Int. J. Nonlinear Anal. Appl. 2 (2011) No. 2, 86-95
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

Generalization of Darbo's fixed point theorem and application

Asadollah Aghajani[a,*], Mahdi Aliaskari[a]

[a] *School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16846-13114, Iran.*

(Communicated by M. Eshaghi Gordji)

Abstract

In this paper, an attempt is made to present an extension of Darbo's theorem, and its application to study the solvability of a functional integral equation of Volterra type.

Keywords: Fixed point theorem, Measure of noncompactness, Darbo's fixed point theorem.
2010 MSC: Primary 47H10; Secondary 47H08.

1. Introduction and preliminaries

The concept of measures of noncompactness was first devised by Kuratowski [13]. Another measure of noncompactness is called Hausdorff measure of noncompactness which has been introduced by Gohberg, Goldenstein, Markus [12], Sadovskii [15] and Geobel [11]. Moreover, for instance, in [1] general definitions of tow measures of noncompactness (Kuratowski, Hausdorff) and their connections are presented. Kuratowski and Hausdorff measures of noncompactness are useful tools in many branches of nonlinear analysis. However, since we don't know the compactness criterion in all Banach spaces, applying them is sometimes difficult and in some cases impossible [8]. Moreover in different Banach spaces we need to look for equivalent relations for measures of Hausdorff and Kuratowski so that we are able to analyze these measures of noncompactness better [7].

Sadovskii for the first time, presented the measures of noncompactness in the form of axiomatic ways in [14]. Presenting the axiomatic ways of the measures of noncompactness solves the above problems to a great extent [6]. For introducing the axioms of the measures of noncompactness, at the outset, we give notation, definitions and some supporting facts which will be necessary afterward. For this reason, suppose that E is a given Banach space which has the norm $\|.\|$ and zero element

*Corresponding author
Email addresses: aghajani@iust.ac.ir (Asadollah Aghajani), Mahdialiaskari66@gmail.com (Mahdi Aliaskari)

Received: August 2013 *Revised:* April 2014

θ. If the closed ball in E is centered at x and has radius r, we show it by $B(x,r)$. In order to show $B(\theta,r)$, we write B_r. If X is a subset of E, in that case, we can show the closure and the closed convex hull of X with the symbols $\overline{X}$ and $ConvX$ respectively. Also $X+Y$ and λX ($\lambda \in \mathbb{R}$) are used to show the algebraic operation on sets. Furthermore $\mathfrak{M}_E$ is used to denote the family of all nonempty bounded subsets of E and $\mathfrak{N}_E$ denote its subfamily includes all relatively compact sets.

Definition 1.1 ([7]). *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be measure of noncompactness in E if it satisfies the following conditions*

(1) *The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$.*

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(3) $\mu(\overline{X}) = \mu(X)$.

(4) $\mu(ConvX) = \mu(X)$.

(5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.

(6) *If (X_n) is a nested sequence of closed sets from $\mathfrak{M}_E$ such that $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.*

Observe that the intersection set X_∞ from axiom (6) is a member of the kerμ. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we have that $\mu(X_\infty) = 0$. This yields that $X_\infty \in ker\mu$ (see [4]).

Definition 1.2 ([9]). *A measure μ is called sublinear, if satisfies the following tow conditions*

(1) $\mu(\lambda X) = |\lambda|\mu(Y)$ *for* $\lambda \in \mathbb{R}$,

(2) $\mu(X+Y) \leq \mu(X) + \mu(Y)$,

where $X, Y \in \mathfrak{M}_E$.

Definition 1.3 ([9]). *A measure μ satisfying the condition $\mu(X \cup Y) = max\{\mu(X), \mu(Y)\}$, will be referred to as a measure with maximum property.*

Other properties of measures of noncompactness may be found in [7].

In 1995, Darbo used the concept of measures of noncompactness and concluded the existence of a fixed point for condensing operators. The Darbo theorem is stated as follows.

Theorem 1.4 ([10]). *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that*

$$\mu(TX) \leq k\mu(X)$$

for any nonempty subset X of Ω, where μ is a measure of noncompactness defined in E. Then T has a fixed point in set Ω.

In fact we can consider the Darbo theorem, an extension of Schauder fixed point theorem which can be stated as follows

Theorem 1.5 ([2]). *If Ω is a nonempty, convex and compact subset of a Banach space E and $F : \Omega \to \Omega$ is continuous on the set Ω, then the operator F has at least one fixed point in the set Ω.*

In this paper, we are going to present an integral type extension of the Darbo theorem in Banach spaces. Then we will have a look at its application, using a number of remarks and examples.

