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An equivalent representation for weighted supremum norm on the upper half-plane

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Abstract

In this paper, firstly, we obtain some inequalities which estimates complex polynomials on the circles. Then, we use these estimates and a Moebius transformation to obtain the dual of this estimates for the lines in upper half-plane. Finally, for an increasing weight υ on the upper half-plane with certain properties and holomorphic functions f on the upper half-plane we obtain an equivalent representation for weighted supremum norm.

Keywords: Weighted spaces, holomorphic functions, de-la-valle Poussion kernel , upper half-plane.
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1. Introduction

In [3], W. Lusky used convolution with de-la-valle Poussion kernel on a certain sequence of integers to obtain a representation equivalent to the weighted supremum norm $\|f\|_\nu$, for holomorphic or harmonic functions f from unit disc into complex plane. In this paper, we obtain an equivalent representation for weighted supremum norm for holomorphic functions form upper half-plane into complex plane whenever our weights satisfy certain properties. Paper is organized as follows: in section two we present some necessary notations and definitions. Section three is devoted to some technical lemmas which we need for the proof of the main result of the paper in Theorem 3.8.

Definition 1.1. *Suppose $x \in \mathbb{C}$ and $y \in \mathbb{R}$ $(y > 0)$, $x + \gamma_y$ denotes the circle with center x and radius y in $x \in \mathbb{C}$.*

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Definition 1.2. *By $x+\gamma_y \subseteq x'+\gamma_{y'}$ we mean that the circle $x+\gamma_y$ is inside of the the circle $x'+\gamma_{y'}$*

Definition 1.3. *Suppose $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ is a complex function. We define $M_\infty(f,\Omega) = \sup_{z\in\Omega} \mid f(z) \mid$*

Definition 1.4. *$\mathbb{D} = \{z \in \mathbb{C} : \mid z \mid < 1\}$ and $G = \{\omega \in \mathbb{C} : \ Im\ \omega > 0\}$ are unit disc and upper half-plane respectively.*

Definition 1.5. *For any $\delta \geq 0$ we define $L_\delta := \{\omega \in G : \ Im\ \omega = \delta\}$ and $G_\delta := \{\omega \in \mathbb{C} : Im\ \omega \geq \delta\}$. In particular $L_0 := \{\omega \in \mathbb{C} : Im\ \omega = 0\}$ is the real line.*

Definition 1.6. *Define $\alpha : \mathbb{D} \longrightarrow G$ by $\alpha(z) = \frac{1+z}{1-z}i$, so $\alpha^{-1} : G \longrightarrow \mathbb{D}$ is $\alpha^{-1}(\omega) = \frac{\omega-i}{\omega+i}$. If $\omega \in L_\delta$ then $\mid \alpha^{-1}(\omega) - \frac{\delta}{\delta+1} \mid = \frac{1}{1+\delta}$. So $\alpha^{-1}(\omega)$ maps the line L_δ to the circle $\frac{\delta}{1+\delta} + \gamma_{\frac{1}{1+\delta}} \setminus \{(1,0)\}$.*

Definition 1.7. *A continuous function $\upsilon : G \longrightarrow (0,+\infty)$ is called a weight. We say a weight ν satisfies $(*)$ if $a = \sup_{n\in\mathbb{N}\cup\{0\}} \frac{\upsilon(2^{-n}i)}{\upsilon(2^{-n-1}i)} < \infty$.*

Remark 1.8. *From now on, we always assume weight ν is increasing, satisfies $(*)$, depends only on the imaginary part, that is $\nu(\omega) = \nu(Im\ \omega\ i)$ and $\lim_{t\to 0} \nu(ti) = 0$.*

Definition 1.9. *For a function $f : G \longrightarrow \mathbb{C}$ we consider, the weighted sup-norm*

$$\|f\|_\upsilon := \sup_{z\in G} \mid f(z) \mid \upsilon(z)$$

and the spaces

$$H(G) := \{f \mid f : G \longrightarrow \mathbb{C}, f \text{ is holomorphic}\}$$

$$H_\upsilon(G) := \{f \mid f : G \longrightarrow \mathbb{C}, f \text{ is holomorphic and } \|f\|_\upsilon < \infty\}$$

2. Inequalities and Estimations

Lemma 2.1. *Suppose $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) = \sum_{k=m}^{n} \alpha_k z^k$ (where $m, n \in \mathbb{N}$ and $m < n$). Also $r, s \in \mathbb{R}$, $0 < r < s, a, b \in \mathbb{C}$ such that $a + \gamma_r \subseteq b + \gamma_s \subset \Omega$ and $0 \notin b + \gamma_s$ then*

$$M_\infty(f, a+\gamma_r) \leq \left(\frac{r+\mid a \mid}{s-\mid b \mid}\right)^m M_\infty(f, b+\gamma_s)$$

Proof . Firstly note that $z \in a+\gamma_r$ and $\omega \in b+\gamma_s$ imply $\mid z \mid \leq r+ \mid a \mid$ and $\frac{1}{|\omega|} \leq \frac{1}{s-|b|}$ respectively. Define $g : \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$ by $g(z) = \sum_{k=0}^{n-m} \alpha_{k+m} z^k$ so $f(z) = z^m g(z)$.

$$M_\infty(f, a+\gamma_r) = \sup_{z\in a+\gamma_r} \mid f(z) \mid = \sup_{z\in a+\gamma_r} \mid z \mid^m \mid g(z) \mid$$

$$\leq \sup_{z\in a+\gamma_r} \mid z \mid^m \sup_{z\in a+\gamma_r} \mid g(z) \mid \leq (r+\mid a \mid)^m \sup_{z\in a+\gamma_r} \mid g(z) \mid .$$

Now, using maximum modulus principle we have

$$M_\infty(f, a+\gamma_r) \leq (r+\mid a \mid)^m \sup_{z\in b+\gamma_s} \mid g(z) \mid$$

Since $\mid g(z) \mid$ is a holomorphic function, there is a $\omega \in b+\gamma_s$ such that $\mid g(z) \mid$ attains its supremum in the point ω. So

$$M_\infty(f, a+\gamma_r) \leq (r+\mid a \mid)^m \mid g(\omega) \mid = \frac{(r+\mid a \mid)^m}{\mid \omega \mid^m} \mid \omega \mid^m \mid g(\omega) \mid$$

$$\leq (\frac{r+\mid a \mid}{s-\mid b \mid})^m \mid f(\omega) \mid \leq (\frac{r+\mid a \mid}{s-\mid b \mid})^m M_\infty(f, b+\gamma_s)$$

□

Lemma 2.2. *Suppose* $f : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ *defined by* $f(z) = \sum_{k=0}^n \alpha_k z^k$ *and* $0 < r < s$, $a, b \in \mathbb{C}$ *such that* $a+\gamma_r \subseteq b+\gamma_s \subset \Omega$. *Also* $(0,0) \notin a+\gamma_r$ *and* $(0,0) \notin b+\gamma_s$ *then*

$$M_\infty(f, b+\gamma_s) \leq (\frac{s+\mid b \mid}{r-\mid a \mid})^n M_\infty(f, a+\gamma_r)$$

