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A Multidimensional Discrete Hilbert-Type Inequality

Bicheng Yang

Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, P. R. China.

Dedicated to the Memory of Charalambos J. Papaioannou

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Abstract

In this paper, by using the way of weight coefficients and technique of real analysis, a multidimensional discrete Hilbert-type inequality with a best possible constant factor is given. The equivalent form, the operator expression with the norm are considered.

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1. Introduction

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+), ||f||_p = \{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}} > 0$, $||g||_q > 0$, we have the following Hardy-Hilbert's integral inequality (cf. [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1.1}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q$, $||a||_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0, ||b||_q > 0$, then we have the following discrete Hilbert's inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{1.2}$$

Email address: bcyang@gdei.edu.cn; bcyang818@163.com (Bicheng Yang)

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Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [3], [11], [17], [14], [19], [20]).

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [18] gave an extension of (1.1) at $p = q = 2$. For improving the results of [18], Yang gave some extensions of (1.1) and (1.2) as follows (cf. [17]):

If $\lambda_1, \lambda_2, \lambda \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1 - 1}dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}, \psi(x) = x^{q(1-\lambda_2)-1}, f(x), g(y) \geq 0,$

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{1.3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x > 0 (y > 0)$, then for $a_m, b_n \geq 0$,
a$\in l_{p,\phi} = \left\{ a; ||a||_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty \right\}$,
$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have (cf. [14])

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{1.4}$$

with the best possible constant factor $k(\lambda_1)$.

Clearly, for $\lambda = 1, k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (1.3) reduces to (1.1), while (1.4) reduces to (1.2). Some other results including multidimensional Hilbert-type inequalities are provided by [23]-[10].

On half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [3]. But they did not prove that the the constant factors are the best possible. However, Yang [21] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$by introducing a variable and proved that the constant factor is the best possible. In 2011 Yang [22] gave the following half-discrete Hardy-Hilbert's inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx < B(\lambda_1, \lambda_2) ||f||_{p,\phi}||a||_{q,\psi}, \tag{1.5}$$

where $\lambda_1, \lambda_2 > 0$, $0 \leq \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $\mathrm{B}(u,v) = \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1} dt (u, v > 0)$is the beta function. Zhong et *al* ([29]–[37]) investigated several half-discrete Hilbert-type inequalities with particular kernels.

Applying the way of weight functions and the techniques of discrete and integral Hilbert-type inequalities with some additional conditions on the kernel, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ is obtained as follows:

$$\int_0^\infty f(x) \sum_{n=1}^\infty k_\lambda(x,n)a_n dx < k(\lambda_1)||f||_{p,\phi}||a||_{q,\psi}, \tag{1.6}$$

which is an extension of (1.5) (see Yang and Chen [24]). At the same time, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [15].

In this paper, by using the way of weight coefficients and technique of real analysis, a multi-dimensional discrete Hilbert-type inequality with parameters and a best possible constant factor is given, which is a multidimensional extension of (1.4) for $k_\lambda(m,n)=\frac{\ln(m/n)}{m^\lambda-n^\lambda}$. The equivalent form, the operator expression with the norm are also considered.

2. Some lemmas

If $i_0, j_0 \in \mathbf{N}$($\mathbf{N}$ is the set of positive integers), $\alpha,\beta>0$, we put

$$||x||_\alpha \ :\ = \left(\sum_{k=1}^{i_0}|x_k|^\alpha\right)^{\frac{1}{\alpha}} \ (x=(x_1,\cdots,x_{i_0})\in \mathbf{R}^{i_0}), \tag{2.1}$$

$$||y||_\beta \ :\ = \left(\sum_{k=1}^{j_0}|y_k|^\beta\right)^{\frac{1}{\beta}} \ (y=(y_1,\cdots,y_{j_0})\in \mathbf{R}^{j_0}). \tag{2.2}$$

