(1-p)}. \tag{2.27}$$

In order to show that the functions $D_n(x) = 2^n f(\frac{x}{2^n})$ form a convergent sequence, we use the Cauchy convergence criterion. Indeed, replace x by $\frac{x}{2^m}$ and multiply by 2^m in (2.27), where m is an arbitrary positive integer. We find that

$$\|2^{m+n} f(\frac{x}{2^{m+n}}) - 2^m f(\frac{x}{2^m})\| \leq 2\theta\|x\|^p \sum_{i=m}^{m+n-1} 2^{i(1-p)}$$

for all positive integers. Hence, by the Cauchy criterion the limit $D(x) = \lim_{n\to\infty} D_n(x)$ exists for each $x \in A$. By taking the limit as $n \to \infty$ in (2.27) we see that

$$\|D(x) - f(x)\| \leq 2\theta\|x\|^p \sum_{i=0}^{\infty} 2^{i(1-p)}$$

and (2.25) holds for all $x \in A$. Thus, we have

$$\|D(\frac{\mu x + y + z}{4}) + D(\frac{3\mu x - y - 4z}{4}) + D(\frac{4\mu x + 3z}{4}) - 2\mu D(x)\|$$

$$= \lim_{n\to\infty} 2^n \|f(\frac{2^{-n}\mu x + 2^{-n}y + 2^{-n}z}{4}) + f(\frac{3.2^{-n}\mu x - 2^{-n}y - 4.2^{-n}z}{4})$$

$$+f(\frac{4.2^{-n}\mu x + 3.2^{-n}z}{4}) - 2\mu f(2^{-n}x)\|$$

$$\leq \lim_{n\to\infty} 2^n\theta(\|2^{-n}x\|^{p_1}\|2^{-n}y\|^{p_2}\|2^{-n}\|^{p_3} + \|2^{-n}x\|^p + \|2^{-n}y\|^p + \|2^{-n}z\|^p)$$

$$= \lim_{n\to\infty} 2^{n(1-p_1+p_2+p_3)}\theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{P_3}) + \lim_{n\to\infty} 2^{n(1-p)}\theta(\|x\|^p + \|y\|^p + \|z\|^P)$$

$$= 0$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Hence,

$$D(\frac{\mu x + y + z}{4}) + D(\frac{3\mu x - y - 4z}{4}) + D(\frac{4\mu x + 3z}{4}) = 2\mu D(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. So by Lemma 2.2, D is $\mathbb{C}$-linear.

Also put $\lambda = 1$ in (2.18) to obtain

$$\|g(x + y) - g(x) - g(y)\| \leq \theta(\|x\|^p + \|y\|^P) \tag{2.28}$$

fix $x \in A$. Replacing y by x and then replacing x by $\frac{x}{2}$ in (2.28), we get

$$\|2g(\frac{x}{2}) - g(x)\| \leq 2\theta\|\frac{x}{2}\|^p$$

one can use the induction to show that

$$\|2^{m+n}g(\frac{x}{2^{m+n}}) - 2^m g(\frac{x}{2^m})\| \leq \theta\|x\|^p \sum_{i=m+1}^{m+n} 2^{(i)(1-p)} \tag{2.29}$$

for all $x \in A$. It follows from the convergence of series (2.29) that the sequence $\{2^n g(\frac{x}{2^n})\}$ is cauchy. Hence the limit $\sigma(x) = \lim_{n\to\infty} 2^n g(\frac{x}{2^n})$ exists for all $x \in A$. we easily prove that by (2.18) that $\sigma(\lambda x + \lambda y) = \lambda\sigma(x) + \lambda\sigma(y)$ and by (2.29) that

$$\|g(x) - \sigma(x)\| \leq \theta\|x\|^p \sum_{i=1}^{\infty} 2^{(i)(1-p)}$$

and (2.22) holds for all $x \in A$. Similarly there exist linear mappings τ and ξ on A such that (2.23) and (2.24) hold for all $x \in A$. Thus, we have

$$\|D([xyz]_A) - [D(x)\tau(y)\xi(z)]_X - [\sigma(x)D(y)\xi(z)]_X - [\sigma(x)\tau(y)D(z)]_X\|$$

$$= \lim_{n\to\infty} 8^n\|(f[(2^{-n}x)(2^{-n}y)(2^{-n}z)]_A) - [f(2^{-n}x)h(2^{-n}y)k(2^{-n}z)]_X$$

$$-[g(2^{-n}x)f(2^{-n}y)k(2^{-n}z)]_X - [g(2^{-n}x)h(2^{-n}y)f(2^{-n}z)]_X\|$$

$$\leq \lim_{n\to\infty} 8^n\theta\|\frac{x}{2^n}\|^p\|\frac{y}{2^n}\|^p\|\frac{z}{2^n}\|^p = \lim_{n\to\infty} \theta 8^{n(1-p)}\|x\|^p\|y\|^p\|z\|^p$$

$$= 0$$

for all $x \in A$, which means that

$$D([xyz]_A) = [D(x)\tau(y)\xi(z)]_X + [\sigma(x)D(y)\xi(z)]_X + [\sigma(x)\tau(y)D(z)]_X.$$