Proof . Again we have $\mid z \mid \leq s+\mid b \mid$ and $\frac{1}{|\omega|} \leq \frac{1}{1-|a|}$. Define $g : \Omega \setminus \{(0,0)\} = \Omega_1 \subset \mathbb{C} \longrightarrow \mathbb{C}$ by $g(z) = \sum_{k=0}^n \alpha_k (\frac{1}{z})^{n-k} = \sum_{k=0}^n \alpha_k z^{k-n}$. So $f(z) = z^n g(z)$ $\forall z \in \Omega_1$.
Now, maximum modulus principle implies that

$$M_\infty(g, b+\gamma_s) \leq M_\infty(g, a+\gamma_r)$$

$$M_\infty(f, b+\gamma_s) = \sup_{z\in b+\gamma_s} \mid f(z) \mid = \sup_{z\in b+\gamma_s} \mid z \mid^n \mid g(z) \mid$$

$$\leq \sup_{z\in b+\gamma_s} \mid z \mid^n \sup_{z\in b+\gamma_s} \mid g(z) \mid \leq (s+\mid b \mid)^n M_\infty(g, b+\gamma_s)$$

$$\leq (s+\mid b \mid)^n M_\infty(g, a+\gamma_r)$$

Since $\mid g \mid$ is holomorphic, there exists a $\omega \in a+\gamma_r$ such that $\mid g \mid$ attains its maximum in ω. So

$$M_\infty(f, b+\gamma_s) \leq (s+\mid b \mid)^n \mid g(\omega) \mid = \frac{(s+\mid b \mid)^n}{\mid \omega \mid^n} \mid \omega \mid^n \mid g(\omega) \mid \leq$$

$$(\frac{s+\mid b \mid}{\mid r-\mid a \mid})^n \mid f(\omega) \mid \leq (\frac{(s+\mid b \mid)}{\mid r-\mid a \mid})^n M_\infty(f, a+\gamma_r)$$

□

Lemma 2.3. *Suppose* $0 < \delta_1 < \delta_2$ *and* $f : \boldsymbol{G} \longrightarrow \mathbb{C}$ *is defined by* $f(\omega) = \sum_{k=0}^\infty \alpha_k (\frac{\omega-i}{\omega+i})^k$. *Then* $M_\infty(f, L_{\delta_1}) \geq M_\infty(f, L_{\delta_2})$

Proof . We recall that if $\omega \in L_{\delta_1}$ and $\omega \in L_{\delta_2}$ then $\alpha^{-1}(\omega) \in \frac{\delta_1}{1+\delta_1} + \gamma_{\frac{1}{1+\delta_1}} \setminus \{(1,0)\} = C_1$ and $\alpha^{-1}(\omega) \in \frac{\delta_2}{1+\delta_2} + \gamma_{\frac{1}{1+\delta_2}} \setminus \{(1,0)\} = C_2$ respectively.
Since $\delta_1 < \delta_2$, $C_2 \subseteq C_1 \subseteq \mathbb{D}$. Define $f_1 : \mathbb{D} \longrightarrow \mathbb{C}$ by $f_1(\omega') = \sum_{k=0}^\infty \alpha_k (\omega')^k$ thus $M_\infty(f, L_{\delta_1}) = \sup_{\omega\in L_{\delta_1}} \mid f(\omega) \mid = \sup_{\omega'\in C_1} \mid f_1(\omega') \mid$ Using maximum modulus principle we have

$$\sup_{\omega'\in C_1} \mid f_1(\omega') \mid \geq \sup_{\omega'\in C_2} \mid f_1(\omega') \mid$$

Therefore,

$$M_\infty(f, L_{\delta_1}) \geq \sup_{\omega'\in C_2} \mid f_1(\omega') \mid = \sup_{\omega\in L_{\delta_2}} \mid f(\omega) \mid = M_\infty(f, L_{\delta_2})$$

□

Lemma 2.4. *Suppose $f : G \longrightarrow \mathbb{C}$ is defined by $f(\omega) = \sum_{k=0}^{\infty} \alpha_k (\frac{\omega - i}{\omega + i})^k$. If $0 < \delta_1 < \delta_2 < 1$ or $1 < \delta_1 < \delta_2$, then for any $n \in \mathbb{N}$*

$$M_\infty(f, L_{\delta_1}) \leq (\frac{1+\delta_2}{1-\delta_2})^n M_\infty(f, L_{\delta_2}) \tag{2.1}$$

Proof . Let C_1 and C_2 be as in the Lemma 2.3. Note that if $\omega' \in C_1$ then $\mid \omega' \mid \leq 1$ and if $\omega \in C_2$ then $\frac{1}{|\omega|} \leq \frac{1+\delta_2}{1-\delta_2}$.
Now, define $f_1 : C_1 \subseteq \mathbb{D} \longrightarrow \mathbb{C}$ by $f_1(\omega) = \sum_{k=0}^{\infty} \alpha_k \omega^k$ and $g_1 : C_1 \subseteq \mathbb{D} \longrightarrow \mathbb{C}$ by $g_1(\omega) = \sum_{k=0}^{\infty} \alpha_k (\frac{1}{\omega})^{k-n}$ (since $\delta_1 \neq 1$, $(0,0) \notin C_1$ and g_1 is well-defined) therefore, $f_1(\omega) = \omega^n g_1(\omega)$

$$M_\infty(f, L_{\delta_1}) = \sup_{\omega \in L_{\delta_1}} \mid f(\omega) \mid = \sup_{\omega' = \frac{\omega - i}{\omega + i}} \mid f_1(\omega') \mid$$

$$= \sup_{\omega' \in C_1} \mid (\omega')^n \mid\mid g_1(\omega') \mid \leq \sup_{\omega' \in C_1} \mid (\omega')^n \sup_{\omega' \in C_1} \mid g_1(\omega') \mid \leq \sup_{\omega' \in C_1} \mid g_1(\omega') \mid$$

thus, $M_\infty(f, L_{\delta_1}) \leq M_\infty(g_1, C_1)$. Since g_1 defined by variable $\frac{1}{\omega}$ instead of ω, maximum modulus principle implies that $M_\infty(g_1, C_1) \leq M_\infty(g_1, C_2)$.
Again, g_1 is a holomorphic function so there exists a $\omega \in C_2$ such that g_1 attains its maximum on C_2 in ω. Therefore,

$$M_\infty(f, L_{\delta_1}) \leq M_\infty(g_1, C_1) \leq M_\infty(g_1, C_2) = \mid g_1(\omega) \mid$$

but

$$\mid g_1(\omega) \mid = \frac{1}{\mid \omega \mid^n} \mid \omega \mid^n \mid g_1(\omega) \mid \leq (\frac{1+\delta_2}{1-\delta_2})^n \mid f_1(\omega) \mid$$

so

$$M_\infty(f, L_{\delta_1}) \leq (\frac{1+\delta_2}{1-\delta_2})^n \mid f_1(\omega) \mid \leq (\frac{1+\delta_2}{1-\delta_2})^n M_\infty(f_1, C_2)$$

$$\leq (\frac{1+\delta_2}{1-\delta_2})^n M_\infty(f, L_{\delta_2})$$

$\square$

Remark 2.5. *Indeed, if relation 2.1 is true for $n = 1$, then it's true for all $n > 1$.*

Lemma 2.6. *Suppose $h : G \longrightarrow \mathbb{C}$ is defined by $h(\omega) = \sum_{k=m}^{n} \alpha_k (\frac{\omega - i}{\omega + i})^k$ where $m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}$ and $0 < \delta < \tau, \delta \neq 1$. Then for any fixed point $\omega_0 \in L_\tau$ we have*

$$\mid h(\omega_0) \mid \leq \mid \frac{\omega_0 - i}{\omega_0 + i} \mid^m (\frac{1+\delta}{1-\delta})^m M_\infty(h, L_\delta)$$

