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Fixed points for Banach and Kannan contractions in modular spaces with a graph

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Abstract

In this paper, we discuss the existence and uniqueness of fixed points for Banach and Kannan contractions defined on modular spaces endowed with a graph. We do not impose the Δ_2-condition or the Fatou property on the modular spaces to give generalizations of some recent results. The given results play as a modular version of metric fixed point results.

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1. Introduction and preliminaries

To control the pathological behavior of a modular in modular spaces the conditions Δ_2 and Fatou property are usually assumed (see, e.g., [1, 5, 7, 8, 11, 12]. For instance, in [1], Banach fixed point theorem is given in modular spaces that their modular satisfy both the Δ_2-condition and the Fatou property. In [7], Khamsi established some fixed point theorems for quasi-contractions in modular spaces satisfying only the Fatou property.

In [6], Jachymski investigated Banach fixed point theorem in metric spaces with a graph and his idea followed by the authors in uniform spaces (see, e.g., [2, 3]).

In this paper motivated by the ideas given in [1, 6], we aim to discuss the fixed points of Banach and Kannan contractions in modular spaces endowed with a graph without Δ_2-condition and Fatou property. We also clarify the independence of these contractions in modular spaces.

We first commence some basic concepts about modular spaces as formulated by Musielak and Orlicz [10]. For more details, the reader is referred to [9].

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Definition 1.1. *A real-valued function ρ defined on a real vector space X is called a modular on X if it satisfies the following conditions:*

M1) $\rho(x) \geq 0$ *for all* $x \in X$*;*
M2) $\rho(x) = 0$ *if and only if* $x = 0$*;*
M3) $\rho(x) = \rho(-x)$ *for all* $x \in X$*;*
M4) $\rho(ax + by) \leq \rho(x) + \rho(y)$ *for all* $x, y \in X$ *and all* $a, b \geq 0$ *with* $a + b = 1$*.*

If ρ satisfies (M1)-(M4), then the pair (X, ρ), shortly denoted by X, is called a modular space.

The modular ρ is called convex if Condition (M4) is strengthened by replacing with

M4′) $\rho(ax + by) \leq a\rho(x) + b\rho(y)$ for all $x, y \in X$ and all $a, b \geq 0$ with $a + b = 1$.

It is easy to obtain the following two immediate consequences of Condition (M4) which we need in the sequel:

- If a and b are real numbers with $|a| \leq |b|$, then $\rho(ax) \leq \rho(bx)$ for all $x \in X$;
- If $a_1, \ldots, a_n$ are nonnegative numbers with $\sum_{i=1}^{n} a_i = 1$, then

$$\rho\Big(\sum_{i=1}^{n} a_i x_i\Big) \leq \sum_{i=1}^{n} \rho(x_i) \qquad (x_1, \ldots, x_n \in X).$$

Definition 1.2. *Let (X, ρ) be a modular space.*

1. *A sequence $\{x_n\}$ in X is said to be ρ-convergent to a point $x \in X$, denoted by $x_n \xrightarrow{\rho} x$, if $\rho(x_n - x) \to 0$ as $n \to \infty$.*
2. *A sequence $\{x_n\}$ in X is said to be ρ-Cauchy if $\rho(x_m - x_n) \to 0$ as $m, n \to \infty$.*
3. *The modular space X is called ρ-complete if each ρ-Cauchy sequence in X is ρ-convergent to a point of X.*
4. *The modular ρ is said to satisfy the Δ_2-condition if $2x_n \xrightarrow{\rho} 0$ as $n \to \infty$ whenever $x_n \xrightarrow{\rho} 0$ as $n \to \infty$.*
5. *The modular ρ is said to have the Fatou property if*

$$\rho(x - y) \leq \liminf_{n\to\infty} \rho(x_n - y_n)$$

whenever

$$x_n \xrightarrow{\rho} x \quad \textit{and} \quad y_n \xrightarrow{\rho} y \quad \textit{as} \quad n \to \infty.$$

Conditions (M2) and (M4) ensure that each sequence in a modular space can be ρ-convergent to at most one point. In other words, the limit of a ρ-convergent sequence in a modular space is unique.

We next review some notions in graph theory. All of them can be found in, e.g., [4].

