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On Invariant Sets Topology

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Abstract

In this paper we introduce and study a new topology related to a self mapping on a nonempty set. Let X be a nonempty set and let f be a self mapping on X. Then the set of all invariant subsets of X related to f, i.e. $\tau_f := \{A \subseteq X : f(A) \subseteq A\} \subseteq \mathcal{P}(X)$ is a topology on X. Among other things, we find the smallest open sets contains a point $x \in X$. Moreover, we find the relations between f and τ_f. For instance, we find the conditions on f to show that whenever τ_f is T_0, T_1 or T_2.

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1. Introduction

Suppose that f is a function from a nonempty set X into itself. Define $\tau_f = \{A \subseteq X : f(A) \subseteq A\} \subseteq \mathcal{P}(X)$ which is called the set of *invariant sets related to the function* f. The paper is based on some elementary results about τ_f which is a topology on X. This topology have some interesting results with new applications.

First it is shown that (X, τ_f) is a topological space. Also some basic properties are investigated. Second using the concept of *orbit* for $x \in X$, we introduce the separation concepts of T_0, T_1 and T_2. In this section we give some conditions for topological space (X, τ_f) and f which are equivalent to the separation concepts of T_0, T_1 and T_2. In the last section as an application we prove some fixed point theorems related to the separation concepts and orbits.

It is known that for function f, an element $x \in X$ is a *fixed point* if $f(x) = x$. The proof of some results in the following section are easy and then we can omit them. We mainly used [2] and [3] in this paper.

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2. Basic Results

Proposition 2.1. *(X, τ_f) is a topological space.*

Proof. It is easy to see that $\emptyset, X \in \tau_f$.

(a) For $U_\alpha \in \tau_f$ with $\alpha \in I$, $f(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f(U_\alpha) \subseteq \bigcup_\alpha (U_\alpha)$.

(b) For $i = 1, ..., n$ and $U_i \in \tau_f$, $f(\bigcap_{i=1}^n U_i) \subseteq \bigcap_{i=1}^n f(U_i) \subseteq \bigcap_{i=1}^n U_i$.

Proposition 2.2. In topological space (X, τ_f), every intersection (resp. union) of closed sets is closed.

Proposition 2.3. Consider the topological space (X, τ_f). Then

(a) if $G \in \tau_f$, then $G \subseteq f^{-1}(G)$.

(b) if $F^c \in \tau_f$, then $f^{-1}(F) \subseteq F$.

Proposition 2.4. *Consider the topological space (X, τ_f). For any set $A \subseteq X$ we have $x \in \bar{A}$ if for any $G \in \tau_f$ which contains x, there is $y \in A$ such that $f(y) \in G$. The converse is also true for a set A if any $y \in A$ be a fixed point of f.*

Proof. Suppose that $x \in \overline{A}$ and $x \in G \in \tau_f$. If for any $y \in A$, $f(y) \notin G$ then $f(y) \in G^c$. Using Proposition 2.3 we have $y \in f^{-1}(G^c) \subseteq G^c$. So $x \in \overline{A} \subseteq G^c$ which contradicts $x \in G$.
For the converse suppose that $x \notin \overline{A}$. Hence according to the definition of closure in topological spaces, there is a closed set F such that $A \subseteq F$ and $x \notin F$. So $A \cap F^c = \emptyset$ and F^c is an open set including x. This implies that $y \notin F^c$ for any $y \in A$. But this means that $f(y) \notin F^c$ which is a contradiction.

The continuity of a function $g : (X, \tau_f) \to (X, \tau_f)$ has the usual definition. Indeed for any $x \in X$, the relation $f(x) \in B \in \tau_f$ necessitates that there is $A \in \tau_f$ including x such that $g(A) \subseteq B$.