Proof . Firstly, note that since $\delta \neq 1$, $(0,1) \notin L_\delta$. Thus for each $\omega \in L_\delta$ $\alpha^{-1}(\omega) = \frac{\omega - i}{\omega + i} \neq 0$. Define $g : G \longrightarrow \mathbb{C}$ by $g(\omega) = \sum_{k=m}^{n} \alpha_k (\frac{\omega - i}{\omega + i})^{k-m}$ so $h(\omega) = (\frac{\omega_0 - i}{\omega_0 + i})^m g(\omega)$. We have

$$\mid g(\omega_0) \mid \leq \sup_{\omega \in L_\tau} \mid g(\omega) \mid = M_\infty(g, L_\tau) \leq M_\infty(g, L_\delta)$$

The last inequality is a consequence of Lemma 2.3. Now, since g is holomorphic, there exists a point in L_δ (call it again ω) such that g attains its maximum on L_δ in ω, thus $\mid g(\omega_0) \mid \leq \mid g(\omega) \mid$.

$$\mid h(\omega_0) \mid = \mid \frac{\omega_0 - i}{\omega_0 + i} \mid^m \mid g(\omega_0) \mid \leq \mid \frac{\omega_0 - i}{\omega_0 + i} \mid^m \mid g(\omega) \mid$$

$\mid h(\omega_0) \mid \leq \mid \frac{\omega_0 - i}{\omega_0 + i} \mid^m \mid \frac{\omega - i}{\omega + i} \mid^m \mid h(\omega) \mid$. Therefore,

$$\mid h(\omega_0) \mid \leq \mid \frac{\omega_0 - i}{\omega_0 + i} \mid^m \mid \frac{\omega - i}{\omega + i} \mid^m \; M_\infty(h, L_\delta) \tag{2.2}$$

Now, we make an upper bound for the factor $\mid \frac{\omega - i}{\omega + i} \mid^m$. Since ω is a point in L_δ, there exists a $x \in \mathbb{R}$ such that $\omega = x + i\delta$. $\mid \omega + i \mid^2 = (\omega + i)\overline{(\omega - i)} = (\omega + i)(\overline{\omega} - i) = \mid \omega \mid^2 - i\omega + i\overline{\omega} + 1 = x^2 + (1+\delta)^2$. Similarly, $\mid \omega - i \mid^2 = x^2 + (1-\delta)^2$. So $\frac{|\omega+i|^2}{|\omega-i|^2} = \frac{x^2+(1+\delta)^2}{x^2+(1-\delta)^2} = f(t) = \frac{t+a}{t+b}$ where $t = x^2, (1+\delta)^2 = a$ and $(1-\delta)^2 = b$.
$\frac{|\omega+i|^2}{|\omega-i|^2} = \frac{x^2+(1+\delta)^2}{x^2+(1-\delta)^2} = f(t) = \frac{t+a}{t+b}$ where $t = x^2, (1+\delta)^2 = a$ and $(1-\delta)^2 = b$. Since $f'(t) = \frac{b-a}{(t+b)^2} < 0$, $f : [0, \infty) \longrightarrow \mathbb{R}$ is a decreasing function and $\max f = f(0) = \frac{a}{b}$. This gives $\frac{|\omega+i|^2}{|\omega-i|^2} \leq \frac{(1+\delta)^2}{(1-\delta)^2}$. Therefore,

$$\mid \frac{\omega - i}{\omega + i} \mid^m \leq \left(\frac{1+\delta}{1-\delta}\right)^m \tag{2.3}$$

By inserting relation 2.3 in relation 2.2 we are done. □

Put $m_1 = 1$, since ν is an increasing function and $\lim_{t\to 0} \nu(ti) = 0$, we can find an integer m_2 such that m_2 is the smallest integer larger than m_1 for which $\frac{\nu(\frac{1}{2^{m_2}}i)}{\nu(\frac{1}{2^{m_1}}i)} \leq \frac{1}{2}$. Hence, by induction, we can define a sequence of integers $\{m_n\}$ such that m_{n+1} is the smallest integer larger than m_n, for which we have $\frac{\nu(\frac{1}{2^{m_{n+1}}}i)}{\nu(\frac{1}{2^{m_n}}i)} \leq \frac{1}{2}$.

Clearly in the above construction we can begin with $m_1 = 0,\ m_1 = 2$ or any other integers.

Remark 2.7. *From now on , we always assume sequence $\{m_n\}$ has constructed such that*

$$\frac{\nu(\frac{1}{2^{m_{n+1}}}i)}{\nu(\frac{1}{2^{m_n}}i)} \leq \frac{1}{2}.$$

Lemma 2.8. *Let $0 < \tau \leq \frac{1}{2}$ be given. If $f : G \longrightarrow \mathbb{C}$ is defined by $f(\omega) = \sum_{k'=2^{m_n}}^{2^{m_{n+1}}} \alpha_{k'} (\frac{\omega - i}{\omega + i})^{k'}$ where $m_n, m_{n+1} \in \{m_n\}$, then there exists a universal constant $C > 0$ such that*

$$\mid f(i\tau) \mid \nu(\tau i) \leq C M_\infty(f, L_{\frac{1}{2^{m_{n+1}-1}}}) \nu(\frac{1}{2^{m_{n+1}-1}} i)$$

Proof . Firstly, note that, we can find $m_k \in \{m_n\}$ such that $\frac{1}{2^{m_{k+1}}} \leq \tau \leq \frac{1}{2^{m_k}}$. We prove the lemma in two cases.

Case 1: $(\tau > \frac{1}{2^{m_{n+1}-1}})$: Now, by using Lemma 2.6 for $\omega_0 = i\tau, h = f$ and $\delta = \frac{1}{2^{m_{n+1}-1}}$, we have

$$\mid f(i\tau) \mid \leq \mid \frac{1-\tau}{1+\tau} \mid^{2^{m_n}} \left(\frac{1+\delta}{1-\delta}\right)^{2^{m_n}} M_\infty(f, L_\delta)$$

Before continuing the proof we recall that for all nonnegative x we have

$$1 + x \leq e^x \;\&\; 1 - x \leq e^{-x}.$$

Since $0<\delta<1$ and $\frac{1+\delta}{1-\delta}=1+\frac{2\delta}{1-\delta}$, $(\frac{1+\delta}{1-\delta})^{2^{m_n}}\leq e^{\frac{2\delta}{1-\delta}2^{m_n}}$. Also $\frac{2\delta}{1-\delta}=\frac{2}{2^{m_{n+1}-1}-1}$. Hence, $e^{\frac{2\delta}{1-\delta}2^{m_n}}=e^{\frac{2}{2^{m_{n+1}-1}-1}2^{m_n}}=e^{\frac{2}{2^{m_{n+1}-m_n-1}-2^{-m_n}}}$. Since $\{m_n\}$ is an increasing sequence, $m_{n+1}-m_n-1\geq 0$. So $2^{m_{n+1}-m_n-1}\geq 2^0$ which implies that $\frac{2}{2^{m_{n+1}-m_n-1}}\leq\frac{2}{2^0-2^{-m_n}}$. consequently, $e^{\frac{2}{2^{m_{n+1}-m_n-1}-2^{-m_n}}}\leq e^{\frac{2}{2^0-2^{-m_n}}}$.
$m_n\geq 1\Rightarrow\frac{1}{2^0-2^{-m_n}}\leq\frac{1}{2^0-2^1}$. Hence, $e^{\frac{2}{2^0-2^{-m_n}}}\leq e^{2^0-2^1}=e^4$ and we have

$$(\frac{1+\delta}{1-\delta})^{2^{m_n}}\leq e^4.$$

$\frac{1-\tau}{1+\tau}=1-\frac{2\tau}{1+\tau}\Rightarrow\mid\frac{1-\tau}{1+\tau}\mid^{2^{m_n}}\leq e^{-\frac{2\tau}{1+\tau}2^{m_n}}$. Since, $\tau\geq\frac{1}{2^{m_{k+1}}}$, $1+\frac{1}{\tau}\leq 1+2^{m_{k+1}}\Rightarrow\frac{-2\tau}{1+\tau}\leq\frac{-2}{1+2^{m_{k+1}}}$. Thus, $e^{-\frac{2\tau}{1+\tau}2^{m_n}}\leq e^{\frac{-2\ 2^{m_n}}{1+2^{m_{k+1}}}}$.