Let X be a modular space. Consider a directed graph G with $V(G) = X$ and $E(G) \supseteq \{(x, x) : x \in X\}$, i.e., $E(G)$ contains all loops. Suppose further that G has no parallel edges. With these assumptions, we may denote G by the pair $(V(G), E(G))$. In this way, the modular space X is endowed with the graph G. The notation $\widetilde{G}$ is used to denote the undirected graph obtained from G by deleting the directions of the edges of G. Thus,

$$V(G) = X \qquad E(G) = \big\{(x, y) \in X \times X : (x, y) \in E(G) \ \vee \ (y, x) \in E(G)\big\}.$$

By a path in G from a vertex x to a vertex y, it is meant a finite sequence $(x_s)_{s=0}^{N}$ of vertices of G such that $x_0 = x$, $x_N = y$, and $(x_{s-1}, x_s) \in E(G)$ for $s = 1, \ldots, N$. A graph G is called weakly connected if there exists a path in $\widetilde{G}$ between each two vertices of G, i.e., there exists an undirected path in G between its each two vertices.

2. Main results

Let X be a modular space endowed with a graph G and $f : X \to X$ be any mapping. The set of all fixed points for f is denoted by $\mathrm{Fix}(f)$, and by C_f, we mean the set of all elements x of X such that $(f^n x, f^m x) \in E(\widetilde{G})$ for $m, n = 0, 1, \ldots$.

We begin with introducing Banach and Kannan G-ρ-contractions.

Definition 2.1. *Let X be a modular space with a graph G and $f : X \to X$ be a mapping. We call f a Banach G-ρ-contraction if*

B1) *f preserves the edges of G, i.e., $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$;*
B2) *there exist positive numbers k, a and b with $k < 1$ and $a < b$ such that*

$$\rho\big(b(fx - fy)\big) \leq k\rho\big(a(x - y)\big)$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

The numbers k, a and b are called the constants of f. And we call f a Kannan G-ρ-contraction if

K1) *f preserves the edges of G;*
K2) *there exist positive numbers k, l, a_1, a_2 and b with $k + l < 1$, $a_1 \leq \frac{b}{2}$ and $a_2 \leq b$ such that*

$$\rho\big(b(fx - fy)\big) \leq k\rho\big(a_1(fx - x)\big) + l\rho\big(a_2(fy - y)\big)$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

The numbers k, l, a_1, a_2 and b are called the constants of f.

It might be valuable if we discuss these contractions a little. Our first proposition follows immediately from Condition (M3) and Definition 2.1.

Proposition 2.2. *Let X be a modular space with a graph G. If a mapping from X into itself satisfies* (B1) *(respectively,* (B2)*) for G, then it satisfies* (B1) *(respectively,* (B2)*) for $\widetilde{G}$. In particular, a Banach G-ρ-contraction is also a Banach $\widetilde{G}$-ρ-contraction. Similar statements are true for Kannan G-ρ-contractions provided that $a_2 \leq \frac{b}{2}$.*

We also have the following remark about Kannan G-ρ-contractions.

Remark 2.3. *For a Kannan $\widetilde{G}$-ρ-contraction $f : X \to X$, we can interchange the roles of x and y in (K2) since $E(\widetilde{G})$ is symmetric. Having done this, we find*

$$\begin{aligned}
\rho\big(b(fx - fy)\big) &= \rho\big(b(fy - fx)\big) \\
&\leq k\rho\big(a_1(fy - y)\big) + l\rho\big(a_2(fx - x)\big) \\
&= l\rho\big(a_2(fx - x)\big) + k\rho\big(a_1(fy - y)\big).
\end{aligned}$$

Therefore, no matter $a_1 \leq \frac{b}{2}$ or $a_2 \leq \frac{b}{2}$ whenever we are faced with Kannan $\widetilde{G}$-ρ-contractions. Nevertheless, both a_1 and a_2 must be not more than b.

We now give some examples.

Example 2.4. *Let X be a modular space with any arbitrary graph G. Since $E(G)$ contains all loops, each constant mapping $f : X \to X$ is both a Banach and a Kannan G-ρ-contraction. In fact, $E(G)$ should contain all loops if we want any constant mapping to be either a Banach or a Kannan G-ρ-contraction.*

Example 2.5. *Let X be a modular space and G_0 be the complete graph $(X, X \times X)$. Then Banach (Kannan) G_0-ρ-contractions are precisely the Banach (Kannan) contractions in modular spaces.*

Example 2.6. *Let $\preceq$ be a partial order on a modular space X and consider a poset graph G_1 by $V(G_1) = X$ and $E(G_1) = \{(x, y) \in X \times X : x \preceq y\}$. Then Banach G_1-ρ-contractions are precisely the nondecreasing ordered ρ-contractions. A similar statement is true for Kannan G_1-ρ-contractions.*

Finally, we show that Banach and Kannan G-ρ-contractions are independent of each other. More precisely, we construct two mappings on $\mathbb{R}$ such that one of them satisfies (B2) but not (K2), and the other, (K2) but not (B2) for the complete graph G_0.