Proposition 2.5. *The function $f : (X, \tau_f) \to (X, \tau_f)$ is continuous.*

Proof. Suppose that $f(x) \in B \in \tau_f$. So $x \in f^{-1}(B)$. From Proposition 2.3, $f(f^{-1}(B)) \subseteq B \subseteq f^{-1}(B)$. Hence $f^{-1}(B) \in \tau_f$. Now set $A = f^{-1}(B)$. Then $x \in A$ and $f(A) = f(f^{-1}(B)) \subseteq B$.

Proposition 2.6. *Consider two functions $f, g : X \to X$. If the function $h : (X, \tau_f) \to (X, \tau_g)$ is continuous, then the relation $h(x) \in B \in \tau_g$ implies that there is $A \in \tau_f$ including x such that $(g \circ h \circ f)(A) \subseteq B$.*

Proof. Consider $x \in X$ with $h(x) \in B \in \tau_g$. By continuity of h, there is $A \in \tau_f$ including x such that $h(A) \subseteq B$. Since $g(B) \subseteq B$ and $f(A) \subseteq A$, then

$$g \circ h \circ f(A) \subseteq g \circ h(A) \subseteq g(B) \subseteq B.$$

Corollary 2.7. *Suppose that $g : (X, \tau_f) \to (X, \tau_f)$ is continuous. for any $x \in X$ the relation $f(x) \in B \in \tau_f$ implies that there is $A \in \tau_f$ including x such that $(g \circ f)(A) \subseteq B$.*

Definition 2.8. *Consider the topological space (X, τ_f). The orbit of $x \in X$ is defined as the following:*

$$O(x) = \{f^n(x);\ n = 0, 1, 2, \cdots\}.$$

Since $f(O(x)) \subseteq O(x)$, then $O(x) \in \tau_f$. This means that for any $x \in X$, $O(x)$ is an open set including x. Also $O(x)$ is included in any open set including x.

Proposition 2.9. *Consider the topological space (X, τ_f).*

(a) For any $x \in X$, $O(x)$ is the smallest open set including x. That is $O(x)$ is intersection of all open sets including x.

(b) The set $\{O(x); x \in X\}$ is a base for τ_f.

(c) If $x \in X$ is a fixed point of the function f, then $O(x) = \{x\}$.

Proof. It is easy consequence of Definition 2.8.

Definition 2.10. *Consider the topological space (X, τ_f). For any set $A \subseteq X$ we define the orbit hull of A as the following:*

$$O_h(A) = \bigcup_{x \in A} O(x).$$

Proposition 2.11. *Consider the topological space (X, τ_f) and two set $A, B \subseteq X$. Then*

(a) $A \subseteq O_h(A)$.

(b) $O_h(A) \in \tau_f$.

(c) If $A \subseteq B$, then $O_h(A) \subseteq O_h(B)$.

(d) If $A \in \tau_f$, then $O_h(A) = A$.

(e) $O_h(A) = \bigcap\{G; A \subseteq G \text{ and } G \in \tau_f\}$.

Proof. (a), (b) and (c) are obvious. For assertion (d), Since $A \in \tau_f$, then for any $x \in A$, $O(x) \subseteq A$. So $O_h(A) \subseteq A \subseteq O_h(A)$. By Definition 2.8 and the fact $A \subseteq O_h(A) \in \tau_f$ we have $O_h(A) \subseteq \bigcap\{G; A \subseteq G \text{ and } G \in \tau_f\} \subseteq O_h(A)$.

3. Separation Theorems

The purpose of this section is establishing equivalent conditions for famous separations axioms related to the function f. We begin with definition of separation axioms.

Definition 3.1. *According to the classical definitions in topological spaces we define that (X, τ_f) has the property of:*

(a) T_0 if for any $x, y \in X$ with $x \neq y$ there is $G \in \tau_f$ such that either $x \in G$ and $y \in X \setminus G$ or $y \in G$ and $x \in X \setminus G$.

(b) T_1 if for any $x, y \in X$ with $x \neq y$ there are $G, U \in \tau_f$ such that $x \in G$, $y \in X \setminus G$ and $y \in U$ and $x \in X \setminus U$.

