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Remarks on some recent M. Borcut's results in partially ordered metric spaces

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Abstract

In this paper, some recent results established by Marin Borcut [M. Borcut, Tripled fixed point theorems for monotone mappings in partially ordered metric spaces, Carpathian J. Math. 28, 2 (2012), 207–214] and [M. Borcut, Tripled coincidence theorems for monotone mappings in partially ordered metric spaces, Creat. Math. Inform. 21, 2 (2012), 135–142] are generalized and improved, with much shorter proofs. Also, examples are given to support these improvements.

Keywords: Tripled coincidence point; g-monotone property; partially ordered set.
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1. Introduction and preliminaries

It is well known that the metric fixed point theory is still very actual, important and useful in all areas of Mathematics. It can be applied, for instance in variational inequalities, optimization, approximation theory, etc.

Fixed point results in partially ordered metric spaces play a major role in proving the existence and uniqueness of solutions for some differential and integral equations. One of the first such theorems was obtained by J. J. Nieto and R. R. Lopez [7] who applied it to linear and nonlinear differential equations. Afterwards, several authors obtained a lot of interesting fixed point results in ordered metric spaces.

We begin with some notation and preliminaries. Throughout the paper, $(\mathcal{X}, d, \preceq)$ always denotes a partially ordered metric space, i.e., a triple where $(\mathcal{X}, \preceq)$ is a partially ordered set and $(\mathcal{X}, d)$ is a metric space.

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Definition 1.1. [3, 4, 9] Let $F : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be two mappings.

(1) We say that the mapping F is g-monotone if $F(x, y, z)$ is g-nondecreasing in all three variables, that is, for all $x, y, z \in \mathcal{X}$,

$$\begin{aligned} x_1, x_2 \in \mathcal{X},\ g(x_1) \preceq g(x_2) \ &\Rightarrow\ F(x_1, y, z) \preceq F(x_2, y, z), \\ y_1, y_2 \in \mathcal{X},\ g(y_1) \preceq g(y_2) \ &\Rightarrow\ F(x, y_1, z) \preceq F(x, y_2, z), \\ z_1, z_2 \in \mathcal{X},\ g(z_1) \preceq g(z_2) \ &\Rightarrow\ F(x, y, z_1) \preceq F(x, y, z_2). \end{aligned}$$

In particular, if $g = i_{\mathcal{X}}$, F is said to be monotone.

(2) An element $(x, y, z) \in \mathcal{X}^3$ is called a tripled coincidence point of F and g if $F(x, y, z) = gx$, $F(y, x, z) = gy$ and $F(z, y, x) = gz$. Moreover, (x, y, z) is called a tripled common fixed point of F and g if

$$F(x, y, z) = gx = x, \quad F(y, x, z) = gy = y \text{ and } F(z, y, x) = gz = z. \tag{1.1}$$

In particular, for $g = i_{\mathcal{X}}$, the element (x, y, z) is called a tripled fixed point of F.

(3) The mappings F and g are called commutative if $g(F(x, y, z)) = F(gx, gy, gz)$ holds for all $x, y, z \in \mathcal{X}$.

(4) The mappings F and g are called compatible if

$$\begin{aligned} \lim_{n\to\infty} d(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n)) &= 0, \\ \lim_{n\to\infty} d(gF(y_n, x_n, z_n), F(gy_n, gx_n, gz_n)) &= 0, \text{ and} \\ \lim_{n\to\infty} d(gF(z_n, y_n, x_n), F(gz_n, gy_n, gx_n)) &= 0, \end{aligned}$$

hold whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in $\mathcal{X}$ such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n, y_n, z_n) &= \lim_{n\to\infty} gx_n, \quad \lim_{n\to\infty} F(y_n, x_n, z_n) = \lim_{n\to\infty} gy_n \\ \text{and} \quad & \lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} gz_n. \end{aligned}$$

Remark 1.2. It will be clear in the sequel that part (2) of the previous definition can be modified in several ways. In fact, any three combinations of elements x, y, z can be taken instead of (x, y, z), (y, x, z) and (z, y, x) in (1.1), with the only condition that the first entry of each triple matches the right-hand side. In particular, the "cyclic" case, i.e., the condition

$$F(x, y, z) = gx = x, \quad F(y, z, x) = gy = y \text{ and } F(z, x, y) = gz = z$$

can be considered. It will also be clear which modifications should be made to the results that follows, so we will not state them explicitly.