Example 2.7. *Let ρ be the usual Euclidean norm on $\mathbb{R}$, i.e., $\rho(x) = |x|$ for all $x \in \mathbb{R}$. Define a mapping $f : \mathbb{R} \to \mathbb{R}$ by $fx = \frac{x}{3}$ for all $x \in \mathbb{R}$. Then f is a Banach G_0-ρ-contraction with the constants $k = \frac{2}{3}$, $a = \frac{1}{2}$ and $b = 1$. Indeed, given any $x, y \in \mathbb{R}$, we have*

$$\rho\big(b(fx - fy)\big) = \frac{1}{3}|x - y| = k\rho\big(a(x - y)\big).$$

On the other hand, if k, l, a_1, a_2 and b are any arbitrary positive numbers satisfying $k+l < 1$, $a_1 \le \frac{b}{2}$ and $a_2 \le b$, then for $y = 0$ and any $x \neq 0$ we see that

$$\rho\big(b(fx - f0)\big) = \frac{b|x|}{3} > \frac{2a_1k|x|}{3} = k\rho\big(a_1(fx - x)\big) + l\rho\big(a_2(f0 - 0)\big).$$

Therefore, (K2) fails to hold and f is not a Kannan G_0-ρ-contraction.

Example 2.8. *It is easy to verify that the function $\rho(x) = x^2$ defines a modular on $\mathbb{R}$ and $(\mathbb{R}, \rho)$ is ρ-complete because $(\mathbb{R}, |\cdot|)$ is a Banach space. Now, consider a mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $fx = \frac{1}{2}$ if $x \neq 1$ and $f1 = \frac{1}{10}$. Then f is a G_0-ρ-Kannan contraction with the constants $k = \frac{64}{81}$, $l = \frac{16}{81}$, $a_1 = \frac{1}{2}$ and $a_2 = b = 1$. Indeed, given any $x, y \in \mathbb{R}$, we have the following three possible cases:*

Case 1: *If $x = y = 1$ or $x, y \neq 1$, then (K2) holds trivially since $fx = fy$;*

Case 2: *If $x = 1$ and $y \neq 1$, then*

$$\rho\big(b(fx - fy)\big) = \frac{4}{25} \le \frac{4}{25} + \frac{16}{81}\Big(\frac{1}{2} - y\Big)^2 = k\rho\big(a_1(fx - x)\big) + l\rho\big(a_2(fy - y)\big);$$

Case 3: *Finally, if $x \neq 1$ and $y = 1$, then*

$$\rho\big(b(fx - fy)\big) = \frac{4}{25} \le \frac{16}{81}\Big(\frac{1}{2} - x\Big)^2 + \frac{4}{25} = k\rho\big(a_1(fx - x)\big) + l\rho\big(a_2(fy - y)\big).$$

Note that $k + l = \frac{80}{81} < 1$, $a_1 \le \frac{b}{2}$ and $a_2 \le b$. But f is not a Banach G_0-ρ-contraction; for if k, a and b are any arbitrary positive numbers satisfying $k < 1$ and $a < b$, then putting $x = 1$ and $y = \frac{3}{5}$ yields

$$\rho\big(b(fx - fy)\big) = \frac{4b^2}{25} > \frac{4a^2k}{25} = k\rho\big(a(x - y)\big).$$

Now we are going to prove our fixed point results. The first one is about the existence and uniqueness of fixed points for Banach $\widetilde{G}$-ρ-contractions.

Theorem 2.9. *Let X be a ρ-complete modular space endowed with a graph G and the triple (X,ρ,G) have the following property:*

(∗) *If $\{x_n\}$ is a sequence in X such that $\beta x_n \xrightarrow{\rho} \beta x$ for some $\beta > 0$ and $(x_n, x_{n+1}) \in E(\widetilde{G})$ for all $n \geq 1$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $(x_{n_i}, x) \in E(\widetilde{G})$ for all $i \geq 1$.*

Then a Banach $\widetilde{G}$-ρ-contraction $f : X \to X$ has a fixed point if and only if $C_f \neq \emptyset$. Moreover, this fixed point is unique if G is weakly connected.

