Int. J. Nonlinear Anal. Appl. 5 (2014) No.1, 64-70
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

A fixed point result for a new class of set-valued contractions

Asieh Sadeghi-Hafshejani[a], Alireza Amini-Harandi[a,*]

[a]*Department of Mathematics, University of Shahrekord, Shahrekord, 88186-34141, Iran.*

Dedicated to the Memory of Charalambos J. Papaioannou

(Communicated by M. B. Ghaemi)

Abstract

In this paper, we introduce a new class of set-valued contractions and obtain a fixed point theorem for such mappings in complete metric spaces. Our main result generalizes and improves many well-known fixed point theorems in the literature.

Keywords: Fixed point, Set-valued contraction.
2010 MSC: Primary 47H10.

1. Introduction and preliminaries

Let (X, d) be a metric space. We denote the family of all nonempty closed and bounded subsets of X by $CB(X)$. Let $\mathcal{H}$ denotes the Hausdorff metric on $CB(X)$ induced by d, that is,

$$\mathcal{H}(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}, \quad \text{for all} \quad A, B \in CB(X),$$

where $d(x, B) = \inf_{y\in B} d(x, y)$.
In 1989, Mizoguchi and Takahashi [10] proved the following generalization of Nadler's fixed point theorem [12].

Theorem 1.1. *([10]) Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a set-valued mapping. Assume that*

$$\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \quad \textit{for all} \quad x, y \in X,$$

where $\alpha : [0, \infty) \to [0, 1)$ satisfies $\limsup_{s\to t^+} \alpha(s) < 1$ for each $t \in [0, \infty)$. Then T has a fixed point.

*Corresponding author
Email addresses: sadeghi_math2007@yahoo.com (Asieh Sadeghi-Hafshejani), aminih_a@yahoo.com (Alireza Amini –

Received: June 2013 *Revised:* February 2014

In 2011, the second author [1] gave the following fixed point theorem for set-valued quasi-contraction mappings in metric spaces.

Theorem 1.2. *([1]) Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a k-set-valued quasi-contraction mapping with $k < \frac{1}{2}$, that is,*

$$\mathcal{H}(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for any $x, y \in X$. Then T has a fixed point.

The he raised the following question.

Question 1.3. *Does the conclusion of Theorem 1.2 remain true for any $k \in [\frac{1}{2}, 1)$?*

Up to our knowledge, this question is still open.

Theorem 1.4. *([8]) Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a set-valued mapping such that for any $x, y \in X$,*

$$\mathcal{H}(Tx, Ty) \leq k \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{1.1}$$

where $0 < k < 1$. Then T has a fixed point.

In recent years, the existence of fixed points for various set-valued contractive mappings have been studied by many authors under different conditions, see [1-12] and references therein.
In this paper, we introduce a new class of set-valued contractions and then we give a fixed point result for such mappings.

2. Main results

We denote by Φ the set of all functions $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ satisfy the following conditions:

(C_1) $\phi(t_1, t_2, t_3, t_4, t_5)$ is non-decreasing in t_2, t_3, t_4 and t_5,

(C_2) $t_{n+1} \leq \phi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0)$ implies $\sum_{n=1}^{\infty} t_n < \infty$, for each positive sequence $\{t_n\}$,

(C_3) If $t_n, s_n \to 0$ and $u_n \to \gamma$ for some $\gamma > 0$ as $n \to \infty$, then $\limsup_{n\to\infty} \phi(t_n, s_n, \gamma, u_n, t_{n+1}) < \gamma$.

Now, we are ready to state our main result.

Theorem 2.1. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a set-valued map which satisfying:*

$$\mathcal{H}(Tx, Ty) < \phi(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)), \tag{2.1}$$

for each $x, y \in X$ with $x \neq y$, where $\phi \in \Phi$. Then, T has a fixed point.