2. Main results

In this section, using the technique of measure of noncompactness, we prove the main result of this paper.

Theorem 2.1. *Let E be a Banach space, and Ω be a nonempty, closed, bounded and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator which satisfies the following inequality*

$$\int_0^{\mu(TX)} \varphi(t)dt \le \psi\Big(\int_0^{\mu(X)} \varphi(t)dt\Big)$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function, such that $\lim_{n\to\infty} \psi^n(t) = 0$ for any $t \ge 0$. Also, $\varphi : [0+\infty[\to [0+\infty]$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0+\infty[$ and for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Then T has at least one fixed point in Ω.

Proof . We define the sequence Ω_n as follows

$$\Omega_0 = \Omega \quad \text{and} \quad \Omega_n = ConvT\Omega_{n-1}, \quad n \ge 1.$$

If there exists a natural number such as n_0 such that $\mu(\Omega_{n_0}) = 0$, then Ω_{n_0} is compact and in such case, by using Theorem 1.5, T in Ω has at least one fixed point. So without loss of generality, we assume for every $n \ge 1$, $\mu(\Omega_n) > 0$. Taking into account such assumption, we have

$$\begin{aligned}
\int_0^{\mu(\Omega_{n+1})} \varphi(t)dt &= \int_0^{\mu(convT\Omega_n)} \varphi(t)dt \\
&= \int_0^{\mu(T\Omega_n)} \varphi(t)dt \\
&\leqslant \psi\Big(\int_0^{\mu(\Omega_n)} \varphi(t)dt\Big) \\
&\leqslant \psi^2\Big(\int_0^{\mu(\Omega_{n-1})} \varphi(t)dt\Big) \\
&\leqslant \ldots \\
&\leqslant \psi^n\Big(\int_0^{\mu(\Omega)} \varphi(t)dt\Big).
\end{aligned}$$

Now regarding the fact that for every $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$ we can conclude that $\mu(\Omega_n) \to 0$ as $n \to \infty$. Now since Ω_n is a nested sequence, in view of axiom 6 of Definition 1.1, we deduce that $\Omega_\infty = \bigcap_{n-1}^{\infty} \Omega_n$ is a nonempty, closed and convex subset of the set Ω. Moreover, we know that Ω_∞ is a member of $Ker\mu$. So Ω_0 is compact. Keeping in mind that T, maps Ω_∞ into itself it is possible to apply Schauder fixed point theorem. Therefore the operator T, has at least one fixed point in Ω_∞. Since $\Omega_\infty \subset \Omega$, the proof of the theorem is completed. □

More, in this section, there are some remarks and examples of Theorem 2.1.

Remark 2.2. *By letting*

$$\varphi(t) = 1 \quad \textit{and} \quad \psi(t) = k \quad 0 \le k < 1\,,$$

in Theorem 2.1, then we have

$$\int_0^{\mu(TX)} \varphi(t)dt = \mu(TX) \le k\mu(x) = \psi(\int_0^{\mu(X)} \varphi(t)dt).$$

So in such case the Darbo theorem is obtained.

Now, we present an application of Theorem 2.1 in fixed point problems in metric spaces.

Corollary 2.3. *Let Ω be a nonempty, bounded, closed and convex subset of the Banach space E and let $T : \Omega \to \Omega$ be an operator sech that for each $x, y \in \Omega$,*

$$\int_0^{\|T_x - T_y\|} \varphi(t)dt \leq \psi\Big(\int_0^{\|x-y\|} \varphi(t)dt\Big),$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function with $\lim_{n\to\infty} \psi^n(t) = 0$ for every $t > 0$ and $\varphi : [0 + \infty[\to [0 + \infty]$ is a Lebesque-integrable mapping which is summable on each compact subset of $[0 + \infty[$. Also, for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Then T has at least one fixed point in Ω.