Up to now, we have shown

$$\mid f(i\tau)\mid\leq e^4e^{\frac{-2\ 2^{m_n}}{1+2^{m_{k+1}}}}M_\infty(f,L_\delta)\ where\ \delta=\frac{1}{2^{m_{n+1}-1}}.$$

Above relation implies that

$$\mid f(i\tau)\mid\nu(\tau i)\leq e^4e^{\frac{-2\ 2^{m_n}}{1+2^{m_{k+1}}}}M_\infty(f,L_\delta)\nu(\frac{1}{2^{m_{n+1}-1}}i)\frac{\nu(\tau i)}{\nu(\frac{1}{2^{m_{n+1}-1}}i)}.$$

Since ν is increasing and $\tau\leq\frac{1}{2^{m_k}}$, we have

$$\mid f(i\tau)\mid\nu(\tau i)\leq e^4M_\infty(f,L_\delta)\nu(\frac{1}{2^{m_{n+1}-1}}i)e^{\frac{-2\ 2^{m_n}}{1+2^{m_{k+1}}}}\frac{\nu(\frac{1}{2^{m_k}}i)}{\nu(\frac{1}{2^{m_{n+1}-1}}i)}\tag{2.4}$$

By the construction of the sequence $\{m_n\}$, we have $\frac{\nu(\frac{1}{2^{m_{n+1}}}i)}{\nu(\frac{1}{2^{m_n}}i)}\leq\frac{1}{2}$ $\forall n\in\mathbb{N}$. Thus, $\frac{\nu(\frac{1}{2^{m_{n+1}-1}}i)}{\nu(\frac{1}{2^{m_n}}i)}>\frac{1}{2}$ $\forall n\in\mathbb{N}$ and this gives

$$\frac{\nu(\frac{1}{2^{m_n}}i)}{\nu(\frac{1}{2^{m_{n+1}-1}}i)}<2\quad\forall n\in\mathbb{N}\tag{2.5}$$

Clearly,

$$\frac{\nu(\frac{1}{2^{m_k}}i)}{\nu(\frac{1}{2^{m_{n+1}-1}}i)}=\frac{\nu(\frac{1}{2^{m_k}}i)}{\nu(\frac{1}{2^{m_{k+1}}}i)}\frac{\nu(\frac{1}{2^{m_{k+1}}}i)}{\nu(\frac{1}{2^{m_{k+2}}}i)}\cdots\frac{\nu(\frac{1}{2^{m_n}}i)}{\nu(\frac{1}{2^{m_{n+1}-1}}i)}\tag{2.6}$$

Now, we estimate each factor of the right hand side of relation 2.6. Obviously,

$$\frac{\nu(\frac{1}{2^{m_k}}i)}{\nu(\frac{1}{2^{m_{k+1}}}i)}=\frac{\nu(\frac{1}{2^{m_k}}i)}{\nu(\frac{1}{2^{m_{k+1}-1}}i)}\frac{\nu(\frac{1}{2^{m_{k+1}-1}}i)}{\nu(\frac{1}{2^{m_{k+1}}}i)}$$

Since, ν satisfies $(*)$ (see Definition 1.7 and Remark 1.8) and relation relation 2.5 holds, we have

$$\frac{\nu(\frac{1}{2^{m_k}}i)}{\nu(\frac{1}{2^{m_{k+1}}}i)}\leq 2a$$

Similarly, each factor of the right hand side of relation 2.6 is less than or equal to $2a$. Thus,

$$\frac{\nu(\frac{1}{2^{m_k}}i)}{\nu(\frac{1}{2^{m_{n+1}-1}}i)} \leq (2a)^{n-k+1} \tag{2.7}$$

Clearly, we can find a nonnegative constant M such that $a \leq e^M$. Hence,

$$(2a)^{n-k+1} \leq e^{n-k+1}e^{Mn-Mk+M} = e^{n-k+1+Mn-Mk+M} \tag{2.8}$$

Relations 2.7 and 2.8 imply that

$$e^{\frac{-2\ 2^{m_n}}{1+2^{m_{k+1}}}} \frac{\nu(\frac{1}{2^{m_k}}i)}{\nu(\frac{1}{2^{m_{n+1}-1}}i)} \leq e^{\frac{-2\ 2^{m_n}}{1+2^{m_{k+1}}}+n-k+1+Mn-Mk+M} := D \tag{2.9}$$

Now, we make an upper bound for $\frac{-2\ 2^{m_n}}{1+2^{m_{k+1}}}$
$\frac{-2\ 2^{m_n}}{1+2^{m_{k+1}}} = \frac{-2\ 2^{m_n-m_{k+1}}}{2^{-m_{k+1}}+1}$.
$m_n - m_{k+1} \geq n-k-1 \Rightarrow 2^{m_n-m_{k+1}} \geq 2^{n-k-1} \Rightarrow -2\ 2^{m_n-m_{k+1}} \geq -2^{n-k}$. Also $\frac{1}{1+2^{-m_{k+1}}} \leq 1$. Thus,

$$D \leq e^{-2^{n-k}+n-k+1+Mn-Mk+M} := e^E$$

Since, M and k are fixed, for large enough n, $E \leq 1$ and $D \leq e$.
Finally, relation 2.4 and relation 2.9 imply that

$$|\ f(i\tau)\ |\ \nu(\tau i) \leq e^5 M_\infty(f, L_\delta)\nu(\delta i) \ where\ \delta = \frac{1}{2^{m_{n+1}-1}}$$

which completes the proof in **Case 1**.
Case 2: ($\tau \leq \frac{1}{2^{m_{n+1}-1}}$): Note that, $|\ f(i\tau)\ | \leq M_\infty(f, L_\delta)$. Since $\tau \leq \frac{1}{2^{m_{n+1}-1}}$, Using Lemma 2.4, we have

$$M_\infty(f, L_\tau) \leq (\frac{1+\delta}{1-\delta})^{2^{m_{n+1}}} M_\infty(f, L_\delta)\ where \delta = \frac{1}{2^{m_{n+1}-1}}$$

Thus,

$$\begin{aligned}|\ f(i\tau)\ |\ \nu(\tau i) &\leq (\frac{1+\delta}{1-\delta})^{2^{m_{n+1}}} M_\infty(f, L_\delta)\ \nu(\tau i)\\ &\leq (\frac{1+\delta}{1-\delta})^{2^{m_{n+1}}} M_\infty(f, L_\delta)\ \nu(\delta i)\end{aligned}$$

$\frac{1+\delta}{1-\delta} = 1 + \frac{2\delta}{1-\delta} \Rightarrow \ \frac{1+\delta}{1-\delta}^{2^{m_{n+1}}} \leq e^{\frac{2\delta}{1-\delta}2^{m_{n+1}}}$. Also $\frac{2\delta}{1-\delta}2^{m_{n+1}} = \frac{2}{2^{-1}-\frac{1}{2^{m_{n+1}}}}$.
Since, $m_{n+1} \geq 2$, $\frac{2}{2^{-1}-\frac{1}{2^{m_{n+1}}}} \leq 8$. Hence, $(\frac{1+\delta}{1-\delta})^{2^{m_{n+1}}} \leq e^8$. Therefore,

$$|\ f(i\tau)\ |\ \nu(\tau i) \leq e^8\ M_\infty(f, L_\delta)\ \nu(\delta i) \ \ where\ \delta = \frac{1}{2^{m_{n+1}-1}}.$$

Above relation proves the lemma in **Case 2**. Now, put $C = e^8$, we are done. $\square$
We say two real factors A and B are equivalent and we write $A \sim B$ iff there are universal constants a and b such that $aA \leq B \leq bB$.
We conclude this section with the following three corollaries.

