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New results for fractional evolution equations using Banach fixed point theorem

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Abstract

In this paper, we study the existence of solutions for fractional evolution equations with nonlocal conditions. These results are obtained using Banach contraction fixed point theorem. Other results are also presented using Krasnoselskii theorem.

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1. Introduction

The theory of fractional differential equations is a new branch of mathematics by which many physical phenomena in various fields of science and engineering can be modeled. Significant development in this area has been achieved for the last few years. For details, we refer to [4, 6]. Moreover, the study of fractional evolution equations is also of great importance [1, 3, 8, 9].
The main aim of this paper is to establish new existence results for evolution equations in Banach spaces by using the fractional derivatives and fixed point theorems. So, let us consider the following problem:

$$\begin{aligned} D^{\alpha}x\left(t\right) &= Ax\left(t\right)+f\left(t,x\left(t\right)\right), \quad t\neq t_i, t\in J, 0<\alpha<1, \\ \Delta x|_{t=t_i} &= I_i(x(t_i)), i=1,2,...,m, x\left(0\right)=g\left(x\right), \end{aligned} \tag{1.1}$$

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where D^α is the Caputo derivative, $J=[0,b]$, $A:D(A)\to X$ is a nondensely defined operator, X is a real Banach space, $(t_i)_{i=1,\ldots,m}$ are fixed points, with $0<t_1<t_2<\ldots<t_i<\ldots<t_m<1$, m is fixed in $\mathbb{N}^*$, and $\Delta x|_{t=t_i}=x(t_i^+)-x\left(t_i^-\right)$, such that $x(t_i^+)$ and $x\left(t_i^-\right)$ represent the right-hand limit and left-hand limit of $x(t)$ at $t=t_i$, respectively, and f,g and I_i are appropriate functions to be specified later.

2. Preliminaries

We introduce some definitions and properties which will be used in this paper:

Definition 2.1. *A real valued function f is said to be in the space $C_\mu([0,\infty[),\mu\in\mathbb{R}$ if there exists a real number $r>\mu$, such that $f(t)=t^r f_1(t)$, where $f_1\in C([0,\infty))$.*

Definition 2.2. *A function f is said to be in the space $C_\mu^n([0,\infty[),n\in\mathbb{N}$, if $f^{(n)}\in C_\mu([0,\infty[)$.*

Definition 2.3. *The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$, for a function $f\in C_\mu([0,\infty[),\mu\geq -1$, is defined as*

$$\begin{aligned} J^\alpha f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau; \quad \alpha>0, t>0 \\ J^0 f(t) &= f(t). \end{aligned} \tag{2.1}$$

The fractional derivative of $f\in C_{-1}^n$ in the sense of Caputo is defined as

$$D^\alpha f(t)=\begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, & n-1<\alpha<n, n\in N^*, \\ \frac{d^n}{dt^n}f(t), & \alpha=n. \end{cases} \tag{2.2}$$

For more details, see [2, 10].
We need the following lemma [5]:

Lemma 2.4. *For $\alpha>0$, the general solution of the fractional differential equation*

$$D^\alpha x(t)=0$$

is given by

$$x(t)=c_0+c_1t+c_2t^2+\ldots+c_{n-1}t^{n-1},$$

where $c_i\in\mathbb{R}$ are arbitrary real constants for $i=0,1,2,\ldots,n-1$, $n=[\alpha]+1$.

We prove also the following auxiliary result:

Lemma 2.5. *Let $f(t,x)\in X$ and $A:D(A)\to X$ is a nondensely defined operator. A solution of the problem (1.1) is given by*

$$x(t)=\begin{cases} g(x)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}Ax(\tau)\,d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,x)\,d\tau, t\in[0,t_1], \\ g(x)+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^i\int_{t_{j-1}}^{t_j}(t_j-\tau)^{\alpha-1}f(\tau,x)\,d\tau \\ +\frac{1}{\Gamma(\alpha)}\int_{t_i}^t (t-\tau)^{\alpha-1}f(\tau,x)\,d\tau+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^i\int_{t_{j-1}}^{t_j}(t-\tau)^{\alpha-1}Ax(\tau)\,d\tau \\ +\frac{1}{\Gamma(\alpha)}\int_{t_i}^t (t-\tau)^{\alpha-1}Ax(\tau)\,d\tau+\sum_{j=1}^i I_j(x(t_j)), \\ t\in[t_i,t_{i+1}], i=1,\ldots m, 0<\alpha<1. \end{cases} \tag{2.3}$$

Proof . Assume x satisfies (1.1). If $t\in[0,t_1]$, then $D^{\alpha}x\,(t)=Ax\,(t)+f(t,x\,(t))$. Using Lemma 2.4, we can write

$$x\,(t)=g\,(x)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}Ax\,(\tau)\,d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau.$$