Therefore, we conclude that D is a ternary (σ, τ, ξ)-derivation. Suppose that there exists another ternary (σ, τ, ξ)-derivation $D^{'} : A \to X$ satisfying (2.25) Since $D^{'}(x) = 2^n D^{'}(\frac{x}{2^n})$, we see that

$$\begin{aligned}
\|D(x) - D^{'}(x)\| &= 2^n\|D(\tfrac{x}{2^n}) - D^{'}(\tfrac{x}{2^n})\| \\
&\leq 2^n(\|f(\tfrac{x}{2^n}) - D(\tfrac{x}{2^n})\| + \|f(\tfrac{x}{2^n}) - D^{'}(\tfrac{x}{2^n})\|) \\
&\leq 4\theta\tfrac{2^p}{2^p-2}2^{n(1-p)}\|x\|^p ,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. Hence, $D^{'} = D$ as claimed and similarly we can prove that σ, τ and ξ are unique on A and proof of theorem is complete. □

We are going to investigate the Hyers–Ulam–Rassias stability problem for functional equation (1.5).

Corollary 2.6. *Let $P \in (-\infty, 1) \cup (1, \infty)$, $\theta > 0$. Suppose $f : A \to X$ is a mapping for which there exist mappings g, h, k on A whit $g(0) = h(0) = k(0) = 0$ such that*

$$\|f(\frac{\mu x + y + z}{4}) + f(\frac{3\mu x - y - 4z}{4}) + f(\frac{4\mu x + 3z}{4}) - 2\mu f(x)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p),$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$,

$$\|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

$$\|h(\lambda x + \lambda y) - \lambda h(x) - \lambda h(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

$$\|k(\lambda x + \lambda y) - \lambda k(x) - \lambda k(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$,

$$\|f([xyz]_A) - [f(x)h(y)k(z)]_X - [g(x)f(y)k(z)]_X - [g(x)h(y)f(z)]_X\| \leq \theta\|x\|^p\|y\|^p\|z\|^p$$

for all $x, y, z \in A$. Then there exist unique linear mappings σ, τ and k on A and a unique ternary (σ,τ,ξ)-derivation $D : A \to X$ satisfying

$$\|g(x) - \sigma(x)\| \leq \theta \frac{2}{|2^p - 2|}\|x\|^p$$

$$\|h(x) - \tau(x)\| \leq \theta \frac{2}{|2^p - 2|}\|x\|^p$$

$$\|k(x) - \xi(x)\| \leq \theta \frac{2}{|2^p - 2|}\|x\|^p$$

$$\|D(x) - f(x)\| \leq 2\theta \frac{2^p}{|2^p - 2|}\|x\|^p$$

for all $x \in A$.

By Theorems 2.4 and 2.5 we solve the following Hyers–Ulam stability problem for functional equation (1.5).

Corollary 2.7. *Let θ be a positive real number. Suppose $f : A \to X$ is a mapping for which there exist mappings g, h, k on A whit $g(0) = h(0) = k(0) = 0$ such that*

$$\|f(\frac{\mu x + y + z}{4}) + \mu f(\frac{3\mu x - y - 4z}{4}) + f(\frac{4\mu x + 3z}{4}) - 2\mu f(x)\| \leq \theta,$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$,

$$\|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| \leq \theta$$

$$\|h(\lambda x + \lambda y) - \lambda h(x) - \lambda h(y)\| \leq \theta$$

$$\|k(\lambda x + \lambda y) - \lambda k(x) - \lambda k(y)\| \leq \theta$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$,

$$\|f([xyz]_A) - [f(x)h(y)k(z)]_X - [g(x)f(y)k(z)]_X - [g(x)h(y)f(z)]_X\| \leq \theta$$

for all $x, y, z \in A$, Then there exist unique linear mappings σ, τ and k on A and a unique ternary (σ,τ,ξ)-derivation $D : A \to X$ satisfying

$$\|g(x) - \sigma(x)\| \leq \frac{\theta}{2}$$

$$\|h(x) - \tau(x)\| \leq \frac{\theta}{2}$$

$$\|k(x) - \xi(x)\| \leq \frac{\theta}{2}$$

$$\|D(x) - f(x)\| \leq \theta$$

for all $x \in A$.

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