Proof . ($\Rightarrow$) It is trivial since $\mathrm{Fix}(f) \subseteq C_f$.

($\Leftarrow$) Let k, a and b be the constants of f and let $\alpha > 1$ be the exponential conjugate of $\frac{b}{a}$, i.e., $\frac{a}{b} + \frac{1}{\alpha} = 1$. Choose an $x \in C_f$ and keep it fixed. We are going to show that the sequence $\{bf^n x\}$ is ρ-Cauchy in X. To this end, note first if n is a positive integer, then by (B2) we have

$$\begin{aligned}\rho\big(a(f^n x - x)\big) &= \rho\big(a(f^n x - fx) + a(fx - x)\big)\\ &= \rho\Big(\frac{a}{b} b\big(f^n x - fx\big) + \frac{1}{\alpha}\alpha a\big(fx - x\big)\Big)\\ &\leq \rho\big(b(f^n x - fx)\big) + \rho\big(\alpha a(fx - x)\big)\\ &\leq k\rho\big(a(f^{n-1} x - x)\big) + r,\end{aligned}$$

where $r = \rho(\alpha a(fx - x))$. Hence using the mathematical induction, we get

$$\begin{aligned}\rho\big(a(f^n x - x)\big) &\leq k\rho\big(a(f^{n-1} x - x)\big) + r\\ &\leq k\Big[k\rho\big(a(f^{n-2} x - x)\big) + r\Big] + r\\ &= k^2\rho\big(a(f^{n-2} x - x)\big) + kr + r\\ &\ \ \vdots\\ &\leq k^{n-1}\rho\big(a(fx - x)\big) + k^{n-2}r + \cdots + r\end{aligned}$$

for all $n \geq 1$. Since $\alpha > 1$, it follows that $\rho(a(fx - x)) \leq \rho(\alpha a(fx - x)) = r$ and therefore,

$$\rho\big(a(f^n x - x)\big) \leq k^{n-1} r + \cdots + r = \frac{(1-k^n)r}{1-k} \leq \frac{r}{1-k} \qquad n = 1, 2, \dots . \tag{2.1}$$

Now using (B2) once more, we find

$$\begin{aligned}\rho\big(b(f^m x - f^n x)\big) &\leq k\rho\big(a(f^{m-1} x - f^{n-1} x)\big)\\ &\leq k\rho\big(b(f^{m-1} x - f^{n-1} x)\big)\\ &\ \ \vdots\\ &\leq k^n\rho\big(a(f^{m-n} x - x)\big)\end{aligned} \tag{2.2}$$

for all m and n with $m > n \geq 1$. Consequently, by combining (2.1) and (2.2), it is seen that for all $m > n \geq 1$ we have

$$\rho\big(b(f^m x - f^n x)\big) \leq k^n\rho\big(a(f^{m-n} x - x)\big) \leq \frac{k^n r}{1-k}.$$

Therefore, $\rho(b(f^m x - f^n x)) \to 0$ as $m, n \to \infty$, and so $\{bf^n x\}$ is a ρ-Cauchy sequence in X and because X is ρ-complete, it is ρ-convergent. On the other hand, X is a real vector space and $b > 0$. Thus, there exists an $x^* \in X$ such that $bf^n x \xrightarrow{\rho} bx^*$.

We next show that x^* is a fixed point for f. Since $x \in C_f$, it follows that $(f^n x, f^{n+1}x) \in E(\widetilde{G})$ for all $n \geq 0$, and so by Property $(*)$, there exists a strictly increasing sequence $\{n_i\}$ of positive integers such that $(f^{n_i}x, x^*) \in E(\widetilde{G})$ for all $i \geq 1$. Hence using (B2) we get

$$\begin{aligned}
\rho\Big(\frac{b}{2}\big(fx^* - x^*\big)\Big) &= \rho\Big(\frac{b}{2}\big(fx^* - f^{n_i+1}x\big) + \frac{b}{2}\big(f^{n_i+1}x - x^*\big)\Big) \\
&\leq \rho\big(b(fx^* - f^{n_i+1}x)\big) + \rho\big(b(f^{n_i+1}x - x^*)\big) \\
&= \rho\big(b(f^{n_i+1}x - fx^*)\big) + \rho\big(b(f^{n_i+1}x - x^*)\big) \\
&\leq k\rho\big(a(f^{n_i}x - x^*)\big) + \rho\big(b(f^{n_i+1}x - x^*)\big) \\
&\leq k\rho\big(b(f^{n_i}x - x^*)\big) + \rho\big(b(f^{n_i+1}x - x^*)\big) \to 0
\end{aligned}$$

as $i \to \infty$. So $\rho(\frac{b}{2}(fx^* - x^*)) = 0$, and since $b > 0$, it follows that $fx^* - x^* = 0$ or equivalently, $fx^* = x^*$, i.e., x^* is a fixed point for f.