Proof . Let $x_1 \in X$ and $x_2 \in Tx_1$. If $x_1 = x_2$, then $x_1 \in T(x_1)$ and we are done. So, we may assume $x_1 \neq x_2$. Then, by (2.1) and (C_1), we have

$$\begin{aligned} d(x_2, Tx_2) &\leq \mathcal{H}(Tx_1, Tx_2) \\ &< \phi(d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), d(x_1, Tx_2), d(x_2, Tx_1)) \end{aligned}$$

$$\leq \phi(d(x_1,x_2), d(x_1,x_2), d(x_2,Tx_2), d(x_1,x_2)+d(x_2,Tx_2), 0).$$

Thus there exists $x_3 \in Tx_2$, such that

$$d(x_2,x_3) \leq \phi(d(x_1,x_2), d(x_1,x_2), d(x_2,Tx_2), d(x_1,x_2)+d(x_2,Tx_2), 0)$$

$$\leq \phi(d(x_1,x_2), d(x_1,x_2), d(x_2,x_3), d(x_1,x_2)+d(x_2,x_3), 0).$$

Therefore, by induction we can find a sequence $\{x_n\}$ in X such that for each $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$ and

$$\begin{aligned} &d(x_{n+1},x_{n+2}) \\ &\leq \phi(d(x_n,x_{n+1}), d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), d(x_n,x_{n+1})+d(x_{n+1},x_{n+2}), 0) \end{aligned} \tag{2.2}$$

Then from (C_2) and (2.2), we have $\sum_{n=1}^{\infty} d(x_n,x_{n+1}) < \infty$. Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. We show that x^* is a fixed point of T. Assume that $x^* \notin Tx^*$, that is, $d(x^*,Tx^*) > 0$. Then, by (2.1) and (C_1), we have (without loss of generality, we may assume that $x_n \neq x^*$ for each $n \in \mathbb{N}$)

$$\begin{aligned} &d(x_{n+1},Tx^*) \leq \mathcal{H}(Tx_n,Tx^*) \\ &< \phi(d(x_n,x^*), d(x_n,Tx_n), d(x^*,Tx^*), d(x_n,Tx^*), d(x^*,Tx_n)) \\ &\leq \phi(d(x_n,x^*), d(x_n,x_{n+1}), d(x^*,Tx^*), d(x_n,Tx^*), d(x^*,x_{n+1})). \end{aligned} \tag{2.3}$$

Then, from (2.3) and (C_3), we get

$$d(x^*,Tx^*) = \lim_{n\to\infty} d(x_{n+1},Tx^*)$$

$$\leq \limsup_{n\to\infty} \phi(d(x_n,x^*), d(x_n,x_{n+1}), d(x^*,Tx^*), d(x_n,Tx^*), d(x^*,x_{n+1}))$$

$$< d(x^*,Tx^*),$$

a contradiction. This implies that $d(x^*,Tx^*) = 0$, and since Tx^* is closed, then we have $x^* \in Tx^*$. □

Remark 2.2. *In 2011, Chen obtained a fixed point theorem ([6], Theorem 4) for a class of set-valued mappings satisfy a contractive condition similar to (2.1) under some different conditions. But the proof of his main result seems to be incorrect. Indeed, in page 3, line 15, he used the inequality* $d(x_{m_k},x_{n_k}) \leq \mathcal{H}(Tx_{m_k-1},Tx_{n_k-1})$, *where* $x_{m_k} \in Tx_{m_k-1}$ *and* $x_{n_k} \in Tx_{n_k-1}$, *which is false in general.*

Now, we get the following generalization of the above mentioned Theorem 1.1 of Mizoguchi and Takahashi [10], Theorem 4 of Berinde and Berinde [5] and Theorem 2.2 of Berinde [4].

Theorem 2.3. *Let* (X,d) *be a complete metric space and* $T : X \to CB(X)$ *be a set-valued mapping such that for all* $x,y \in X$,

$$\mathcal{H}(Tx,Ty) \leq$$

$$\alpha(d(x,y)).\max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\Big\} + Ld(y,Tx),$$

where $L \geq 0$ *and* $\alpha : [0,\infty) \to [0,1)$ *satisfies* $\limsup_{s\to t^+} \alpha(s) < 1$ *for all* $t \in [0,\infty)$. *Then,* T *has a fixed point in* X.