Proof . We define

$$\mu : \mathfrak{M}_E \to \mathbb{R}_+ \qquad \mu(X) = diam X, \quad X \in M_E\,.$$

Checking axioms 1-6 of Definition 1.1, we can understand the above defined μ, is a measure of noncompactness on the space E (cf. [5]). Moreover, under the above assumption, we have

$$\int_0^{\sup_{x,y\in X} \|T_x - T_y\|} \varphi(t)dt \leq \psi\Big(\int_0^{\sup_{x,y\in X} \|x-y\|} \varphi(t)dt\Big).$$

So, we can get

$$\int_0^{\mu(TX)} \varphi(t)dt \leq \psi\Big(\int_0^{\mu(X)} \varphi(t)dt\Big).$$

Now Theorem 2.1 guarantees the existence of a fixed point for the operator T. □

3. Application

In this section, we use Theorem 2.1, to prove the existence of a solution for the integral equation of Voltera type

$$x(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \qquad t \in \mathbb{R}_+. \tag{3.1}$$

We work on the Banach space $BC(\mathbb{R}_+)$ which consists of all defined, bounded and continuous functions on $\mathbb{R}_+$.

We endow the space $BC(\mathbb{R}_+)$ with the standard norm

$$\|x\| = \sup\{|x(t)| : t \geq 0\}.$$

Let us fix a nonempty and bounded subset X of $BC(\mathbb{R}_+)$ and a positive number $L > 0$ for $x \in X$ and $\varepsilon \geq 0$, we denote the modules of continuity of the function x on the interval $[0, L]$ by $W^L(x, \varepsilon)$ and define it as follows

$$\omega^L(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, L], |t - s| \leq \varepsilon\}.$$

Moreover, let us put

$$\omega^L(X, \varepsilon) = \sup\{\omega^L(x, \varepsilon) : x \in X\},$$
$$\omega_0^L(X) = \lim_{\varepsilon\to 0} \omega^L(X, \varepsilon),$$
$$\omega_0(X) = \lim_{L\to\infty} \omega_0^L(X).$$

Furthermore, for a fixed real number $t \in \mathbb{R}$, we put

$$X(t) = \{x(t) : x \in X\}.$$

Now, we define the function μ on family $M_{BC}(\mathbb{R}_+)$ as follows

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} diam X(t),$$

where $diam X(t)$ is understood as

$$diam X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

The function μ is a sublinear measure of noncompactness which has maximum property on the space $BC(\mathbb{R}_+)$ (see[3]) and $Ker\mu$ consists of nonempty bounded sets X such that members of X on $\mathbb{R}_+$ are locally continuous and tend to zero in infinity.

Let Ψ be the family of all functions such as $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which are nondecreasing on $\mathbb{R}_+$ and also for each $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$.

Now assume that the functions f, g in equation 3.1 satisfy the following conditions.

(I) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Moreover, $t \to f(t, 0)$ is a member of the space $BC(\mathbb{R}_+)$.

(II) There exists an upper semicontinuous function $\psi \in \Psi$ and a Lebesque-integrable mapping $\varphi : [0, +\infty[\to [0, +\infty]$ which is summable on every compact subset of $[0, +\infty[$ and for every $\varepsilon > 0$, $\int_0^\varepsilon \varphi(\rho)d\rho > 0$ we have that

$$\int_0^{|f(t,x)-f(t,y)|} \varphi(\rho)d\rho \le \psi\Big(\int_0^{|x-y|} \varphi(\rho)d\rho\Big), \quad t \in \mathbb{R}_+, \quad x, y \in \mathbb{R}.$$

Moreover, we assume that ψ is superadditive i.e., for each $t, s, \in \mathbb{R}_+, \psi(t) + \psi(s) \leqslant \psi(t + s)$.

(III) $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist continuous functions $c, d : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} c(t) \int_0^t d(s)ds = 0$ and $|g(t, s, x)| \le c(t)d(s)$ for $t, s \in \mathbb{R}_+$ such that $s \le t$, and for each $x \in \mathbb{R}$.

(IV) The inequality $\psi\Big(\int_0^r \varphi(\rho)d\rho\Big) + q \le \int_0^r \varphi(\rho)d\rho$ has a positive solution r_0 in which q is constant and defined as

$$q = \sup\Big\{\int_0^{|f(t,0)|+c(t)\int_0^t d(s)ds} \varphi(\rho)d\rho : t \ge 0\Big\}.$$

To prove Theorem 3.2, we need the below lemma.

Lemma 3.1 ([3]). *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then the following two conditions are equivalent*

(1) $\lim_{n\to\infty} \psi^n(t) = 0$ *for each* $t > 0$.

(2) $\psi(t) < t$ *for any* $t > 0$.