Lemma 2.1. *If $s\in\mathbf{N}$,$\gamma, M>0, \Psi(u)$ is a non-negative measurable function in $(0,1]$, and $D_M := \left\{x\in\mathbf{R}_+^s;\sum_{i=1}^s x_i^\gamma \le M^\gamma\right\}$,then we have (cf. [16])*

$$\int\cdots\int_{D_M}\Psi\left(\sum_{i=1}^s\left(\frac{x_i}{M}\right)^\gamma\right)dx_1\cdots dx_s \tag{2.3}$$

$$= \frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})}\int_0^1\Psi(u)u^{\frac{s}{\gamma}-1}du. \tag{2.4}$$

Lemma 2.2. *For $s\in\mathbf{N}$,$\gamma>0$, $\varepsilon>0$, we have*

$$\sum_m ||m||_\gamma^{-s-\varepsilon} = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})}+O(1)(\varepsilon\to 0^+). \tag{2.5}$$

Proof . For $M>s^{1/\gamma}$, we set $\Psi(u)=\begin{cases} 0, 0<u<\frac{s}{M^\gamma}, \\ (Mu^{1/\gamma})^{-s-\varepsilon}, \frac{s}{M^\gamma}\le u\le 1.\end{cases}$ Then by the decreasing property and (2.4), it follows $\sum_m ||m||_\gamma^{-s-\varepsilon} \ge \int_{\{x\in\mathbf{R}_+^s;x_i\ge 1\}}||x||_\gamma^{-s-\varepsilon}dx$

$$= \lim_{M\to\infty}\int\cdots\int_{D_M}\Psi\left(\sum_{i=1}^s\left(\frac{x_i}{M}\right)^\gamma\right)dx_1\cdots dx_s \tag{2.6}$$

$$= \lim_{M\to\infty}\frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})}\int_{s/M^\gamma}^1(Mu^{1/\gamma})^{-s-\varepsilon}u^{\frac{s}{\gamma}-1}du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})}, \tag{2.7}$$

$$\begin{aligned}\sum_m ||m||_\gamma^{-s-\varepsilon} &= a+\sum_{\{m\in\mathbf{N}^s;m_i\ge 2\}}||m||_\gamma^{-s-\varepsilon} \\ &\le a+\int_{\{u\in\mathbf{R}_+^s;u_i\ge 1\}}||u||_\gamma^{-s-\varepsilon}du = a+\frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})}.\end{aligned}$$

Then we have (2.5). □ □

Definition 2.3. *For* $\alpha, \beta > 0, \lambda > 0, 0 < \lambda_1 \leq i_0, 0 < \lambda_2 \leq j_0, \lambda_1 + \lambda_2 = \lambda,\quad m = (m_1, \cdots, m_{i_0}) \in \mathbf{N}^{i_0}$, $n = (n_1, \cdots, n_{j_0}) \in \mathbf{N}^{j_0}$, *define two weight coefficients* $w_\lambda(\lambda_2, n)$ *and* $W_\lambda(\lambda_1, m)$ *as follows:*

$$w_\lambda(\lambda_2, n) \ :\ = \sum_m \frac{\ln(||m||_\alpha/||n||_\beta)}{||m||_\alpha^\lambda - ||n||_\beta^\lambda} \frac{||n||_\beta^{\lambda_2}}{||m||_\alpha^{i_0-\lambda_1}}, \tag{2.8}$$

$$W_\lambda(\lambda_1, m) \ :\ = \sum_n \frac{\ln(||m||_\alpha/||n||_\beta)}{||m||_\alpha^\lambda - ||n||_\beta^\lambda} \frac{||m||_\alpha^{\lambda_1}}{||n||_\beta^{j_0-\lambda_2}}, \tag{2.9}$$

where, $\sum_m = \sum_{m_{i_0}=1}^\infty \cdots \sum_{m_1=1}^\infty$ *and* $\sum_n = \sum_{n_{j_0}=1}^\infty \cdots \sum_{n_1=1}^\infty$.