Proof . Define $\phi : \mathbb{R}^5_+ \to \mathbb{R}_+$ by

$$\phi(t_1, t_2, t_3, t_4, t_5) = \alpha(t_1). \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\} + Lt_5.$$

We claim that $\phi \in \Phi$. Indeed (C_1) obviously holds. To show (C_2), let $\{t_n\}$ be a positive sequence such that $t_{n+1} \le \phi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0)$
$= \alpha(t_n). \max\Big\{t_n, t_n, t_{n+1}, \frac{1}{2}(t_n + t_{n+1})\Big\}$
$= \alpha(t_n). \max\{t_n, t_{n+1}\}$ for all n. If for some $n_0 \in \mathbb{N}$, $t_{n_0+1} \ge t_{n_0}$, then from the above $t_{n_0+1} \le \alpha(t_{n_0})t_{n_0+1} < t_{n_0+1}$, a contradiction. Hence, $t_{n+1} \le t_n$, for all $n \in \mathbb{N}$. So

$$t_{n+1} \le \alpha(t_n)t_n, \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus $\{t_n\}_{n\in\mathbb{N}}$ is a non-negative non-increasing sequence and so, is convergent. Let $\lim_{n\to\infty} t_n = r_0$. Since $\limsup_{t\to r_0^+} \alpha(t) < 1$, there exist $0 < k < 1$ and $N \in \mathbb{N}$ such that $\alpha(t_n) < k$, for all $n \ge N$. Consequently,

$$t_{n+1} \le kt_n, \qquad n \ge N,$$

and so $\sum_{n=1}^{\infty} t_n < \infty$. To show (C_3), assume that $t_n, s_n \to 0$ and $u_n \to \gamma$ for some $\gamma > 0$ as $n \to \infty$. Then

$$\limsup_{n\to\infty} \phi(t_n, s_n, \gamma, u_n, t_{n+1})$$

$$= \limsup_{n\to\infty}\Big(\alpha(t_n). \max\{t_n, s_n, \gamma, \frac{1}{2}(u_n + t_{n+1})\} + Lt_{n+1}\Big)$$

$$= \limsup_{n\to\infty} \alpha(t_n)\gamma < \gamma.$$

Hence all of the assumptions of Theorem 2.1 are satisfied and so, T has a fixed point. □

Remark 2.4. *If we define the function $\phi : \mathbb{R}^5_+ \to \mathbb{R}_+$ by*

$$\phi(t_1, t_2, t_3, t_4, t_5) = k \max\{t_1, t_2, t_3, t_4, t_5\}, \quad \textit{where} \quad 0 < k < \frac{1}{2},$$

then $\phi \in \Phi$ and Theorem 2.1 reduces to the above mentioned Theorem 1.2 of the second author [1].

Corollary 2.5. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a set-valued mapping such that for any $x, y \in X$*

$$\mathcal{H}(Tx, Ty) \le k \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\},$$

where $k \in (0, 1)$. Then, T has a fixed point in X.

Proof . Define the function $\phi : \mathbb{R}^5_+ \to \mathbb{R}_+$ by

$$\phi(t_1, t_2, t_3, t_4, t_5) = k \max\{t_1, t_2, t_3, t_5\},$$

and apply Theorem 2.1. □

Corollary 2.6. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a set-valued mapping such that for any $x, y \in X$,*

$$\mathcal{H}(Tx, Ty) \leq k \max\Big\{d(x, y), d(x, Tx), \frac{1}{2}(d(y, Ty) + d(x, Ty)), d(y, Tx)\Big\},$$

where $0 < k < \frac{2}{3}$. Then, T has a fixed point in X.