Theorem 3.2. *Under the assumptions (I) to (IV) the integral equation 3.1 has at least one solution in the space $BC(\mathbb{R}_+)$.*

Proof . We define the operator T on the space $BC(\mathbb{R}_+)$ as follows

$$(Tx)(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \quad t \in \mathbb{R}_+.$$

With regard to the above assumptions, the function Tx is a continuous function on $\mathbb{R}_+$ for any $x \in BC(\mathbb{R}_+)$. For an arbitrary fixed function $x \in BC(\mathbb{R}_+)$, we have

$$\begin{aligned}
\int_0^{|(Tx)(t)|} \varphi(\rho)d\rho &\leq \int_0^{|f(t,x(t))-f(t,0)|+|f(t,0)|+\int_0^t |g(t,s,x(s))|ds} \varphi(\rho)d\rho \\
&\leq \psi\Big(\int_0^{|x(t)|} \varphi(\rho)d\rho\Big) + \int_0^{|f(t,0)|+\int_0^t |g(t,s,x(s))|ds} \varphi(\rho)d\rho \\
&\leq \psi\Big(\int_0^{|x(t)|} \varphi(\rho)d\rho\Big) + \int_0^{|f(t,0)|+c(t)\int_0^t d(s)ds} \varphi(\rho)d\rho
\end{aligned}$$

So, we get

$$\int_0^{\|T_x\|} \varphi(\rho)d\rho \leq \psi\Big(\int_0^{\|x(t)\|} \varphi(\rho)d\rho\Big) + q,$$

in which q is a constant, defined in assumption (IV). So T maps the space $BC(\mathbb{R}_+)$ in to itself. Moreover in view of assumption (IV), we deduce that T maps the ball B_{r_0} into itself in which r_0 is a constant appearing in assumption (IV). Now we show that operator T is continuous on the ball B_{r_0}. To do so, fix an arbitrary $\varepsilon > 0$ and $x, y \in B_{r_0}$ such that $\|x - y\| \leq \varepsilon$. So we can conclude

$$\begin{aligned}
\int_0^{|(Tx)(t)-(Ty)(t)|} \varphi(\rho)d\rho &\leq \psi\Big(\int_0^{|x(t)-y(t)|} \varphi(\rho)d\rho\Big) \\
&+ \int_0^{\int_0^t |g(t,s,x(s))-g(t,s,y(s))|ds} \varphi(\rho)d\rho \\
&\leq \psi\Big(\int_0^{|x(t)-y(t)|} \varphi(\rho)d\rho\Big) \\
&+ \int_0^{\int_0^t |g(t,s,x(s))|ds} \varphi(\rho)d\rho \\
&+ \int_0^{\int_0^t |g(t,s,y(s))|ds} \varphi(\rho)d\rho \\
&\leq \psi\Big(\int_0^{\varepsilon} \varphi(\rho)d\rho\Big) + \int_0^{2k(t)} \varphi(\rho)d\rho,
\end{aligned} \tag{3.2}$$

where we denoted

$$k(t) = c(t) \int_0^t d(s)ds.$$

Further, in view of assumption (III), we deduce that there exists a number $L > 0$ such that

$$2k(t) = 2c(t) \int_0^t d(s)ds \leq \varepsilon, \tag{3.3}$$

for each $t \geq L$. Thus, taking into account Lemma 3.1 and linking 3.3 and 3.2, for an arbitrary $t \geq L$ we get

$$\begin{aligned}
\int_0^{|(Tx)(t)-(Ty)(t)|} \varphi(\rho)d\rho &\leq \int_0^{\varepsilon} \varphi(\rho)d\rho + \int_0^{\varepsilon} \varphi(\rho)d\rho \\
&\leq \int_0^{2\varepsilon} \varphi(\rho)d\rho.
\end{aligned}$$

So, for $t \geq L$, we get

$$|(Tx)(t) - (Ty)(t)| \leq 2\varepsilon. \tag{3.4}$$

Now, we define the quantity $\omega^L(g,\varepsilon)$ as follows

$$\omega^L(g,\varepsilon)=\sup\{\int_0^{|g(t,s,x)-g(t,s,y)|}\varphi(\rho)d\rho : t,s\in[0,L], x,y\in[-r_0,r_0], \|x-y\|\le\varepsilon\}.$$