Lemma 2.4. *As the assumptions of Definition 1, then (i) we have*

$$w_\lambda(\lambda_2, n) \ <\ K_2(n \in \mathbf{N}^{j_0}), \tag{2.10}$$

$$W_\lambda(\lambda_1, m) \ <\ K_1(m \in \mathbf{N}^{i_0}), \tag{2.11}$$

where,

$$K_1 \ =\ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \left[\frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}\right]^2, \tag{2.12}$$

$$K_2 \ =\ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \left[\frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}\right]^2; \tag{2.13}$$

(ii) for $p > 1$, $0 < \varepsilon < p\lambda_1$, setting $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}, \widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$, we have

$$0 < \widetilde{K}_2(1 - \widetilde{\theta}_\lambda(n)) < w_\lambda(\widetilde{\lambda}_2, n), \tag{2.14}$$

where,

$$\widetilde{\theta}_\lambda(n) \ =\ \left[\frac{1}{\pi}\sin(\frac{\pi\widetilde{\lambda}_1}{\lambda})\right]^2 \int_0^{i_0^{\lambda/\alpha}/||n||_\beta^\lambda} \frac{(\ln v)v^{(\widetilde{\lambda}_1/\lambda)-1}}{v-1}dv \tag{2.15}$$

$$=\ O\left(\frac{1}{||n||_\beta^{\widetilde{\lambda}_1 - \frac{\lambda_1}{2}}}\right), \tag{2.16}$$

$$\widetilde{K}_2 \ =\ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \left[\frac{\pi}{\lambda \sin(\frac{\pi\widetilde{\lambda}_1}{\lambda})}\right]^2. \tag{2.17}$$

Proof . *Proof.* By the decreasing property and (2.4), it follows

$$w_\lambda(\lambda_2, n) < \int_{\mathbf{R}_+^{i_0}} \frac{\ln(||x||_\alpha/||n||_\beta)}{||x||_\alpha^\lambda - ||n||_\beta^\lambda} \frac{||n||_\beta^{\lambda_2}}{||x||_\alpha^{i_0-\lambda_1}}dx$$

$$= \lim_{M\to\infty} \int_{\mathbf{D}_M} \frac{\ln(M[\sum_{i=1}^{i_0}(\frac{x_i}{M})^\alpha]^{\frac{1}{\alpha}}/||n||_\beta)}{M^\lambda[\sum_{i=1}^{i_0}(\frac{x_i}{M})^\alpha]^{\frac{\lambda}{\alpha}} - ||n||_\beta^\lambda} \frac{||n||_\beta^{\lambda_2}dx_1\cdots dx_{i_0}}{M^{i_0-\lambda_1}[\sum_{i=1}^{j_0}(\frac{x_i}{M})^\alpha]^{\frac{i_0-\lambda_1}{\alpha}}}$$

$$= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{\ln(Mu^{\frac{1}{\alpha}}/||n||_\beta)}{M^\lambda u^{\frac{\lambda}{\alpha}} - ||n||_\beta^\lambda} \frac{||n||_\beta^{\lambda_2}u^{\frac{i_0}{\alpha}-1}}{M^{i_0-\lambda_1}u^{\frac{i_0-\lambda_1}{\alpha}}}du$$

$$= \lim_{M\to\infty} \frac{M^{\lambda_1}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \frac{\ln(Mu^{\frac{1}{\alpha}}/||n||_\beta)||n||_\beta^{\lambda_2}}{M^\lambda u^{\frac{\lambda}{\alpha}} - ||n||_\beta^\lambda} u^{\frac{\lambda_1}{\alpha}-1} du$$

$$u = ||n-\sigma||_\beta^\alpha M^{-\alpha} v^{\alpha/\lambda} = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\lambda^2\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_0^\infty \frac{(\ln v)v^{(\lambda_1/\lambda)-1}}{v-1} dv$$

$$= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \left[\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\right]^2 = K_2.$$

Hence, we have (2.10). By the same way, we have (2.11).

Since $\lim_{v\to0^+} \frac{v^{\lambda_1/(2\lambda)}\ln v}{v-1} = 0$, there exists a constant $M > 0$, such that

$$\frac{v^{\lambda_1/(2\lambda)}\ln v}{v-1} \le M(v \in (0, i_0^{\lambda(\frac{1}{\alpha}-\frac{1}{\beta})}]).$$