Finally, to prove the uniqueness of the fixed point, suppose that G is weakly connected and $y^* \in X$ is a fixed point for f. Then there exists a path $(x_s)_{s=0}^N$ in $\widetilde{G}$ from x^* to y^*, i.e., $x_0 = x^*$, $x_N = y^*$, and $(x_{s-1}, x_s) \in E(\widetilde{G})$ for $s = 1, \ldots, N$. Thus, by (B1), we have

$$(f^n x_{s-1}, f^n x_s) \in E(\widetilde{G}) \qquad (n \geq 0 \text{ and } s = 1, \ldots, N).$$

And using (B2) and the mathematical induction we get

$$\begin{aligned}
\rho\Big(\frac{b}{N}\big(x^* - y^*\big)\Big) &= \rho\Big(\frac{b}{N}\big(x^* - f^n x_1\big) + \cdots + \frac{b}{N}\big(f^n x_{N-1} - y^*\big)\Big) \\
&\leq \rho\big(b(x^* - f^n x_1)\big) + \cdots + \big(b(f^n x_{N-1} - y^*)\big) \\
&= \sum_{s=1}^{N} \rho\big(b(f^n x_{s-1} - f^n x_s)\big) \\
&\leq k \sum_{s=1}^{N} \rho\big(a(f^{n-1} x_{s-1} - f^{n-1} x_s)\big) \\
&\leq k \sum_{s=1}^{N} \rho\big(b(f^{n-1} x_{s-1} - f^{n-1} x_s)\big) \\
&\;\vdots \\
&\leq k^n \sum_{s=1}^{N} \rho\big(b(x_{s-1} - x_s)\big) \to 0
\end{aligned}$$

as $n \to \infty$. So $\frac{b}{N}(x^* - y^*) = 0$, and since $b > 0$, it follows that $x^* = y^*$. Consequently, the fixed point of f is unique. □

Setting $G = G_0$ and $G = G_1$, we get the following consequences of Theorem 2.9 in modular and partially ordered modular spaces, respectively.

Corollary 2.10. *Let X be a ρ-complete modular space and a mapping $f : X \to X$ satisfies*

$$\rho\big(b(fx - fy)\big) \leq k\rho\big(a(x - y)\big) \qquad (x, y \in X),$$

where $0 < k < 1$ and $0 < a < b$. Then f has a unique fixed point $x^* \in X$ and $bf^n x \xrightarrow{\rho} bx^*$ for all $x \in X$.

Corollary 2.11. *Let $\preceq$ be a partial order on a ρ-complete modular space X such that the triple $(X, \rho, \preceq)$ has the following property:*

(∗∗) *If $\{x_n\}$ is a sequence in X with successive comparable terms such that $\beta x_n \xrightarrow{\rho} \beta x$ for some $\beta > 0$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \preceq x$ for all $i \geq 1$.*

Assume that a nondecreasing mapping $f : X \to X$ satisfies

$$\rho\big(b(fx - fy)\big) \leq k\rho\big(a(x - y)\big) \qquad (x, y \in X \text{ and } x \preceq y),$$

where $0 < k < 1$ and $0 < a < b$. Then f has a fixed point if and only if there exists an $x \in X$ such that $T^n x$ is comparable to $T^m x$ for all $m, n \geq 0$. Moreover, this fixed point is unique if the following condition holds:

label= *For all $x, y \in X$, there exists a finite sequence $(x_s)_{s=0}^N$ in X with comparable successive terms such that $x_0 = x$ and $x_N = y$.*

Corollary 2.12. *Let (X, ρ) be a ρ-complete modular space endowed with a graph G, where ρ is a convex modular, and the triple (X, ρ, G) have Property $(*)$. Assume that $f : X \to X$ is a mapping which preserves the edges of $\widetilde{G}$ and satisfies*

$$\rho\big(b(fx - fy)\big) \leq k\rho\big(a(x - y)\big) \qquad \big(x, y \in X \text{ and } (x, y) \in E(\widetilde{G})\big),$$

where k, a and b are positive numbers with $b > \max\{a, ak\}$. Then f has a fixed point if and only if $C_f \neq \emptyset$. Moreover, this fixed point is unique if G is weakly connected.