Now with regard to the fact that the function $g(t,s,x)$ is uniformly continuous on the set $[0,L]\times[0,L]\times[-r_0,r_0]$, so

$$\lim_{\varepsilon\to 0}\omega^L(g,\varepsilon)=0.$$

Now by considering 3.2 for an arbitrary fixed $t\in[0,L]$, we conclude that

$$\begin{aligned}\int_0^{|(Tx)(t)-(Ty)(t)|}\varphi(\rho)d\rho &\le \psi(\int_0^{\varepsilon}\varphi(\rho)d\rho)+\int_0^{\int_0^L\omega^L(g,\varepsilon)ds}\varphi(\rho)d\rho\\ &= \psi(\int_0^{\varepsilon}\varphi(\rho)d\rho)+\int_0^{L\omega^L(g,\varepsilon)}\varphi(\rho)d\rho.\end{aligned}\tag{3.5}$$

Combining 3.4 and 3.5, it is possible to conclude that the operator T is continuous on the ball B_{r_0}. Now, let X be an arbitrary nonempty subset of the ball B_{r_0}. Fix numbers $\varepsilon>0$ and $L>0$. Next, choose $t,s\in[0,L]$ such that $\|t-s\|\le\varepsilon$. Without loss of generality, we assume that $s<t$. Then, for $x\in X$ we conclude

$$\begin{aligned}&\int_0^{|Tx)(t)-(Tx)(s)|}\varphi(\rho)d\rho\le\int_0^{|f(t,x(t))-f(s,x(s))|+|\int_0^t g(t,\tau,x(\tau))d\tau-\int_0^s g(s,\tau,x(\tau))d\tau|}\varphi(\rho)d\rho\\
&\le\int_0^{|f(t,x(t))-f(s,x(t))|+|f(s,x(t))-f(s,x(s))|+|\int_0^t g(t,\tau,x(\tau))d\tau-\int_0^t g(s,\tau,x(\tau))d\tau|+\int_0^t g(s,\tau,x(\tau))d\tau-\int_0^s g(s,\tau,x(\tau))d\tau|}\varphi(\rho)d\rho\\
&\le\int_0^{\omega_1^L(f,\varepsilon)}\varphi(\rho)d\rho\\
&+\psi(\int_0^{|x(t)-x(s)|}\varphi(\rho)d\rho+\int_0^{\int_0^t|g(t,\tau,x(\tau))-g(s,\tau,x(\tau))|d\tau}\varphi(\rho)d\rho+\int_0^{\int_s^t|g(s,\tau,x(\tau))|d\tau}\varphi(\rho)d\rho\\
&\le\int_0^{\omega_1^L(f,\varepsilon)}\varphi(\rho)d\rho\\
&+\psi\Big(\int_0^{\omega^L(x,\varepsilon)}\varphi(\rho)d\rho\Big)+\int_0^{\int_0^t\omega_1^L(g,\varepsilon)d\tau}\varphi(\rho)d\rho+\int_0^{c(s)\int_s^t d(\tau)d\tau}\varphi(\rho)d\rho\\
&\le\int_0^{\omega_1^L(f,\varepsilon)}\varphi(\rho)d\rho\\
&+\psi\Big(\int_0^{\omega^L(x,\varepsilon)}\varphi(\rho)d\rho\Big)+\int_0^{L\omega_1^L(g,\varepsilon)}\varphi(\rho)d\rho+\int_0^{\varepsilon\sup\{c(s)d(t):t,s\in[0,L]\}}\varphi(\rho)d\rho\end{aligned}\tag{3.6}$$

where we denote

$$\begin{aligned}\omega_1^L(f,\varepsilon)&=\sup\{|f(t,x)-f(s,x)| : t,s\in[0,L], x\in[-r_0,r_0], |t-s|<\varepsilon\},\\ \omega_1^L(g,\varepsilon)&=\sup\{|g(t,t,x)-g(s,t,x)| : t,s,t\in[0,L], x\in[-r_0,r_0], |t-s|<\varepsilon\}.\end{aligned}$$

Now with regard to the fact that f is uniformly continuous on the set $[0,L]\times[-r_0,r_0]$ and g is uniformly continuous on the set $[0,L]\times[0,L]\times[-r_0,r_0]$, we can conclude $\int_0^{\omega_1^L(f,\varepsilon)}\varphi(\rho)d\rho\to 0$ and$\int_0^{\omega_1^L(g,\varepsilon)}\varphi(\rho)d\rho\to 0$ as $\varepsilon\to 0$. Moreover, since $c=c(t)$ and $d=d(t)$ are continuous on $\mathbb{R}_+$, the quantity $\sup\{c(s)d(t) : t,s\in[0,L]\}$ is finite. From 3.6, we conclude

$$\int_0^{\omega_0^L(TX)}\varphi(\rho)d\rho\le\lim_{\varepsilon\to 0}\psi\Big(\int_0^{\omega^L(X,\varepsilon)}\varphi(\rho)d\rho\Big).$$