By the decreasing property and the same way as obtaining (2.7), we have

$$w_\lambda(\widetilde{\lambda}_2, n) > \int_{\{x\in\mathbf{R}_+^{i_0}; x_i\ge1\}} \frac{\ln(||x||_\alpha/||n||_\beta)}{||x||_\alpha^\lambda - ||n||_\beta^\lambda} \frac{||n||_\beta^{\widetilde{\lambda}_2}}{||x||_\alpha^{i_0-\widetilde{\lambda}_1}} dx$$

$$= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\lambda^2\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_{i_0^{\lambda/\alpha}/||n||_\beta^\lambda}^\infty \frac{(\ln v)v^{(\widetilde{\lambda}_1/\lambda)-1}}{v-1} dv = \widetilde{K}_2(1-\widetilde{\theta}_\lambda(n)) > 0,$$

$$0 < \widetilde{\theta}_\lambda(n) = \left[\frac{1}{\pi}\sin(\frac{\pi\widetilde{\lambda}_1}{\lambda})\right]^2 \int_0^{i_0^{\lambda/\alpha}/||n||_\beta^\lambda} \frac{(\ln v)v^{(\widetilde{\lambda}_1/\lambda)-1}dv}{v-1}$$

$$\le \left[\frac{1}{\pi}\sin(\frac{\pi\widetilde{\lambda}_1}{\lambda})\right]^2 M \int_0^{i_0^{\lambda/\alpha}/||n||_\beta^\lambda} v^{\frac{\widetilde{\lambda}_1}{\lambda}-\frac{\lambda_1}{2\lambda}-1} dv$$

$$= \frac{\lambda M\sin^2(\frac{\pi\widetilde{\lambda}_1}{\lambda})}{(\widetilde{\lambda}_1-\frac{\lambda_1}{2})\pi^2} \frac{i_0^{(\widetilde{\lambda}_1-\frac{\lambda_1}{2})/\alpha}}{||n||_\beta^{\widetilde{\lambda}_1-\frac{\lambda_1}{2}}}.$$

The lemma is proved. □ □

3. Main Results and Operator Expressions

Setting $\Phi(m) := ||m||_\alpha^{p(i_0-\lambda_1)-i_0} (m \in \mathbf{N}^{i_0})$ and $\Psi(n) := ||n||_\beta^{q(j_0-\lambda_2)-j_0}$ $(n \in \mathbf{N}^{j_0})$, we have

Theorem 3.1. *If $\alpha, \beta > 0$, $\lambda > 0, 0 < \lambda_1 \le i_0, 0 < \lambda_2 \le j_0, \lambda_1+\lambda_2 = \lambda$, then for $p > 1, \frac{1}{p}+\frac{1}{q} = 1$ $a_m, b_n \ge 0, 0 < ||a||_{p,\Phi}, ||b||_{q,\Psi} < \infty$, we have the following inequality*

$$I := \sum_n \sum_m \frac{\ln(||m||_\alpha/||n||_\beta)}{||m||_\alpha^\lambda - ||n||_\beta^\lambda} a_m b_n < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi} ||b||_{q,\Psi}, \tag{3.1}$$

where the constant factor

$$K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \left[\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\right]^2 \tag{3.2}$$

is the best possible.

Proof . By Hölder's inequality (cf. [8]), we have

$$I = \sum_n \sum_m \frac{\ln(||m||_\alpha/||n||_\beta)}{||m||_\alpha^\lambda - ||n||_\beta^\lambda} \left[\frac{||m||_\alpha^{(i_0-\lambda_1)/q}}{||n||_\beta^{(j_0-\lambda_2)/p}} a_m\right] \left[\frac{||n||_\beta^{(j_0-\lambda_2)/p}}{||m||_\alpha^{(i_0-\lambda_1)/q}} b_n\right]$$

$$\leq \left\{\sum_m W_\lambda(\lambda_1, m)||m||_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p\right\}^{\frac{1}{p}}$$

$$\times \left\{\sum_n w_\lambda(\lambda_2, n)||n||_\beta^{q(j_0-\lambda_2)-j_0} b_n^q\right\}^{\frac{1}{q}}.$$

Then by (2.10) and (2.11), we have (3.1).