Corollary 2.9. *Suppose* $f : G \longrightarrow \mathbb{C}$ *is defined by* $f(\omega) = \sum_{k=2^{m_n}}^{2^{m_{n+1}}} \alpha_k(\frac{\omega-i}{\omega+i})^k$, *where* $m_n, m_{n+1} \in \{m_n\}$ *and* $\delta = \frac{1}{2^{m_{n+1}-1}}$. *Then* $M_\infty(f, L_0) \sim M_\infty(f, L_\delta)$

Proof . Using Lemmas 2.3 and 2.4 for $\delta_1 = 0 < \delta_2 = \delta$ we have the following relations.
$M_\infty(f, L_\delta) \leq M_\infty(f, L_0)$ and $M_\infty(f, L_\delta) \leq (\frac{1+\delta}{1-\delta})^{2^{m_{n+1}}} M_\infty(f, L_\delta)$.
Now similar to the proof of the Lemma 2.8 again, $(\frac{1+\delta}{1-\delta})^{2^{m_{n+1}}} \leq e^8$. Therefore,

$$M_\infty(f, L_\delta) \leq M_\infty(f, L_0) \leq e^8 M_\infty(f, L_\delta)$$

□

Corollary 2.10. *$f : G \longrightarrow \mathbb{C}$ is defined by $f(\omega) = \sum_{k=2^{m_n}}^{2^{m_{n+1}}} \alpha_k (\frac{\omega - i}{\omega + i})^k$, where $m_n, m_{n+1} \in \{m_n\}$ and $\delta_1 = \frac{1}{2^{m_{n+1}}}$ and $\delta_2 = \frac{1}{2^{m_{n+1}-1}}$. Then $M_\infty(f, L_{\delta_1})\ \nu(\delta_1 i) \sim M_\infty(f, L_{\delta_2})\ \nu(\delta_2 i)$*

Proof . Lemma 2.3 implies $M_\infty(f, L_{\delta_1}) \geq M_\infty(f, L_{\delta_2})$. Since ν satisfies $(*)$, $\nu(\delta_2 i) \leq a\nu(\delta_1 i)$. Hence,

$$a\ M_\infty(f, L_{\delta_1})\ \nu(\delta_1 i) \geq M_\infty(f, L_{\delta_2})\ \nu(\delta_2 i) \tag{2.10}$$

Using Lemma 2.4 and the argument of Lemma 2.8 we have $M_\infty(f, L_{\delta_1}) \leq e^8 M_\infty(f, L_{\delta_2})$.
ν is increasing so

$$M_\infty(f, L_{\delta_1})\ \nu(\delta_1 i) \leq e^8 M_\infty(f, L_{\delta_2})\ \nu(\delta_2 i) \tag{2.11}$$

Now, relation 2.10 and relation 2.11 prove the corollary. □

Corollary 2.11. *Under the assumptions of Corollary 2.10*

$$M_\infty(f, L_{\delta_3})\ \nu(\delta_3 i) \leq 2\ M_\infty(f, L_{\delta_2})\ \nu(\delta_2 i) \leq 2a\ M_\infty(f, L_{\delta_1})\ \nu(\delta_1 i)$$

where $\delta_3 = \frac{1}{2^{m_n}}$.

Proof . Since $\delta_2 \leq \delta_3$, Lemma 2.3 implies that $M_\infty(f, L_{\delta_2}) \geq M_\infty(f, L_{\delta_3})$. Relation relation 2.5 in Lemma 2.8 gives

$$M_\infty(f, L_{\delta_3})\ \nu(\delta_3 i) \leq 2 M_\infty(f, L_{\delta_2})\ \nu(\delta_2 i)$$

Now, use relation 2.10 in Corollary 2.10 to conclude the proof. □

3. Main result

For arriving to the main result of this paper, we need to introduce following concepts.
For a map $f : G \longrightarrow \mathbb{C}$ is defined by $f(\omega) = \sum_{k=0}^{N} \alpha_k (\frac{\omega - i}{\omega + i})^k$ $(N \in \mathbb{N} \cup \{\infty\})$, we denote the composition map $f \circ \alpha$ by $\tilde{f}$ which is defined from $\mathbb{D}$ into $\mathbb{D}$ by

$$(f \circ \alpha)(z) = \tilde{f}(z) = f(\alpha(z)) = \sum_{k=0}^{N} \alpha_k (\frac{\alpha(z) - i}{\alpha(z) + i})^k = \sum_{k=0}^{N} \alpha_k z^k$$

Now, consider $f : G \longrightarrow \mathbb{C}$ defined by $f(\omega) = \sum_{k=0}^{N} \alpha_k (\frac{\omega - i}{\omega + i})^k$ $(N \in \mathbb{N} \cup \{\infty\})$, for any $n \in \mathbb{N}$, we define $R_n f, \tilde{R_n} \tilde{f}$ as follows.

$$(\tilde{R_n} \tilde{f})(z) = \sum_{k=0}^{2^n} \alpha_k z^k + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k z^k$$

$$(R_n f)(\omega) = \sum_{k=0}^{2^n} \alpha_k (\frac{\omega - i}{\omega + i})^k + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k (\frac{\omega - i}{\omega + i})^k$$

$\tilde{R_n}$ is a convolution with a de-la-valle-Poussion kernel on $\mathbb{D}$.

Remark 3.1. *Note that, if* $N < n$ *then* $R_n f = f$ *and* $\tilde{R}_n \tilde{f} = \tilde{f}$*. Also, it is clear that* $\tilde{R}_n \tilde{R}_m \tilde{f} = \tilde{R}_{\min(m,n)}$ *and* $R_n R_m f = R_{\min(m,n)} f$ $\forall$ $m, n \in \mathbb{N}$ *such that* $m \neq n$*.*

Remark 3.2. *For* $f : G \longrightarrow \mathbb{C}$ *defined by* $f(\omega) = \sum_{k'=0}^{N} \alpha_{k'} (\frac{\omega - i}{\omega + i})^{k'}$ $(N \in \mathbb{N} \cup \{\infty\})$ *we have* $(R_{m_{k+1}} - R_{m_k})(f) = \sum_{k'=2^{m_k}}^{2^{m_k+1}} \alpha_{k'} (\frac{k' - 2^{m_k}}{2^{m_k}})(\frac{\omega - i}{\omega + i})^{k'} + \sum_{k'=2^{m_k+1}+1}^{2^{m_{k+1}}} \alpha_{k'} (\frac{\omega - i}{\omega + i})^{k'} + \sum_{k'=2^{m_{k+1}}+1}^{2^{m_{k+1}+1}} \alpha_{k'} \frac{2^{m_{k+1}+1} - k'}{2^{m_{k+1}}} (\frac{\omega - i}{\omega + i})^{k'}$ *where* $m_k, m_{k+1} \in \{m_n\}$*.*
Also, we can rewrite $(R_{m_{k+1}} - R_{m_k})(f)$ *in the following shorter form.*
$(R_{m_{k+1}} - R_{m_k})(f) = \sum_{k'=2^{m_k}}^{2^{m_{k+1}+1}} \beta_{k'} (\frac{\omega - i}{\omega + i})^{k'}$.