If $t\in[t_1,t_2]$, then thanks to Lemma 2.4, we get

$$\begin{aligned}
x\,(t) &= x\left(t_1^+\right)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-\tau)^{\alpha-1}Ax\,(\tau)\,d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau\\
&= \Delta x\mid_{t=t_1}+x\left(t_1^-\right)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-\tau)^{\alpha-1}Ax\,(\tau)\,d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau\\
&= I_1\left(x\left(t_1^-\right)\right)+g\,(x)+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t-\tau)^{\alpha-1}Ax\,(\tau)\,d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-\tau)^{\alpha-1}Ax\,(\tau)\,d\tau.
\end{aligned}$$

If $t\in[t_2,t_3]$, then by Lemma 2.4 again, we have

$$\begin{aligned}
x\,(t) &= x\left(t_2^+\right)+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t(t-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau\\
&= \Delta x\mid_{t=t_2}+x\left(t_2^-\right)+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t(t-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau\\
&= I_2\left(x\left(t_2^-\right)\right)+I_1\left(x\left(t_1^-\right)\right)+g\,(x)+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t(t-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t-\tau)^{\alpha-1}Ax\,(\tau)\,d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t-\tau)^{\alpha-1}Ax\,(\tau)\,d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t2}^t(t-\tau)^{\alpha-1}Ax\,(\tau)\,d\tau.
\end{aligned}$$

And if $t\in[t_i,t_{i+1}]$, $i=1,..m$, with the same arguments as before, we obtain the second quantity in (2.3). Lemma 2.5 is thus proved. □

To establish the existence of solutions of (1.1), we need following conditions:

$(\mathbf{H}_A)$: $A(t)$ is a bounded linear operator on $D\,(A)\subset X$, the function $t\to A(t)$ is continuous in the uniform operator topology, and

$$\max_{t\in J}\|A(t)\|=C.$$

$(\mathbf{H}_1)$: The nonlinear function $f:J\times X\to X$ is continuous and there exist constants $\beta>0$, ß> 0, such that

$$||f(t,x(t))-f(t,y(t))||\leq\beta||x-y||;\ x,y\in X,t\in J$$

and

$$ß=\max_{t\in J}\|f(t,0)\|\,.$$

$(\mathbf{H}_2)$: The functions $I_i : X\to X$ are continuous and there exist constants ϖ_i, such that

$$\| I_i(x)-I_i(y)\|\leq\varpi_i\| x-y\|, i=1,2,...,m,\ \text{for each } x,y\in X,$$

and $\omega=\|I_i(0)\|$.

$(\mathbf{H}_3)$: The function $g : X\to X$ is continuous, and there exist $\lambda>0$ and $M>0$, such that

$$\| g(x)-g(y)\|\leq\lambda||x-y||,\ \text{for } x,y\in X$$

and $M=\|g(0)\|$.

$(\mathbf{H}_4)$: There exists a positive constant $r>0$ such that

$$(m+1)\,\gamma\,(\beta r+ß+Cr)+\sum_{i=1}^{m}\varpi_i r+m\omega+\lambda r+M\leq r,$$

where $\gamma=\frac{b^\alpha}{\Gamma(\alpha+1)}$.

3. Main Results

Our first result is the following theorem:

Theorem 3.1. *If the hypotheses* $(\mathbf{H}_A)$, $(\mathbf{H}_j)_{j=\overline{1,4}}$ *and*

$$0\leq\Lambda:=(m+1)\,\gamma\,(C+\beta)+\sum_{i=1}^{m}\varpi_i+\lambda<1$$

are satisfied, then (1.1) *has a unique solution on* J.

Proof . Let us take $B_r=\{x\in X:\|x\|\leq r\}$. We define the operator T as follows:

$$Tx\,(t)=\begin{cases} g\,(x)+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t_j-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau \\ +\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}f\,(\tau,x)\,d\tau+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t-\tau)^{\alpha-1}Ax\,(\tau)\,d\tau \\ +\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}Ax\,(\tau)\,d\tau+\sum_{j=1}^{i}I_j(x(t_j)). \end{cases}\tag{3.1}$$

(1*) We shall prove that $T(B_r)\subset B_r$. For $x\in B_r$, and for any $t\in J$, we have:

$$\|Tx(t)\| \leq \|g(x)\| + \left\|\tfrac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t_j-\tau)^{\alpha-1}f(\tau,x)\,d\tau\right\|$$
$$+\left\|\tfrac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}f(\tau,x)\,d\tau\right\| + \left\|\tfrac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t-\tau)^{\alpha-1}Ax(\tau)\,d\tau\right\| \qquad (3.2)$$
$$+\left\|\tfrac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}Ax(\tau)\,d\tau\right\| + \left\|\sum_{j=1}^{i}I_j(x(t_j))\right\|.$$