Now with regard to the fact that ψ is upper semicontinuous, so

$$\int_0^{\omega_0^L(TX)}\varphi(\rho)d\rho\le\psi\Big(\int_0^{\omega_0^L(X)}\varphi(\rho)d\rho\Big),$$

and so

$$\int_0^{\omega_0(TX)} \varphi(\rho)d\rho \le \psi\Big(\int_0^{\omega_0(X)} \varphi(\rho)d\rho\Big). \tag{3.7}$$

Now we choose two arbitrary functions $x, y \in X$. Then for $t \in \mathbb{R}$ we have

$$\begin{aligned}
&\textstyle\int_0^{|(Tx)(t)-(Ty)(t)|} \varphi(\rho)d\rho \le \int_0^{|f(t,x(t))-f(t,y(t))|+\int_0^t |g(t,s,x(s))|ds+\int_0^t |g(t,s,y(s))|ds} \varphi(\rho)d\rho \\
&\le \psi\Big(\textstyle\int_0^{|x(t)-y(t)|} \varphi(\rho)d\rho\Big) + \int_0^{2c(t)\int_0^t d(s)ds} \varphi(\rho)d\rho \\
&\le \psi\Big(\textstyle\int_0^{|x(t)-y(t)|} \varphi(\rho)d\rho\Big) + \int_0^{2k(t)} \varphi(\rho)d\rho.
\end{aligned}$$

This estimate allows us to get the following one

$$\int_0^{diam(TX)(t)} \varphi(\rho)d\rho \le \psi\Big(\int_0^{diamX(t)} \varphi(\rho)d\rho\Big) + \int_0^{2k(t)} \varphi(\rho)d\rho.$$

Now with regard to the upper semicontinuity of the functions ψ we obtain

$$\int_0^{\limsup_{t\to\infty} diam(TX)(t)} \varphi(\rho)d\rho \le \psi\Big(\int_0^{\limsup_{t\to\infty} diamX(t)} \varphi(\rho)d\rho\Big). \tag{3.8}$$

So, combining 3.7 and 3.8, we can conclude

$$\begin{aligned}
&\textstyle\int_0^{\omega_0(TX)} \varphi(\rho)d\rho + \int_0^{\limsup_{t\to\infty} diam(TX)(t)} \varphi(\rho)d\rho \\
&\le \psi\Big(\textstyle\int_0^{\omega_0(X(t))} \varphi(\rho)d\rho\Big) + \psi\Big(\int_0^{\limsup_{t\to\infty} diam(TX)(t)} \varphi(\rho)d\rho\Big) \\
&\le \Big(\textstyle\int_0^{\omega_0(X(t))+\limsup_{t\to\infty} diam(TX)(t)} \varphi(\rho)d\rho\Big),
\end{aligned}$$

or, equivalently

$$\int_0^{\mu(TX)} \varphi(\rho)d\rho \le \psi\Big(\int_0^{\mu(X)} \varphi(\rho)d\rho\Big),$$

in which μ is the defined measure of noncompactness on the space $BC(\mathbb{R}_+)$. Now, Theorem 2.1 completes the proof. □

Example 3.3. *Consider the following functional integral equation*

$$x(t) = \frac{t^2}{2+2t^4}\ln(1+|x(t)|) + \int_0^t \frac{se^{-t}\sin x}{3+|\cos x|}ds, \tag{3.9}$$

for $t \in \mathbb{R}_+$. Observe that Eq. (3.9) is a special case of the functional integral equation (3.1) with

$$\begin{aligned}
&f(t,x) = \tfrac{t^2}{2+2t^4}\ln(1+|x|) \\
&g(t,s,x) = \tfrac{se^{-t}\sin x}{3+|\cos x|}
\end{aligned}$$