Lemma 3.3. *Suppose* $f : G \longrightarrow \mathbb{C}$ *defined by* $f(\omega) = \sum_{k'=0}^{\infty} \alpha_{k'} (\frac{\omega - i}{\omega + i})^{k'}$ *and let* $0 < \tau \leq \frac{1}{2}$ *be given. Then there exists a universal constant* $C > 0$ *such that*

$$\mid f(i\tau) \mid \nu(\tau i) \leq C \sup_{k \in \mathbb{N} \cup \{0\}} M_\infty((R_{m_{k+1}} - R_{m_k}) f, L_{\delta_k}) \nu(\delta_k i)$$

where $\delta_k = \frac{1}{2^{m_{k+1}+1}}$ *and* $m_k, m_{k+1} \in \{m_n\}$ *for any* $k \in \mathbb{N}$*.*

Proof . For each $k \in \mathbb{N}$, we define $g_k(\omega) := (R_{m_{k+1}} - R_{m_k})(f)(\omega) = \sum_{k'=2^{m_k}}^{2^{m_{k+1}+1}} \beta_{k'} (\frac{\omega - i}{\omega + i})^{k'}$. Since $0 < \tau \leq \frac{1}{2}$, there exists $l \in \mathbb{N}$ such that $\frac{1}{2^{m_l+1}} \leq \tau \leq \frac{1}{2^{m_l}}$. If we assume $m_0 = 0$ and $R_{m_0} f = 0$, then $\sum_{k=0}^{M} g_k(\omega) = R_{m_{M+1}}(f)(\omega)$ which implies that $\sum_{k=0}^{\infty} g_k(\omega) = f(\omega)$ since $\lim_{M \to \infty}(f) = f$. Hence, in particular $f(i\tau) = \sum_{k=0}^{\infty} g_k(i\tau)$. Now, we have

$$\mid f(i\tau) \mid \leq \sum_{k=0}^{\infty} \mid g_k(i\tau) \mid = \sum_{\delta_k \geq \tau} \mid g_k(i\tau) \mid + \sum_{\delta_k < \tau} \mid g_k(i\tau) \mid$$

$$\leq \sum_{k=0}^{l-2} \mid g_k(i\tau) \mid + \sum_{k=l-1}^{\infty} \mid g_k(i\tau) \mid$$

Firstly, we compute $\sum_{k=l-1}^{\infty} \mid g_k(i\tau) \mid$. Here, $\delta_k < \tau$ $\forall$ $k \geq l-1$. Therefore, by using Lemma 2.6 for $\omega = i\tau$ we have

$$\mid g_k(i\tau) \mid \leq \mid \frac{\tau - 1}{\tau + 1} \mid^{2^{m_k}} (\frac{1 + \delta_k}{1 - \delta_k})^{2^{m_k}} M_\infty(g_k, L_{\delta_k}) \tag{3.1}$$

for all $k \geq l-1$.
Since $\delta_k < \tau < \frac{1}{2^{m_l}}$, $\frac{\nu(\tau i)}{\nu(\delta_k i)} \leq \frac{\nu(\frac{1}{2^{m_l}})}{\nu(\frac{1}{2^{m_{k+1}+1}})} \leq (2a)^{k-l+1}$ (see the proof of the **Case 1** of Lemma 2.8). Thus,

$$\nu(\tau i) \leq (2a)^{k-l+1} \nu(\delta_k i) \ \forall k, \quad k \geq l-1$$

Again, as in the proof of the **Case 1** of Lemma 2.8 there exists a positive M such that

$$(2a)^{k-l+1} \leq e^{(M+1)(k-l+1)}$$

Now, insert the above relations in relation relation 3.1 we have

$$\mid g_k(i\tau) \mid \ \nu(\tau i) \leq \mid \frac{\tau - 1}{\tau + 1} \mid^{2^{m_k}} (\frac{1 + \delta_k}{1 - \delta_k})^{2^{m_k}} M_\infty(g_k, L_{\delta_k}) e^{(M+1)(k-l+1)} \nu(\delta_k i) \tag{3.2}$$

for all $k \geq l-1$.
Now, we find upper bounds for factors $\mid \frac{\tau - 1}{\tau + 1} \mid^{2^{m_k}}$ and $(\frac{1 + \delta_k}{1 - \delta_k})^{2^{m_k}}$.

Since, $0<\tau<1$, $\mid \frac{\tau-1}{\tau+1} \mid^{2^{m_k}}= (\frac{1-\tau}{\tau+1})^{2^{m_k}} \leq e^{-\frac{2\tau}{1+\tau}2^{m_k}}$.
$\tau \geq \frac{1}{2^{m_l+1}} \Rightarrow -\frac{2\tau}{1+\tau}2^{m_k} \leq \frac{-2\ 2^{m_k}}{1+2^{m_l+1}} = \frac{-2\ 2^{m_k-m_l-1}}{2^{-m_l-1}+1}$.
Since $m_k-m_l-1>k-l-1$, $2^{m_k-m_l-1}>2^{k-l-1} \Rightarrow -2\ 2^{m_k-m_l-1}<-2^{k-l} \Rightarrow \frac{-2\ 2^{m_k}}{1+2^{m_l+1}} < \frac{-2^{k-l}}{2^{-m_l-1}+1} < -2^{k-l}$. Thus,

$$\mid \frac{\tau-1}{\tau+1} \mid^{2^{m_k}} \leq e^{-2^{k-l}} \tag{3.3}$$

Also, $(\frac{1+\delta_k}{1-\delta_k})^{2^{m_k}} \leq e^{\frac{2\delta_k}{1-\delta_k}}$ and $\frac{2\delta_k}{1-\delta_k} = \frac{2}{2^{m_{k+1}-m_k+1}-2^{-m_k}}$. Since $m_{k+1}-m_k+1 \geq 2$, $2^{m_{k+1}-m_k+1} \geq 2^2$. $m_k \geq 1 \Rightarrow -2^{-m_k} \geq -2^{-1}$.
Therefore, $\frac{2}{2^{m_{k+1}-m_k+1}-2^{-m_k}} \leq \frac{4}{7}$ which implies that

$$(\frac{1+\delta_k}{1-\delta_k})^{2^{m_k}} \leq e^{\frac{4}{7}} \tag{3.4}$$

Now, put relations 3.3 and 3.4 in relation 3.2 we have

$$\mid g_k(i\tau) \mid \nu(\tau i) \leq e^{-2^{k-l}} e^{\frac{4}{7}} e^{(M+1)(k-l)} e^{(M+1)} M_\infty(g_k, L_{\delta_k})\nu(\delta_k i)$$

for all $k \geq l-1$.
Thus,

$$\sum_{k=l-1}^{\infty} \mid g_k(i\tau) \mid \ \nu(\tau i) \leq e^{\frac{4}{7}} e^{(M+1)} \sup_{k \geq l-1} M_\infty(g_k, L_{\delta_k})\nu(\delta_k i) \sum_{k=l-1}^{\infty} e^{-2^{k-l}+(k-l)(M+1)}$$

Use the root test to see $\sum_{k=l-1}^{\infty} e^{-2^{k-l}+(k-l)(M+1)}$ is convergent (say to C_1). Therefore,

$$\sum_{k=l-1}^{\infty} \mid g_k(i\tau) \mid \nu(\tau i) \leq e^{\frac{4}{7}} e^{(M+1)}\ C_1 \sup_{k \geq l-1} M_\infty(g_k, L_{\delta_k})\nu(\delta_k i)$$