(c) T_2 ((X, τ_f) is Hausdorff) if for any $x, y \in X$ with $x \neq y$ there are $G, U \in \tau_f$ such that $x \in G$ and $y \in U$ with $G \cap U = \emptyset$.

Proposition 3.2. *Topological space (X, τ_f) has the property of T_0 if and only if for any $x \neq y$ in X either $x \neq f^n(y)$, $n = 0, 1, 2, \ldots$ or $y \neq f^n(x)$, $n = 0, 1, 2, \ldots$.*

Proof. Suppose that (X, τ_f) has the property of T_0 and consider $x, y \in X$ with $x \neq y$. Then there is $G \in \tau_f$ such that either $x \in G$, $y \in X \setminus G$ or $y \in G$, $x \in X \setminus G$. These imply that $O(x) \subseteq G$, $y \notin O(x)$ or $O(y) \subseteq G$, $x \notin O(y)$. So for any $n = 0, 1, 2, \cdots$, $y \neq f^n(x)$ or $x \neq f^n(y)$.
For the converse consider $x, y \in X$ with $x \neq y$. By the assumption for any $n = 0, 1, 2, \cdots$, $y \neq f^n(x)$ or $x \neq f^n(y)$. Hence $x \notin O(y)$ or $y \notin O(x)$. Now set $G = O(x)$ or $G = O(y)$.

Proposition 3.3. *The following assertions are equivalent*

(a) (X, τ_f) has the property of T_1.

(b) for any $x \in X$, $\{x\}^c \in \tau_f$.

(c) for any $x \neq y$ in X we have $x \neq f^n(y)$ and $y \neq f^n(x)$, for $n = 0, 1, 2, \ldots$.

Proof.

(a)$\rightarrow$ (b): Consider $x \in X$. If $y \in \{x\}^c$, then there is an open set V_y such that $x \notin V_y$. So $y \notin \overline{\{x\}}$ which implies that $y \in X \setminus \overline{\{x\}} \in \tau_f$. Hence $f(y) \in f(X \setminus \overline{\{x\}}) \subseteq X \setminus \overline{\{x\}}$ for any $y \in Y$. So $f(\{x\}^c) \subseteq X \setminus \overline{\{x\}} \subseteq X \setminus \{x\} = \{x\}^c$.

(b)$\rightarrow$ (c): Suppose that $x, y \in X$, $x \neq y$ and $\{x\}^c, \{y\}^c \in \tau_f$. So $\{x\} = \overline{\{x\}}$ and $\{y\} = \overline{\{y\}}$. It follows that $x \notin \overline{\{y\}}$ and $y \notin \overline{\{x\}}$. Hence there are open sets U including x and V including y such that $y \notin U$ and $x \notin V$. These imply that $y \notin O(x)$ and $x \notin O(y)$. Then $x \neq f^n(y)$ and $y \neq f^n(x)$, for $n = 0, 1, 2, \ldots$.

(c)$\rightarrow$ (a): Consider $x, y \in X$ with $x \neq y$. By the assumption for any $n = 0, 1, 2, \cdots$, $y \neq f^n(x)$ and $x \neq f^n(y)$. Hence $x \notin O(y)$ and $y \notin O(x)$. Now set $U = O(x)$ and $V = O(y)$.

Corollary 3.4. *Suppose that (X, τ_f) has the property of T_1 and $x \in X$ is a fixed point of function f. Then $\{x\}$ is open and closed.*

Proposition 3.5. *The topological space (X, τ_f) has the property of T_2 (being Hausdorff) if and only if for any $x \neq y$ in X we have $f^n(y) \neq f^n(x)$, for all $m, n \in \{0, 1, 2, \ldots\}$.*

Proof. Suppose that (X, τ_f) has the property of T_2 and consider $x, y \in X$ with $x \neq y$. So there are open sets U including x and V including y such that $U \cap V = \emptyset$. Hence $O(x) \cap O(y) = \emptyset$. Then for all $m, n \in \{0, 1, 2, \ldots\}$, $f^n(y) \neq f^n(x)$. For the converse set $O(x) = U$ and $O(y) = V$.