For $0 < \varepsilon < p\lambda_1$, $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$, we set

$$\widetilde{a}_m = ||m||_\alpha^{-i_0+\lambda_1-\frac{\varepsilon}{p}}, \widetilde{b}_n = ||n||_\beta^{-j_0+\lambda_2-\frac{\varepsilon}{q}} (m \in \mathbf{N}^{i_0}, n \in \mathbf{N}^{j_0}).$$

Then by (2.5) and (2.14), we obtain

$$||\widetilde{a}||_{p,\Phi}||\widetilde{b}||_{q,\Psi} = \left\{\sum_m ||m||_\alpha^{p(i_0-\lambda_1)-i_0}\widetilde{a}_m^p\right\}^{\frac{1}{p}} \left\{\sum_n ||n||_\beta^{q(j_0-\lambda_2)-j_0}\widetilde{b}_n^q\right\}^{\frac{1}{q}}$$

$$= \left\{\sum_m ||m||_\alpha^{-i_0-\varepsilon}\right\}^{\frac{1}{p}} \left\{\sum_n ||n||_\beta^{-j_0-\varepsilon}\right\}^{\frac{1}{q}} \quad (3.3)$$

$$= \frac{1}{\varepsilon}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1)\right]^{\frac{1}{p}} \quad (3.4)$$

$$\times \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}, \quad (3.5)$$

$$\widetilde{I} \ : \ = \sum_n \left[\sum_m \frac{\ln(||m||_\alpha/||n||_\beta)}{||m||_\alpha^\lambda - ||n||_\beta^\lambda}\widetilde{a}_m\right]\widetilde{b}_n = \sum_n w_\lambda(\widetilde{\lambda}_2, n)||n||_\beta^{-j_0-\varepsilon} \quad (3.6)$$

$$> \widetilde{K}_2 \sum_n \left(1 - O(\frac{1}{||n||_\beta^{\widetilde{\lambda}_1-\frac{\lambda_1}{2}}})\right)||n||_\beta^{-j_0-\varepsilon} \quad (3.7)$$

$$= \widetilde{K}_2\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \widetilde{O}(1) - O(1)\right]. \quad (3.8)$$

If there exists a constant $K \leq K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$, such that (3.1) is valid as we replace $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$ by K, then we have $\widetilde{K}_2\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\widetilde{O}(1) - \varepsilon O(1)\right] < \varepsilon\widetilde{I}$

$$< \varepsilon K||\widetilde{a}||_{p,\varphi}||\widetilde{b}||_{q,\psi} = K\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1)\right]^{\frac{1}{p}}$$

$$\times \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\widetilde{O}(1)\right]^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, we find $\Gamma^{j_0}(\frac{1}{\beta})\frac{}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}[\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}]^2 \leq K[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}]^{\frac{1}{p}}[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}]^{\frac{1}{q}},}$ and then $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}} \leq K$.

Hence, $K = K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$ is the best possible constant factor of (3.1). □ □

Theorem 3.2. *As the assumptions of Theorem 1, for* $0 < ||a||_{p,\Phi} < \infty$, *we have the following inequality with the best constant factor* $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$:

$$J := \left\{\sum_n ||n||_\beta^{p\lambda_2-j_0}\left(\sum_m \frac{\ln(||m||_\alpha/||n||_\beta)a_m}{||m||_\alpha^\lambda - ||n||_\beta^\lambda}\right)^p\right\}^{\frac{1}{p}} < K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}||a||_{p,\Phi}, \tag{3.9}$$

which is equivalent to (3.1).

Proof . We set b_n as follows: $\mathrm{b}_n := ||n||_\beta^{p\lambda_2-j_0}\left(\sum_m \frac{\ln(||m||_\alpha/||n||_\beta)a_m}{||m||_\alpha^\lambda-||n||_\beta^\lambda}\right)^{p-1}, n \in \mathbf{N}^{j_0}$.Then it follows $J^p = ||b||_{q,\Psi}^q$. If $J = 0$, then (3.9) is trivially valid for $0 < ||a||_{p,\Phi} < \infty$; if $J = \infty$, then it is impossible since the right hand side of (3.9) is finite. Suppose that $0 < J < \infty$. Then by (3.1), we find ——b——$_{q,\Psi}^q = J^p = I < K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}||a||_{p,\Phi}||b||_{q,\Psi}$,namely, $||b||_{q,\Psi}^{q-1} = J < K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}||a||_{p,\Phi}$, and then (3.9) follows.