Now, we estimate $\sum_{k=0}^{l-2} \mid g_k(i\tau) \mid \nu(\tau i)$. Here $\delta_k > \tau$, so Lemma 2.4 implies that

$$\mid g_k(i\tau) \mid \leq (\frac{1+\delta_k}{1-\delta_k})^{2^{m_{k+1}+1}} M_\infty(g_k, L_{\delta_k}) \tag{3.5}$$

for all k, $0 \leq k \leq l-2$. It easy to see that

$$(\frac{1+\delta_k}{1-\delta_k})^{2^{m_{k+1}+1}} \leq e^{\frac{8}{3}} \tag{3.6}$$

$$\frac{\nu(\frac{1}{2^{m_l}}i)}{\nu(\frac{1}{2^{m_{k+1}+1}}i)} = \frac{\nu(\frac{1}{2^{m_{k+2}}}i)}{\nu(\frac{1}{2^{m_{k+1}+1}}i)} \frac{\nu(\frac{1}{2^{m_{k+3}}}i)}{\nu(\frac{1}{2^{m_{k+2}}}i)} \cdots \frac{\nu(\frac{1}{2^{m_l}}i)}{\nu(\frac{1}{2^{m_{l-1}}}i)} \tag{3.7}$$

Remark 2.7 and Remark 1.8 imply that

$$\frac{\nu(\frac{1}{2^{m_{k+2}}}i)}{\nu(\frac{1}{2^{m_{k+1}+1}}i)} = \frac{\nu(\frac{1}{2^{m_{k+2}}}i)}{\nu(\frac{1}{2^{m_{k+1}}}i)} \frac{\nu(\frac{1}{2^{m_{k+1}}}i)}{\nu(\frac{1}{2^{m_{k+1}+1}}i)} \leq \frac{1}{2}a$$

The other factors of the right hand side of relation 3.7 are less than $\frac{1}{2}$, hence

$$\frac{\nu(\frac{1}{2^{m_l}}i)}{\nu(\frac{1}{2^{m_{k+1}+1}}i)} \leq (\frac{1}{2})^{l-k-1}a$$

Since ν is increasing, $\tau < \frac{1}{2^{m_l}}$ and $\delta_k = \frac{1}{2^{m_{k+1}+1}}$, we have

$$\nu(\tau i) \leq (\frac{1}{2})^{l-k-1}a\ \nu(\delta_k i) \tag{3.8}$$

Considering relations 3.5, 3.6 and 3.8 we have

$$\mid g_k(i\tau) \mid \nu(\tau i) \leq e^{\frac{8}{3}}(\frac{1}{2})^{l-k-1}a\ M_\infty(g_k, L_{\delta_k})\nu(\delta_k i)$$

which implies that

$$\sum_{k=0}^{l-2} \mid g_k(i\tau) \mid \nu(\tau i) \leq ae^{\frac{8}{3}} \sup_{0\leq k\leq l-2} M_\infty(g_k, L_{\delta_k})\ \nu(\delta_k i)$$

since $\sum_{k=0}^{l-2}(\frac{1}{2})^{l-k-1} \leq 1$. Now, using Corollary 2.10 $(m_2 - 1)$ times inductively we can find a universal constant C' such that

$$M_\infty(g_0, L_{\delta_0})\ \nu(\delta_0 i) \leq M_\infty(g_1, L_{\delta_1})\ \nu(\delta_1 i)$$

Hence,

$$\sup_{0\leq k\leq l-2} M_\infty(g_k, L_{\delta_k})\nu(\delta_k i) \leq C' \sup_{k\in\mathbb{N}} M_\infty(g_k, L_{\delta_k})\ \nu(\delta_k i)$$

Therefore,

$$\sum_{k=0}^{l-2} \mid g_k(i\tau) \mid \nu(\tau i) \leq C'a\ e^{\frac{8}{3}} \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty(g_k, L_{\delta_k})\ \nu(\delta_k i)$$

Finally, we have

$\mid f(i\tau) \mid \nu(\tau i) \leq \sum_{k=0}^{l-2} \mid g_k(i\tau) \mid \nu(\tau i) + \sum_{k=l-1}^{\infty} \mid g_k(i\tau) \mid \nu(\tau i) \leq C'a\ e^{\frac{8}{3}} \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty(g_k, L_{\delta_k})\nu(\delta_k i) + C_2 \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty(g_k, L_{\delta_k})\nu(\delta_k i)$

Where $C_2 = e^{\frac{4}{7}}e^{(M+1)}\ C_1$. Now, put $C = \max(C'a\ e^{\frac{8}{3}}, C_2)$. Since $g_k = (R_{m_{k+1}} - R_{m_k})(f)$, proof is complete. □

Lemma 3.4. *Let ν be a bounded weight on $G_{\frac{1}{2}}$ and $A := \{z \in G : Im\ z \leq \frac{1}{2}\}$. Then*

$$\|f\|_\nu \sim \sup_{z\in A} \mid f(z) \mid \nu(z)$$

Proof . Obviously, $\sup_{z\in A} \mid f(z) \mid \nu(z) < \|f\|_\nu$. By Lemma 2.3 $\sup_{z\in A} \mid f(z) \mid\geq \sup_{z\in L_\delta} \mid f(z) \mid$ for any $\delta > \frac{1}{2}$. Hence, $\sup_{z\in G} \mid f(z) \mid\leq \sup_{z\in A} \mid f(z) \mid$. Since ν is increasing and bounded, there exists a constant C such that $\nu(z) = \frac{\nu(z)}{\nu(\frac{1}{2}i)}\nu(\frac{1}{2}i) \leq C\nu(\frac{1}{2}i)$ for all $z \in G$. Therefore,

$$\|f\|_\nu \leq C \sup_{z\in A} \mid f(z) \mid \nu(\frac{1}{2}i) \leq C \sup_{z\in A} \mid f(z) \mid \nu(z)$$

□

For any $a \in \mathbb{R}$, we define $T_a : H_\nu(G) \longrightarrow H_\nu(G)$ by $(T_a f)(z) = f(z + a)$. It is easy to see that $\|T_a f\|_\nu = \|f\|_\nu$. Thus, T_a is an isometric isomorphism.

Lemma 3.5. *Suppose $f \in H(G)$ and ν is a bounded weight on $G_{\frac{1}{2}}$. Then there exists a positive universal constant C such that*

$$\|f\|_\nu \leq \sup_{a\in\mathbb{R}} \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty((R_{m_{k+1}} - R_{m_k})T_a f, L_{\delta_k})\nu(\delta_k i)$$

where $\delta_k = \frac{1}{2^{m_{k+1}+1}}$ and $m_k, m_{k+1} \in \{m_n\}$ for any $k \in \mathbb{N}$.