Proof . Set $c = \max\{a, ak\}$ and choose any $a_0 \in (c, b)$. Then by the hypothesis and convexity of ρ, we have

$$\begin{aligned} \rho\big(b(fx - fy)\big) &\leq k\rho\big(a(x - y)\big) \\ &= k\rho\Big(\frac{a}{a_0}a_0\big(x - y\big) + \big(1 - \frac{a}{a_0}\big)0\Big) \\ &\leq \frac{ak}{a_0}\rho\big(a_0(x - y)\big) \end{aligned}$$

for all $x, y \in X$ with $(x, y) \in E(\widetilde{G})$. Since $a_0 < b$, and $\frac{ak}{a_0} < 1$, it follows that f satisfies (B2) for the graph $\widetilde{G}$ with the constants k and a replaced with $\frac{ak}{a_0}$ and a_0, respectively, and b kept fixed. Since f preserves the edges of $\widetilde{G}$, it follows that f is a Banach $\widetilde{G}$-ρ-contraction and the results are concluded immediately from Theorem 2.9. □

Our next result is about the existence and uniqueness of fixed points for Kannan $\widetilde{G}$-ρ-contractions.

Theorem 2.13. *Let X be a ρ-complete modular space endowed with a graph G and the triple (X, ρ, G) have Property $(*)$. Then a Kannan $\widetilde{G}$-ρ-contraction $f : X \to X$ has a fixed point if and only if $C_f \neq \emptyset$. Moreover, this fixed point is unique if $k < \frac{1}{2}$ and X satisfies the following condition:*

($\star$) *For all $x, y \in X$, there exists a $z \in X$ such that $(x, z), (y, z) \in E(\widetilde{G})$.*

Proof . ($\Rightarrow$) It is trivial since $\mathrm{Fix}(f) \subseteq C_f$.

($\Leftarrow$) Let k, l, a_1, a_2 and b be the constants of f. Choose an $x \in C_f$ and keep it fixed. We are going to show that the sequence $\{bf^n x\}$ is ρ-Cauchy in X. Given any integer $n \geq 2$, by (K2) we have

$$\begin{aligned}\rho\big(b(f^n x - f^{n-1}x)\big) &\leq k\rho\big(a_1(f^n x - f^{n-1}x)\big) + l\rho\big(a_2(f^{n-1}x - f^{n-2}x)\big)\\ &\leq k\rho\big(b(f^n x - f^{n-1}x)\big) + l\rho\big(b(f^{n-1}x - f^{n-2}x)\big),\end{aligned}$$

which yields

$$\rho\big(b(f^n x - f^{n-1}x)\big) \leq \delta\rho\big(b(f^{n-1}x - f^{n-2}x)\big),$$

where $\delta = \frac{l}{1-k} \in (0,1)$. Hence using the mathematical induction, we get

$$\rho\big(b(f^n x - f^{n-1}x)\big) \leq \delta^n\rho\big(b(fx - x)\big) \qquad n = 1, 2, \ldots .$$

Now using (K2) once more, we find

$$\begin{aligned}\rho\big(b(f^m x - f^n x)\big) &\leq k\rho\big(a_1(f^m x - f^{m-1}x)\big) + l\rho\big(a_2(f^n x - f^{n-1}x)\big)\\ &\leq k\rho\big(b(f^m x - f^{m-1}x)\big) + l\rho\big(b(f^n x - f^{n-1}x)\big)\\ &\leq k\delta^m\rho\big(b(fx - x)\big) + l\delta^n\rho\big(b(fx - x)\big)\end{aligned}$$

for all $m, n \geq 1$. Therefore, $\rho(b(f^m x - f^n x)) \to 0$ as $m, n \to \infty$, and so $\{bf^n x\}$ is a ρ-Cauchy sequence in X and because X is ρ-complete, it is ρ-convergent. Thus, there exists an $x^* \in X$ such that $bf^n x \xrightarrow{\rho} bx^*$.