Then, we can write

$$\|Tx(t)\| \leq \|g(x)-g(0)\| + \|g(0)\| + \left\|\tfrac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t_j-\tau)^{\alpha-1}\left[f(\tau,x)-f(\tau,0)\right]d\tau\right\|$$
$$+\left\|\tfrac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t_j-\tau)^{\alpha-1}f(\tau,0)\,d\tau\right\| + \left\|\tfrac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\left[f(\tau,x)-f(\tau,0)\right]d\tau\right\|$$
$$+\left\|\tfrac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}f(\tau,0)\,d\tau\right\| + \left\|\tfrac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t-\tau)^{\alpha-1}Ax(\tau)\,d\tau\right\|$$
$$+\left\|\tfrac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}Ax(\tau)\,d\tau\right\| + \left\|\sum_{j=1}^{i}\left[I_j(x(t_j))-I_j(0)\right]\right\| + \left\|\sum_{j=1}^{i}I_j(0)\right\|.$$

Using (H_A), (H_1), (H_2) and (H_3), we obtain

$$\|Tx\| \leq \lambda\|x\| + M + \frac{(m+1)\beta b^{\alpha}}{\Gamma(\alpha+1)}\|x\| + \frac{(m+1)\,ß b^{\alpha}}{\Gamma(\alpha+1)} + \frac{(m+1)Cb^{\alpha}}{\Gamma(\alpha+1)}\|x\|$$
$$+\sum_{i=1}^{m}\varpi_i\|x\| + m\omega. \qquad (3.3)$$

Therefore,

$$\|Tx\| \leq \lambda r + M + (m+1)\beta\gamma r + (m+1)\,ß\gamma + (m+1)C\gamma r$$
$$+\sum_{i=1}^{m}\varpi_i r + m\omega. \qquad (3.4)$$

Thanks to (H_4), we obtain

$$\|Tx\| \leq r. \qquad (3.5)$$

Then $T(B_r) \subset B_r$. Hence, the operator Φ maps B_r into itself.

(2*) Now we prove that T is a contraction mapping on B_r. Let x and $y \in B_r$, then for any $t \in J$, we can write:

$$\|Tx(t) - Ty(t)\| \le \|g(x) - g(y)\|$$
$$+ \left\| \tfrac{1}{\Gamma(\alpha)} \textstyle\sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} (t_j - \tau)^{\alpha-1} \left[f(\tau, x) - f(\tau, y) \right] d\tau \right\|$$
$$+ \left\| \tfrac{1}{\Gamma(\alpha)} \textstyle\int_{t_i}^{t} (t - \tau)^{\alpha-1} \left[f(\tau, x) - f(\tau, y) \right] d\tau \right\| \tag{3.6}$$
$$+ \left\| \tfrac{1}{\Gamma(\alpha)} \textstyle\sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} (t - \tau)^{\alpha-1} A\left(x(\tau) - y(\tau)\right) d\tau \right\|$$
$$+ \left\| \tfrac{1}{\Gamma(\alpha)} \textstyle\int_{t_i}^{t} (t - \tau)^{\alpha-1} A\left(x(\tau) - y(\tau)\right) d\tau \right\| + \left\| \textstyle\sum_{j=1}^{i} I_j(x(t_j)) - I_j(y(t_j)) \right\|.$$

Therefore,

$$\begin{aligned} & \|Tx(t) - Ty(t)\| \\ \le \; & \| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(Ax(\tau) - Ay(\tau) \right) d\tau \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(f(\tau, x(\tau)) - f(\tau, y(\tau)) \right) d\tau \\ & + \sum_{t_i < t} \left(I_i(x(t_i)) - I_i(y(t_i)) \right) + \left(g(x) - g(y) \right) \|. \end{aligned} \tag{3.7}$$

This implies that

$$\|Tx - Ty\| \le \left((m+1)(C\gamma + \beta\gamma) + \sum_{i=1}^{m} \varpi_i + \lambda \right) \|x - y\|,$$

and consequently,

$$\|Tx - Ty\| \le \Lambda \|x - y\|. \tag{3.8}$$

Since $0 \le \Lambda < 1$, then T is a contraction, hence by Banach fixed point theorem, there exists a unique fixed point $x \in B_r$ such that $Tx = x$. Theorem 3.1 is thus proved. □
Our second main result is based on the following fixed point theorem [7]:

Theorem 3.2. *(Krasnoselskii Fixed Point Theorem) Let S be a closed convex and nonempty subset of a Banach space X. Let P, Q be the operators such that:*

$$(i)\; Px + Qy \in S, \;\; \textit{whenever } x, y \in S,$$

(ii) P is compact and continuous,
(iii) Q is a contraction mapping.
Then there exists x^*, such that

$$x^* = Px^* + Qx^*.$$

We prove the following theorem:

Theorem 3.3. *Suppose that the hypotheses* $(\mathbf{H}_A)$ *and* $(\mathbf{H}_j)_{j=\overline{1,4}}$ *are satisfied. If the quantity* $\Upsilon := (m+1)\gamma(\beta + C) < 1$, *then the problem* (1.1) *has at least a solution on* J.