On the other hand, assuming that (3.9) is valid, by Hölder's inequality, we have

$$I = \sum_n (\Psi(n))^{\frac{-1}{q}}\left[\sum_m \frac{\ln(||m||_\alpha/||n||_\beta)a_m}{||m||_\alpha^\lambda - ||n||_\beta^\lambda}\right][(\Psi(n))^{\frac{1}{q}}b_n] \leq J||b||_{q,\Psi}. \tag{3.10}$$

Then by (3.9), we have (3.1). Hence (3.9) and (3.1) are equivalent.

By the equivalency, the constant factor $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$ in (3.9) is the best possible. Otherwise, we can come to a contradiction by (3.10) that the constant factor $K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}$ in (3.1) is not the best possible. □ □

For $p > 1$, we define two real weight normal discrete spaces $\mathbf{l}_{p,\varphi}$ and $\mathbf{l}_{q,\psi}$ as follows:

$$\mathbf{l}_{p,\varphi} \ : \ = \left\{a = \{a_m\}; ||a||_{p,\Phi} = \{\sum_m \Phi(m)a_m^p\}^{\frac{1}{p}} < \infty\right\},$$
$$\mathbf{l}_{q,\psi} \ : \ = \left\{b = \{b_n\}; ||b||_{q,\Psi} = \{\sum_n \Psi(n)b_n^q\}^{\frac{1}{q}} < \infty\right\}.$$

As the assumptions of Theorem 1, in view of $J < K_1^{\frac{1}{p}}K_2^{\frac{1}{q}}||a||_{p,\Phi}$, we have the following definition:

Definition 3.3. *Define a multidimensional Hilbert-type operator* $T : \mathbf{l}_{p,\Phi} \to \mathbf{l}_{p,\Psi^{1-p}}$ *as follows: For* $a \in \mathbf{l}_{p,\Phi}$, *there exists an unique representation* $Ta \in \mathbf{l}_{p,\Psi^{1-p}}$, *satisfying*

$$Ta(n) := \sum_m \frac{\ln(||m||_\alpha/||n||_\beta)a_m}{||m||_\alpha^\lambda - ||n||_\beta^\lambda}(n \in \mathbf{N}^{j_0}). \tag{3.11}$$

For $b \in \mathbf{l}_{q,\Psi}$, *we define the following formal inner product of* Ta *and* b *as follows:*

$$(Ta, b) := \sum_n \sum_m \frac{\ln(||m||_\alpha/||n||_\beta)a_m}{||m||_\alpha^\lambda - ||n||_\beta^\lambda}b_n. \tag{3.12}$$

Then by Theorem 1 and Theorem 2, for $0 < ||a||_{p,\varphi}, ||b||_{q,\psi} < \infty$, we have the following equivalent inequalities:

$$(Ta, b) < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi} ||b||_{q,\Psi}, \quad (3.13)$$

$$||Ta||_{p,\Psi^{1-p}} < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} ||a||_{p,\Phi}. \quad (3.14)$$

It follows that T is bounded with

$$||T|| := \sup_{a(\neq\theta)\in \mathbf{l}_{p,\Phi}} \frac{||Ta||_{p,\Psi^{1-p}}}{||a||_{p,\Phi}} \le K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}. \quad (3.15)$$

Since the constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ in (3.14) is the best possible, we have

Corollary 3.4. *If T is defined by Definition 2, then it follows*

$$||T|| = K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = [\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}]^{\frac{1}{p}} [\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}]^{\frac{1}{q}} [\frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}]^2. \quad (3.16)$$

Remark 3.5. *For $i_0 = j_0 = 1$ in (3.1), we have inequality*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\ln(m/n)}{m^\lambda - n^\lambda} a_m b_n < \left[\frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}\right]^2 ||a||_{p,\phi} ||b||_{q,\psi}. \quad (3.17)$$

Hence, (3.1) is a multidimensional extension of (1.4) for $k_\lambda(m,n) = \frac{\ln(m/n)}{m^\lambda - n^\lambda}$.

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