We next show that x^* is a fixed point for f. Since $x \in C_f$, it follows that $(f^n x, f^{n+1}x) \in E(\widetilde{G})$ for all $n \geq 0$, and so by Property $(*)$, there exists a strictly increasing sequence $\{n_i\}$ of positive integers such that $(f^{n_i}x, x^*) \in E(\widetilde{G})$ for all $i \geq 1$. Hence using (K2), we get

$$\begin{aligned}\rho\Big(\frac{b}{2}\big(fx^* - x^*\big)\Big) &= \rho\Big(\frac{b}{2}\big(fx^* - f^{n_i+1}x\big) + \frac{b}{2}\big(f^{n_i+1}x - x^*\big)\Big)\\ &\leq \rho\big(b(fx^* - f^{n_i+1}x)\big) + \rho\big(b(f^{n_i+1}x - x^*)\big)\\ &\leq \Big[k\rho\big(a_1(fx^* - x^*)\big) + l\rho\big(a_2(f^{n_i+1}x - f^{n_i}x)\big)\Big] + \rho\big(b(f^{n_i+1}x - x^*)\big)\\ &\leq k\rho\Big(\frac{b}{2}\big(fx^* - x^*\big)\Big) + l\rho\big(b(f^{n_i+1}x - f^{n_i}x)\big) + \rho\big(b(f^{n_i+1}x - x^*)\big)\end{aligned}$$

for all $k \geq 1$. Hence

$$\rho\Big(\frac{b}{2}\big(fx^* - x^*\big)\Big) \leq \delta\rho\big(b(f^{n_i+1}x - f^{n_i}x)\big) + \frac{1}{1-k}\rho\big(b(f^{n_i+1}x - x^*)\big) \to 0$$

as $i \to \infty$. So $\rho(\frac{b}{2}(fx^* - x^*)) = 0$, and since $b > 0$, it follows that $fx^* - x^* = 0$ or equivalently, $fx^* = x^*$, i.e., x^* is a fixed point for f.

Finally, to prove the uniqueness of the fixed point, suppose that Condition $(\star)$ holds and $y^* \in X$ is a fixed point for f. We consider the following two cases:

Case 1: (x^*, y^*) is an edge of $\widetilde{G}$.

In this case, using (K2), we find

$$\rho\big(b(x^* - y^*)\big) = \rho\big(b(fx^* - fy^*)\big) \leq k\rho\big(a_1(fx^* - x^*)\big) + l\rho\big(a_2(fy^* - y^*)\big) = 0.$$

Therefore, $\rho(b(x^*-y^*))=0$, and so $x^*=y^*$ because $b>0$.

Case 2: (x^*, y^*) is not an edge of $\widetilde{G}$.
In this case, by Condition $(\star)$, there exists a $z\in X$ such that both (x^*,z) and (y^*,z) are edges of $\widetilde{G}$. So by (K1), we have $(x^*,f^nz),(y^*,f^nz)\in E(\widetilde{G})$ for all $n\geq 0$ since x^* is a fixed point for f. Therefore, by (K2) we find

$$\begin{aligned}
\rho\big(b(f^nz-x^*)\big) &= \rho\big(b(f^nz-f^nx^*)\big)\\
&\leq k\rho\big(a_1(f^nz-f^{n-1}z)\big)+l\rho\big(a_2(f^nx^*-f^{n-1}x^*)\big)\\
&\leq k\rho\Big(\frac{b}{2}(f^nz-f^{n-1}z)\Big)\\
&= k\rho\Big(\frac{b}{2}(f^nz-f^nx^*)+\frac{b}{2}(f^{n-1}x^*-f^{n-1}z)\Big)\\
&\leq k\rho\big(b(f^nz-f^nx^*)\big)+k\rho\big(b(f^{n-1}x^*-f^{n-1}z)\big)\\
&= k\rho\big(b(f^nz-x^*)\big)+k\rho\big(b(f^{n-1}z-x^*)\big)
\end{aligned}$$

for all $n\geq 1$, which yields

$$\rho\big(b(f^nz-x^*)\big)\leq \lambda\rho\big(b(f^{n-1}z-x^*)\big),$$

where $\lambda=\frac{k}{1-k}\in(0,1)$ because $k<\frac{1}{2}$. So by the mathematical induction, we get

$$\rho\big(b(f^nz-x^*)\big)\leq \lambda^n\rho\big(b(z-x^*)\big)\qquad n=0,1,\dots .$$

Since $\lambda<1$, it follows that $bf^nz\xrightarrow{\rho}bx^*$. Similarly, one can show that $bf^nz\xrightarrow{\rho}by^*$, and so $bx^*=by^*$ because the limit of a ρ-convergent sequence in a modular space is unique. Thus, from $b>0$, it follows that $x^*=y^*$.