Definition 3.6. *The topological space (X, τ_f) is called regular if for any $x \in X$ and closed set $F \subseteq X \setminus \{x\}$ with $F \neq \{O(x)\}^c$ we have $O_h(F) \cap O(x) = \emptyset$.*

Theorem 3.7. *The topological space (X, τ_f) is regular if and only if for any $x \in X$ and any open set $U \neq O(x)$ including x, we have $x \in O(x) \subset \overline{O(x)} \subset U$.*

Proof. Suppose that (X, τ_f) is regular and $x \in U \in \tau_f$. Then

$$O_h(X \setminus U) \cap O(x) = \emptyset. \qquad (1)$$

On the other hand always $X \setminus U \subset O_h(X \setminus U)$ and so

$$[O_h(X \setminus U)]^c \subset U. \qquad (2)$$

Now from (1) and (2) we have $x \in O(x) \subset [O_h(X \setminus U)]^c \subset U$. Since $[O_h(X \setminus U)]^c$ is closed then $\overline{O(x)} \subset [O_h(X \setminus U)]^c$. Hence $x \in O(x) \subset \overline{O(x)} \subset X \setminus U$.
For the converse Consider $x \in X$ and closed F with $O(x)^c \neq F \subseteq X \setminus \{x\}$. These imply that $X \setminus F \neq O(x)$ and $x \in X \setminus F \in \tau_f$. According to the assumption $x \in O(x) \subset \overline{O(x)} \subset U$. So $x \in O(x)$ and $F \subset X \setminus \overline{O(x)}$ which imply that $O_h(F) \subset X \setminus \overline{O(x)}$. Since $(X \setminus \overline{O(x)}) \cap O(x) = \emptyset$ then, $O_h(F) \cap O(x) = \emptyset$.

4. Application

As an application of the Sections 1 and 2, we give the following fixed point theorems using separation axioms and the concept of orbits related to the function f. Also we show that if f is a contraction on a metric space (X, d), then for any $x \in X$ there is an open set with respect to the metric d including x such that belongs to τ_f.

Definition 4.1. *a family $\mathcal{A} = \{A_j : j \in J\}$ of open sets in X is called an* (open) cover *of K if $K \subseteq \bigcup_j A_j$.*

Theorem 4.2. *Suppose that (X, τ_f) is a Hausdorff space. The function $f : (X, \tau_f) \to (X, \tau_f)$ has the fixed point if and only if for every cover $\{G_\alpha; \alpha \in \mathfrak{A}\}$ for X, there are $x_0 \in X$ and $\alpha_0 \in \mathfrak{A}$ such that both of x_0 and $f(x_0)$ are included in G_{α_0}.*

Proof. If f has a fixed point then there is $\alpha_0 \in \mathfrak{A}$ such that both of x_0 and $f(x_0)$ included in G_{α_0}. For the converse suppose that f has no fixed point. So $f(x) \neq x$ for any $x \in X$. Since (X, τ_f) is a Hausdorff space then for any $x \in X$ there are U_x including x an W_x including $f(x)$ such that $U_x \cap W_x = \emptyset$. Also according to the Proposition 2.5, we can choose U_x such that $f(U_x) \subseteq W_x$. Since $\{U_x; x \in X\}$ is a cover for X then there is $z \in X$ and $x_0 \in X$ such that both of z and $f(z)$ are included in U_{x_0}. Hence $f(z) \in f(U_{x_0}) \subseteq W_{x_0}$. So $f(z) \in U_{x_0} \cap W_{x_0}$, which implies that $U_{x_0} \cap W_{x_0} \neq \emptyset$.