Proof . Consider a fixed z in the strip $A := \{z \in G : Im\ z \leq \frac{1}{2}\}$. For $\tau = Im\ z$ we have

$$| f(z) | \nu(\tau i) =| (T_{Re\ z} f(i\tau) | \nu(\tau i)$$

By Lemma 3.3, there exists a $C > 0$ such that

$$| (T_{Re\ z} f(i\tau) | \nu(\tau i) \leq C \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty((R_{m_{k+1}} - R_{m_k})T_{Re\ z} f, L_{\delta_k})\nu(\delta_k i)$$

where $\delta_k = \frac{1}{2^{m_{k+1}+1}}$ and $m_k, m_{k+1} \in \{m_n\}$ for any $k \in \mathbb{N}$. Hence,

$$| f(z) | \nu(\tau i) \leq C \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty((R_{m_{k+1}} - R_{m_k})T_{Re\ z} f, L_{\delta_k})\nu(\delta_k i)$$

Since ν depends only on the imaginary part and $z \in A$ is arbitrary , we have

$$\sup_{z\in A} | f(z) | \nu(z) \leq C \sup_{Re\ z} \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty((R_{m_{k+1}} - R_{m_k})T_{Re\ z} f, L_{\delta_k})\nu(\delta_k i)$$

Put $a = Re\ z$. When z runs over A, a runs over $\mathbb{R}$. Therefore,

$$\sup_{z\in A} | f(z) | \nu(z) \leq C \sup_{a\in\mathbb{R}} \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty((R_{m_{k+1}} - R_{m_k})T_a f, L_{\delta_k})\nu(\delta_k i)$$

Now, Lemma 3.4 completes the proof. □

Lemma 3.6. *Suppose $f \in H(G)$ and $a \in \mathbb{R}$ is arbitrary. Then there exists a universal constant C such that*

$$M_\infty((R_{m_{k+1}} - R_{m_k})T_a f, L_{\delta_k}) \leq C M_\infty(T_a f, L_{\delta_k}) \tag{3.9}$$

where $\delta_k = \frac{1}{2^{m_{k+1}+1}}$ and $m_k, m_{k+1} \in \{m_n\}$ for any $k \in \mathbb{N} \cup \{0\}$.

Proof . We prove the equivalent relation with relation 3.9, which is

$$M_\infty((\tilde{R}_{m_{k+1}} - \tilde{R}_{m_k})\tilde{h}, \frac{\delta_k}{1+\delta_k} + \gamma_{\frac{1}{1+\delta_k}}) \leq C M_\infty(\tilde{h}, \frac{\delta_k}{1+\delta_k} + \gamma_{\frac{1}{1+\delta_k}})$$

where $h = T_a f$ and $\tilde{h} = h \circ \alpha$. For each $k \in \mathbb{N}$ $\delta_k < 1$, hence circle $\frac{\delta_k}{1+\delta_k} + \gamma_{\frac{1}{1+\delta_k}}$ includes the origin for each $k \in \mathbb{N}$. Thus, the concentric circle $\gamma_{\frac{1-\delta_k}{1+\delta_k}}$ is dominated by the circle $\frac{\delta_k}{1+\delta_k} + \gamma_{\frac{1}{1+\delta_k}}$. Put $g = (\tilde{R}_{m_{k+1}} - \tilde{R}_{m_k})\tilde{h}$. Since g is a polynomial of $deg = 2^{m_{n+1}+1}$, using Lemma 2.2 we have

$$M_\infty(g, \frac{\delta_k}{1+\delta_k} + \gamma_{\frac{1}{1+\delta_k}}) \leq (\frac{1+\delta_k}{1-\delta_k})^{2^{m_{k+1}+1}} M_\infty(g, \gamma_{\frac{1-\delta_k}{1+\delta_k}})$$

Similar to the proof of Lemma 3.3, $(\frac{1+\delta_k}{1-\delta_k})^{2^{m_{k+1}+1}} \leq e^{\frac{8}{3}}$. It is wellknown that for any $n \in \mathbb{N} \cup \{0\}$ $\|\tilde{R}_n\| \leq 3$ (see [2]). Hence,

$$M_\infty((\tilde{R}_{m_{k+1}} - \tilde{R}_{m_k})\tilde{h}, \gamma_{\frac{1-\delta_k}{1+\delta_k}}) \leq 6\ M_\infty(\tilde{h}, \gamma_{\frac{1-\delta_k}{1+\delta_k}})$$

Therefore, by maximum modulus principle we have

$$M_\infty((\tilde{R}_{m_{k+1}} - \tilde{R}_{m_k})\tilde{h}, \tfrac{\delta_k}{1+\delta_k} + \gamma_{\frac{1}{1+\delta_k}}) \leq 6e^{\frac{8}{3}} M_\infty(\tilde{h}, \gamma_{\frac{1-\delta_k}{1+\delta_k}}) \leq 6e^{\frac{8}{3}} M_\infty(\tilde{h}, \tfrac{\delta_k}{1+\delta_k} + \gamma_{\frac{1}{1+\delta_k}})$$ □

Lemma 3.7. *Suppose $f \in H(G)$ and ν is a bounded weight on $G_{\frac{1}{2}}$. Then there exists a universal constant C such that*

$$\sup_{a\in\mathbb{R}} \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty((R_{m_{k+1}} - R_{m_k})T_a f, L_{\delta_k})\nu(\delta_k i) \leq C\|f\|_\nu$$

where $\delta_k = \frac{1}{2^{m_{k+1}+1}}$ and $m_k, m_{k+1} \in \{m_n\}$ for any $k \in \mathbb{N} \cup \{0\}$.

Proof . Lemma 3.6 implies that there exists a universal constant $C > 0$ such that for any $a \in \mathbb{R}$ and any $k \in \mathbb{N} \cup \{0\}$

$$M_\infty((R_{m_{k+1}} - R_{m_k})T_a f, L_{\delta_k})\nu(\delta_k i) \leq C\ M_\infty(T_a, L_{\delta_k})$$

Since $M_\infty(T_a f, L_{\delta_k}) = M_\infty(f, L_{\delta_k})$,

$$\sup_{a\in\mathbb{R}} \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty((R_{m_{k+1}} - R_{m_k})T_a f, L_{\delta_k})\nu(\delta_k i) \leq C \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty(f, L_{\delta_k})\nu(\delta_k i)$$

(note that $M_\infty(f, L_{\delta_k})$ does not depend on a). But

$$\sup_{k\in\mathbb{N}\cup\{0\}} M_\infty(f, L_{\delta_k})\nu(\delta_k i) \leq \sup_{\delta>0} M_\infty(f, L_\delta)\nu(\delta i) = \|f\|_\nu$$

Therefore, we are done. □

Theorem 3.8. *Suppose $f \in H(G)$ and ν is a bounded weight on $G_{\frac{1}{2}}$. Then*

$$\|f\|_\nu \sim \sup_{a\in\mathbb{R}} \sup_{k\in\mathbb{N}\cup\{0\}} M_\infty((R_{m_{k+1}} - R_{m_k})T_a f, L_{\delta_k})\nu(\delta_k i)$$

where $\delta_k = \frac{1}{2^{m_{k+1}+1}}$ and $m_k, m_{k+1} \in \{m_n\}$ for any $k \in \mathbb{N} \cup \{0\}$ and ν satisfies required conditions.

Proof . Is a consequence of Lemma 3.6 and Lemma 3.7 □
At the end we present some examples of weights which satisfies required conditions in Theorem 3.8

Example 3.9. *Following weights are increasing satisfy condition $(*)$ and are bounded on $G_{\frac{1}{2}}$.*
$\nu_1(\omega) = (Im\ \omega)^\beta$ for any $0 < \beta < 1$, $\nu_2(\omega) = \min((Im\ \omega)^\beta, 1)$ for any $0 < \beta$.

Proof . Clearly, these weights are increasing and bounded on $G_{\frac{1}{2}}$. Also they satisfy condition $(*)$ (see [1]). □

References

[1] M. A. Ardalani and W. Lusky, *Weighted spaces of holomorphic functions on the upper half-plane*, Mathematica Scandinavica 111 (2012) 244–260.

[2] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall Series in Modern Analysis, Englewood Cliffs, (1962).

[3] W. Lusky, *On weighted spaces of harmonic and holomorphic functions*, J. Lond. Math. Soc. 51 (1995) 309–320.