Proof . Define $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ by

$$\phi(t_1, t_2, t_3, t_4, t_5) = k \max\{t_1, t_2, , \frac{1}{2}(t_3 + t_4), t_5\}.$$

If we show that $\phi \in \Phi$ the the conclusion follows from Theorem 2.1. The condition (C_1) and (C_3) obviously hold. To show (C_2), let $\{t_n\}$ be a positive sequence satisfying

$$\begin{aligned} t_{n+1} &\leq \phi(t_n, t_n, t_{n+1}, t_n + t_{n+1}, 0) \\ &= k \max\Big\{t_n, t_n, \frac{1}{2}(t_n + 2t_{n+1}), 0\Big\} \\ &= k \max\Big\{t_n, \frac{1}{2}(t_n + 2t_{n+1})\Big\} \qquad (2.4) \end{aligned}$$

for each $n \in \mathbb{N}$. Let $c = \max\{\frac{1}{2}, \frac{k}{2(1-k)}\}$. Then $0 < c < 1$ (note that $0 < k < \frac{2}{3}$). Now, we prove that

$$t_{n+1} \leq ct_n, \quad \text{for each} \quad n \in \mathbb{N}. \qquad (2.5)$$

If for some $n \in \mathbb{N}$, $\max\{t_n, \frac{1}{2}(t_n + 2t_{n+1})\} = t_n$, then $\frac{1}{2}(t_n + 2t_{n+1}) \leq t_n$ implies $t_{n+1} \leq \frac{1}{2}t_n \leq ct_n$. Now, if $\max\{t_n, \frac{1}{2}(t_n + 2t_{n+1})\} = \frac{1}{2}(t_n + 2t_{n+1})$, then from (2.4), we have $t_{n+1} \leq k\Big(\frac{1}{2}(t_n + 2t_{n+1})\Big)$, and so $t_{n+1} \leq k2(1 - k)t_n \leq ct_n$. From (2.5), we get $\sum_{n=1}^{\infty} t_n < \infty$. □

Now, we illustrate our main result by the following examples.

Example 2.7. *Let $X = [0, 2]$ and $d(x, y) = |x - y|$ for each $x, y \in X$. Define $T : X \to CB(X)$ by $Tx = [1, \frac{5}{4}]$ whenever $x \in [0, \frac{3}{4}]$, $Tx = [\frac{7}{8}, \frac{9}{8}]$ whenever $x \in (\frac{3}{4}, \frac{5}{4})$ and $Tx = [\frac{3}{4}, 1]$ whenever $x \in [\frac{5}{4}, 2]$. T does not satisfy (1.1) for any $0 < k < 1$ (see Example 2.1 in [11]) and so we cannot invoke the above mentioned Theorem 1.4 of Haghi et al [8] to show the existence of a fixed point for T.*
Now, we show that

$$\mathcal{H}(Tx, Ty) \leq \frac{1}{2} \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}, \qquad (2.6)$$

for each $x, y \in X$. Obviously (2.6) holds whenever either $x, y \in [0, \frac{3}{4}]$ or $x, y \in (\frac{3}{4}, \frac{5}{4})$ or $x, y \in [\frac{5}{4}, 2]$. If $x \in [0, \frac{3}{4}]$ and $y \in [\frac{5}{4}, 2]$, then $\mathcal{H}(Tx, Ty) = \frac{1}{4}$ and $d(x, y) \geq \frac{5}{4} - \frac{3}{4} = \frac{1}{2}$. Hence,

$$\begin{aligned} \mathcal{H}(Tx, Ty) &= \frac{1}{4} \\ &\leq \frac{1}{2}d(x, y) \leq \frac{1}{2} \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}. \end{aligned}$$

If $x \in [0, \frac{3}{4}]$ and $y \in (\frac{3}{4}, \frac{5}{4})$, then $\mathcal{H}(Tx, Ty) = \frac{1}{8}$ and $d(x, Tx) \geq \frac{1}{4}$. Hence,

$$\mathcal{H}(Tx, Ty) = \frac{1}{8}$$

$$\leq \frac{1}{2}d(x,Tx) \leq \frac{1}{2}\max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\}.$$

If $x \in [\frac{5}{4}, 2]$ *and* $y \in (\frac{3}{4}, \frac{5}{4})$, *then* $\mathcal{H}(Tx,Ty) = \frac{1}{8}$ *and* $d(x,Tx) \geq \frac{1}{4}$. *Hence,*

$$\mathcal{H}(Tx,Ty) = \frac{1}{8}$$

$$\leq \frac{1}{2}d(x,Tx) \leq \frac{1}{2}\max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)\}.$$