Proof . Let us define the operators R and S as:

$$Rx(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t_j-\tau)^{\alpha-1}f(\tau,x)\,d\tau\\ +\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}f(\tau,x)\,d\tau\\ +\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t-\tau)^{\alpha-1}Ax(\tau)\,d\tau\\ +\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}Ax(\tau)\,d\tau\end{cases} \tag{3.9}$$

and

$$Sx(t)=g(x)+\sum_{t_i<t}I_i(x(t_i)). \tag{3.10}$$

For x, $y\in B_r$, we have

$$\|Rx+Sy\|\le\|Rx\|+\|Sy\|. \tag{3.11}$$

So for every $t\in J$, we can write:

$$\begin{aligned}\|Rx(t)+Sy(t)\|\le&\left\|\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t_j-\tau)^{\alpha-1}f(\tau,x)\,d\tau\right\|\\ &+\left\|\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}f(\tau,x)\,d\tau\right\|\\ &+\left\|\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(t-\tau)^{\alpha-1}Ax(\tau)\,d\tau\right\|\\ &+\left\|\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}Ax(\tau)\,d\tau\right\|\\ &+\|g(y)\|+\left\|\sum_{t_i<1}I_i(y(t_i))\right\|.\end{aligned} \tag{3.12}$$

Using $(H_A),(H_1),(H_2)$ and (H_3), we get

$$\|Rx+Sy\|\le(m+1)\gamma(Cr+\beta r+\text{ß})+\lambda r+M+\sum_{i=1}^{m}\varpi_i r+m\omega. \tag{3.13}$$

By (H_4), we obtain

$$\|Rx+Sy\|\le r.$$

Hence $Rx+Sy\in B_r$.
On other hand, we have

$$
\begin{aligned}
&\|Rx(t)-Ry(t)\| = \| \tfrac{1}{\Gamma(\alpha)} \textstyle\sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} (t_j-\tau)^{\alpha-1}\left[f(\tau,x)-f(\tau,y)\right] d\tau \\
&+\tfrac{1}{\Gamma(\alpha)} \textstyle\int_{t_i}^{t} (t-\tau)^{\alpha-1}\left[f(\tau,x)-f(\tau,x)\right] d\tau \\
&+\tfrac{1}{\Gamma(\alpha)} \textstyle\sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} (t-\tau)^{\alpha-1}\left[Ax(\tau)-Ay(\tau)\right] d\tau \\
&+\tfrac{1}{\Gamma(\alpha)} \textstyle\int_{t_i}^{t} (t-\tau)^{\alpha-1}\left[Ax(\tau)-Ay(\tau)\, d\tau\right] \|.
\end{aligned}
\tag{3.14}
$$

Hence,

$$
\begin{aligned}
\|Rx-Ry\| &\leq ((m+1)\,\gamma\,(\beta+C))\,\|x-y\| \\
&\leq \Upsilon\,\|x-y\|\,.
\end{aligned}
\tag{3.15}
$$

Since $\Upsilon < 1$, then the operator R is a contraction.

Now, we shall prove that the operator S is completely continuous from B_r to B_r.

Since $I_i \in C(X,X)$, then S is continuous on B_r. So, we prove that S is relatively compact as well as equi-continuous on X for every $t \in J$. To prove the compactness of S, we shall prove that $S(B_r) \subseteq X$ is equi-continuous and $S(B_r)(t)$ is relatively compact for any $r > 0, t \in J$.

Let $x \in B_r$ and $t+h \in J$, then we can write

$$
\|Sx(t+h)-Sx(t)\| \leq \|g(x+h)-g(x)\| + \left\| \sum_{0<t_i<t+h} I_i(x(t_i)) - \sum_{t_i<t} I_i(x(t_i)) \right\|. \tag{3.16}
$$

The inequality (3.16) is independent of x, thus S is equi-continous and as $h \to 0$ the right hand side of the above inequality tends to zero; so $S(B_r)$ is relatively compact, and S is compact. Finally by Krasnoselskii theorem, there exists at least a solution of (1.1).

□

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