We show that all the conditions of Theorem 3.2 are satisfied for the functional integral equation (3.9). Conditions (I) is clearly evident. The function $\psi(t) = \ln(1+t)$ is nondecreasing and concave on $\mathbb{R}_+$ and $\psi(t) < t$ for all $t > 0$, and it is also easily seen that $\varphi(\rho) \equiv 2$ satisfies assumption (II), and for every $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(\rho)d\rho > 0$. In addition, for arbitrarily fixed $x, y \in \mathbb{R}_+$ such that $|x| \geq |y|$ and for $t > 0$ we get

$$\begin{array}{rcl} \int_0^{|f(t,x)-f(t,y)|} \varphi(\rho)d\rho & = & \frac{2t^2}{2+2t^4} \ln\left(\frac{1+|x|}{1+|y|}\right) \\ & \leq & \ln\left(1 + \frac{|x|-|y|}{1+|y|}\right) \\ & < & \ln\left(1 + |x-y|\right) \\ & < & \ln\left(1 + 2|x-y|\right) \\ & = & \psi\Big(\int_0^{|x-y|} \varphi(\rho)d\rho\Big) \end{array}$$

The case $|y| \geq |x|$ can be treated in the same way. Thus, keeping in mind Remark 3.1 we conclude that the function f satisfies assumption (II) of Theorem 3.2. In addition, observe that the function g is continuous and maps the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$. Also, we have

$$|g(t, s, x)| \leq e^{-t}s$$

for $t, s \in \mathbb{R}$ and $x \in \mathbb{R}$. So, if we put $c(t) = e^{-t}$, and $d(s) = s$, then we can see that assumption (III) is satisfied. Indeed, we have

$$\lim_{t\to\infty} c(t) \int_0^t d(s)ds = 0$$

Now, let us compute the constant q appearing in assumption (IV). We obtain

$$q = \sup\Big\{ \int_0^{|f(t,0)|+c(t)\int_0^t d(s)ds} \varphi(\rho)d\rho : t \geq 0\Big\} = \sup\{2t^2 e^{-t/2} : t \geq 0\} = 4e^{-2}.$$

Moreover, we can check that the inequality from assumption (IV) takes the form

$$\ln(1 + 2r) + q \leq 2r.$$

Obviously this inequality has a positive solution r_0. For example, $r_0 = 2$. Consequently, all the conditions of Theorem 3.2 are satisfied. Hence the functional integral equation 3.9 has at least one solution in the space $BC(\mathbb{R}_+)$.

References

[1] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Birkhauser Verlag (Basel, 1992).

[2] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2004.

[3] A. Aghajani, J. Banaś and N, Sabzali, *Some generalizations of Darbo fixed point theorem and applictions*, Bull. Belg. Math. Soc. Simon Stevin 20 (2013), 345-358.

[4] J. Banaś, *Measures of noncompactness in the study of solutions of nonlinear differential and integral equations*, Cent. Eur. J. Math. 10(6) (2012), 2003-2011.

[5] J. Banaś, *On measures of noncompactness in Banach spaces*, Comment. Math. Univ. Carolinae 21 (1980), 131-143.

[6] J. Banaś, *Relative measuree of noncompactness in Banach spaces*, Ph.D. Thesis, Lublin 1978 (in Polish).

[7] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker, New York, 1980.

[8] J. Banaś, A. Hajnosz and S. Wedrychowicz, *On the equation x = f(t,x) in Banach spaces*, Comm. Math. Univ. Carol, 23(2) (1982), 233-247.

[9] J. Banaś, J. Rocha Martin and K. Sadarangani, *On of a quadratic integral equation of Hammerstein type*, Math. Comput. Model 43 (2006), 97-104.

[10] G. Darbo, *Punti uniti in transformazioni a condominio non compatto*, Rend. Sem. Math. Univ. Padova, 2(1955), 84-92.

[11] K. Goebel, *Thickness of sets in metric spacea and its applicationa to the fixed point theory*, Habilit.Thesia, Lublin 1970 (in Polish).

[12] I.T. Gohberg, L.S. Goldenstein and A.S. Markus, *Investigation of some properties of bounded linear operators in connection with their q-norma*, Ufien. Zap. Kishinev. Un-ta, 29 (1957), 2-36 (in Russian).

[13] K. Kuratowski, *Sur les espaces completes*, Fund. Math. 15 (1930), 301-309.

[14] B.N. Sadovskii, *Limit compact and condensing operators*, Russian Math. Surveys, 27 (1972), 86-144.

[15] B.N. Sadovskii, *On a fixed point principle*, Funkc. Analiz i ego Pilo. 1(2) (1967), 74-76 (in Russian).