Therefore, (2.6) holds and then by Corollary 2.5, T *has a fixed point.*

Example 2.8. *Let* $X = [0,1]$ *and let* $d(x,y) = |x-y|$ *for each* $x, y \in X$. *Define* $T : X \to CB(X)$ *by* $Tx = [\frac{x}{2}, \frac{3x}{4}]$ *whenever* $x \in [0,1)$, *and* $Tx = \{1\}$ *whenever* $x = 1$. *It is straightforward to show that*

$$\mathcal{H}(Tx,Ty) \leq$$

$$\frac{1}{2}\max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\Big\} + 2d(y,Tx),$$

for each $x, y \in [0,1]$. *Then by theorem 2.3,* T *has a fixed point. Since* $H(T\frac{1}{2}, T1) = \frac{3}{4} > |\frac{1}{2} - 1|$ *then* T *does not satisfy Mizoguchi-Takahashi contractive condition. Now we show that* T *does not satisfy the contractive condition of the above mentioned Theorem 1.4 of Haghi et al. On the contrary, assume that there exists* $0 < k < 1$ *such that*

$$\mathcal{H}(Tx,Ty) \leq k\max\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

for each $x, y \in [0,1]$. *Let* $x \in [0,1)$ *and let* $y = 1$. *Then, we have*

$$\mathcal{H}(Tx,T1) = 1 - \frac{x}{2} \leq k\max\{\frac{x}{4}, 1-x, 1-\frac{3x}{4}\} = k(1-\frac{3x}{4}).$$

Thus $\frac{1-\frac{x}{2}}{1-\frac{3x}{4}} \leq k$ *for each* $x \in [0,1)$. *Letting* $x \to 0$, *we get* $1 \leq k$, *a contradiction.*

Acknowledgments
This work was supported by the Center of Excellence for Mathematics, University of Shahrekord, Iran.

References

[1] A. Amini-Harandi, *Fixed point theory for set-valued quasi-contraction maps in metric spaces*, Appl. Math. Lett., 24 (2011) 1791-1794.

[2] A. Amini-Harandi, *Fixed and coupled fixed points of a new type set-valued contractive mappings in complete metric spaces*, Fixed Point Theory and Appl., vol. 2012, 2012:215.

[3] A. Amini-Harandi, M. Fakhar, H. R. Hajisharifi and P. Petruşel, *Fixed point theorems for multi-valued contractions in distance spaces*, RACSAM., doi: 10.1007/s13398-013-0136-4.

[4] V. Berinde, *Some remarks on fixed point theorem for Ćirić-type almost contractions*, Carpathian J. Math., 25(2) (2009) 157-162.

[5] M. Berinde and V. Berinde, *On a general class of multi-valued weakly Picard mappings*, J. Math. Anal. Appl., 326 (2007) 772-782.

[6] C. M. Chen, *Some new fixed point theorems for set-valued contractions in complete metric spaces*, Fixed Point Theory Appl., vol. 2011, 2011:72.

[7] L. B. Ćirić, *Multi-valued nonlinear contraction mappings*, Nonlinear Anal., 71 (2009) 2716-2723.

[8] R. H. Haghi, Sh. Rezapour and N. Shahzad, *On fixed points of quasi-contraction type multifunctions*, Appl. Math. Lett., 25 (2012) 843–846.

[9] Z. Liu, Y. Lu and S. M. Kang, *Fixed point theorems for multi-valued contractions with w-distance*, Appl. Math. Comput., 224 (2013) 535-552.

[10] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl., 141 (1989) 177-188.

[11] B. Mohammadi, Sh. Rezapour and N. Shahzad, *Some results on fixed points of* $\alpha - \psi$*-Ćirić generalized multifunctions*, Fixed Point Theory Appl., 2013, 2013: 24.

[12] S. B. Nadler, *Multi-valued contractions mappings*, Pacific J. Math., 30 (1969) 475-488.