Lemma 4.3. *Consider $f : (X, \tau_f) \to (X, \tau_f)$. For any $x \in X$, $\overline{O(x)}$ is both of open and closed.*

Proof. It is clear that $\overline{O(x)}$ is closed. Suppose that W_y is an arbitrary open set including $f(y)$. So there is U_y such that $f(U_y) \subset W_y$. Now consider y as a cluster point of $O(x)$. So $U_y \cap O(x) \neq \emptyset$. Hence $f(U_y) \cap O(x) \neq \emptyset$. Which implies that $W_y \cap O(x) \neq \emptyset$. This guarantees that $f(y)$ is a cluster point in $O(x)$. Hence $f(y) \in \overline{O(x)}$ which implies that $f(\overline{O(x)}) \subseteq \overline{O(x)}$.

Theorem 4.4. *Consider the following assertions for function $f : (X, \tau_f) \to (X, \tau_f)$.*

(a) For some $x_0 \in X$, the set $\overline{O(x_0)}$ is compact;

(b) if x is not a fixed point of f, then $x \notin \overline{O(f^2(x))}$.

Then there is a cluster point y in $O(x_0)$ such that $f(y) = y$.

Proof. Consider $\mathcal{M} = \{A \subseteq \overline{O(x_0)}; A \neq \emptyset \text{ and } A, A^c \in \tau_f\}$. Lemma 4.3 guarantees that $\mathcal{M}$ is nonempty. Let $\mathcal{M}$ be partially ordered by the set inclusion and let $\mathcal{N}$ be a totally ordered subfamily of $\mathcal{M}$. Put $\mathcal{M}_0 = \cap\{A; A \in \mathcal{N}\}$. $\mathcal{M}_0$ is closed nonempty subset of $\overline{O(x_0)}$ by the compactness of $\overline{O(x_0)}$ and it is a lower bound of $\mathcal{N}$. Using Zorn's Lemma we can find a subset Ł of $\mathcal{M}$ which is minimal with respect to being nonempty, closed and mapped into itself by f. By the minimality of Ł we have $\overline{T(Ł)} = Ł$.
Let x be an element in Ł and suppose that $x \neq f(x)$. Then $x \notin \overline{O(f^2(x))}$ and so the continuity of f implies that the set $\overline{O(f^2(x))}$ is mapped into itself by f and the minimality of Ł implies that Ł $= \overline{O(f^2(x))}$. On the other hand we have $x \in$ Ł. It follows that $x \in \overline{O(f^2(x))}$ which is desire contradiction. Therefore $x = f(x)$.

Definition 4.5. [1] *Let (X, d) be a metric space. A function $f : X \to X$ is called a contraction if there exists $k < 1$ such that for any $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$.*

Theorem 4.6. *For metric space (X, d), Consider contraction function f and the topological space (X, τ_f). Then for any $x \in X$ we can find the value r such that $B_r(x) \in \tau_f$.*

Proof. We find r such that for any $y \in B_r(x)$, $f(y) \in B_r(x)$. So it is enough to find r such that $d(x, f(y)) < r$. By the triangle inequality, $d(x, f(y)) \leq d(x, f(x)) + d(f(x), f(y))$ and since f is a contraction, $d(f(x), f(y)) \leq kd(x, y)$. If $B_r(x)$ is any ball and $y \in B_r(x)$, so $d(x, y) \leq r$. Hence $d(f(x), f(y)) \leq kr$ and so $d(x, f(y)) \leq d(x, f(x)) + kr$. Then if we choose r so that $d(x, f(x)) + kr < r$, we would find that $d(x, f(y)) < r$ for all $y \in B_r(x)$ and this completes the proof. Thus we consider:

$$\begin{aligned} d(x, f(x)) + kr &< r \\ d(x, f(x)) &< r - kr \\ d(x, f(x)) &< r(1 - k) \\ d(x, f(x))/(1 - k) &< r. \end{aligned}$$

Hence we see that if $d(x, f(x))/(1 - k) < r$, then $f(y) \in B_r(x)$ for all $y \in B_r(x)$.

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