Consequently, the fixed point of f is unique. $\square$

Setting $G=G_0$ and $G=G_1$ once again, we get the following consequences of Theorem 2.13 in modular and partially ordered modular spaces, respectively.

Corollary 2.14. *Let X be a ρ-complete modular space and a mapping $f:X\to X$ satisfies*

$$\rho\big(b(fx-fy)\big)\leq k\rho\big(a_1(fx-x)\big)+l\rho\big(a_2(fy-y)\big)\qquad (x,y\in X),$$

where k,l,a_1,a_2 and b are positive with $k+l<1$, $a_1\leq\frac{b}{2}$ and $a_2\leq b$. Then f has a unique fixed point $x^*\in X$ and $bf^nx\xrightarrow{\rho}bx^*$ for all $x\in X$.

Corollary 2.15. *Let $\preceq$ be a partial order on a ρ-complete modular space X such that the triple $(X,\rho,\preceq)$ has Property $(**)$. Assume that a nondecreasing mapping $f:X\to X$ satisfies*

$$\rho\big(b(fx-fy)\big)\leq k\rho\big(a_1(fx-x)\big)+l\rho\big(a_2(fy-y)\big)\qquad (x,y\in X,\ \text{and either } x\preceq y \text{ or } y\preceq x),$$

where k,l,a_1,a_2 and b are positive with $k+l<1$, $a_1\leq\frac{b}{2}$ and $a_2\leq b$. Then f has a fixed point if and only if there exists an $x\in X$ such that T^nx is comparable to T^mx for all $m,n\geq 0$. Moreover, this fixed point is unique if $k<\frac{1}{2}$ and each pair of elements of X has either an upper or a lower bound.

As another consequence of Theorem 2.13, we have the convex version of it as follows:

Corollary 2.16. *Let (X,ρ) be a ρ-complete modular space endowed with a graph G, where ρ is a convex modular, and the triple (X,ρ,G) have Property $(*)$. Assume that $f : X \to X$ is a mapping which preserves the edges of $\widetilde{G}$ and satisfies*

$$\rho\big(b(fx-fy)\big) \leq k\rho\big(a_1(fx-x)\big) + l\rho\big(a_2(fy-y)\big) \qquad \big(x,y \in X \text{ and } (x,y) \in E(\widetilde{G})\big),$$

where k, l, a_1, a_2 and b are positive numbers with $b > 4\max\{a_1, a_2, a_1k, a_2l\}$. Then f has a fixed point if and only if $C_f \neq \emptyset$. Moreover, this fixed point is unique if X satisfies Condition $(\star)$.

Proof . Set $c = 2\max\{a_1, a_2, a_1k, a_2l\}$ and choose any $a_0 \in (c, \frac{b}{2}]$. Then by the hypothesis and convexity of ρ, we have

$$\begin{aligned} \rho\big(b(fx-fy)\big) &\leq k\rho\big(a_1(fx-x)\big) + l\rho\big(a_2(fy-y)\big) \\ &= k\rho\Big(\frac{a_1}{a_0}a_0\big(fx-x\big) + \big(1-\frac{a_1}{a_0}\big)0\Big) + l\rho\Big(\frac{a_2}{a_0}a_0\big(fy-y\big) + \big(1-\frac{a_2}{a_0}\big)0\Big) \\ &\leq \frac{a_1k}{a_0}\rho\big(a_0(fx-x)\big) + \frac{a_2l}{a_0}\rho\big(a_0(fy-y)\big) \end{aligned}$$

for all $x,y \in X$ with $(x,y) \in E(\widetilde{G})$. Since $a_0 \leq \frac{b}{2} < b$, and $\frac{a_1k}{a_0} + \frac{a_2l}{a_0} < 1$, it follows that f satisfies (K2) for the graph $\widetilde{G}$ with the constants k, l, a_1 and a_2 replaced with $\frac{a_1k}{a_0}$, $\frac{a_2l}{a_0}$, a_0 and a_0, respectively, and b kept fixed. Since f preserves the edges of $\widetilde{G}$, it follows that f is a Kannan $\widetilde{G}$-ρ-contraction and the first assertion is concluded immediately from Theorem 2.13.

On the other hand, since $a_0 > c \geq 2a_1k$, it follows that $\frac{a_1k}{a_0} < \frac{1}{2}$, and because X satisfies Condition $(\star)$, Theorem 2.13 guarantees the uniqueness of the fixed point of f. $\square$

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