It is important to notice that this considerably differs from the case of so-called "mixed-monotone situation", first treated in [2]. Namely, as shown in [10], in this case only some particular combinations are possible (in particular, the cyclic case cannot be treated in this way).

Definition 1.3. The space $(\mathcal{X}, d, \preceq)$ is said to be regular if it has the following properties:

(i) if for a non-decreasing sequence $\{x_n\}$, $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all n;

(iI) if for a non-increasing sequence $\{x_n\}$, $x_n \to x$ as $n \to \infty$, then $x_n \succeq x$ for all n.

In [3], Borcut proved the following results and formulated as Theorems 2.3–2.5.

Theorem 1.4. (i) Let $(\mathcal{X}, d, \preceq)$ be a complete ordered metric space and let $F : \mathcal{X}^3 \to \mathcal{X}$ be a continuous and monotone mapping. Assume that there exist constants $j, k, l \in [0, 1)$ with $j+k+l < 1$ for which

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \leq jd\left(x,u\right)+kd\left(y,v\right)+ld\left(z,w\right), \tag{1.2}$$

whenever $x \succeq u$, $y \succeq v$, $z \succeq w$ or $x \preceq u$, $y \preceq v$, $z \preceq w$. If there exist $x_0, y_0, z_0 \in \mathcal{X}$ such that

$$x_0 \preceq F\left(x_0,y_0,z_0\right), \quad y_0 \preceq F\left(y_0,x_0,z_0\right) \text{ and } z_0 \preceq F\left(z_0,y_0,x_0\right),$$

then there exists a tripled fixed point $(x, y, z) \in \mathcal{X}^3$ of F.

(ii) The previous conclusion remains valid if the condition that F is continuous is replaced by the condition that the space $(\mathcal{X}, d, \preceq)$ is regular.

(iii) By adding the following condition to the previous hypotheses: for every $(x, y, z), (x_1, y_1, z_1) \in \mathcal{X}^3$, there exists $(u, v, w) \in \mathcal{X}^3$ that is comparable with (x, y, z) and (x_1, y_1, z_1), the uniqueness of the tripled fixed point of F follows.

Notice that the condition $x \succeq u$, $y \succeq v$, $z \succeq w$ or $x \preceq u$, $y \preceq v$, $z \preceq w$ in (i) was wrongly stated in [3] as $x \succeq u$, $y \preceq v$, $z \succeq w$.

In [4], Borcut proved the following result and formulated as Theorem 2.3.

Theorem 1.5. Let $(\mathcal{X}, d, \preceq)$ be a complete ordered metric space. Let $F : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be such that F has the g-monotone property. Assume there is a non-decreasing function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ and $\lim_{r\to t^+} \varphi(r) < t$, for each $t > 0$, such that

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \leq \varphi\left(\max\left\{d\left(gx,gu\right),d\left(gy,gv\right),d\left(gz,gw\right)\right\}\right) \tag{1.3}$$

for all $x, y, z, u, v, w \in \mathcal{X}$ with $gx \preceq gu$, $gy \preceq gv$ and $gz \preceq gw$ or $gx \succeq gu$, $gy \succeq gv$ and $gz \succeq gw$. Suppose $F(\mathcal{X}^3) \subseteq g(\mathcal{X})$, g is continuous and commutes with F and also suppose that either F is continuous or the space $\mathcal{X}$ is regular. If there exist $x_0, y_0, z_0 \in \mathcal{X}$ such that

$$x_0 \preceq F\left(x_0,y_0,z_0\right), \quad y_0 \preceq F\left(y_0,x_0,z_0\right) \text{ and } z_0 \preceq F\left(z_0,y_0,x_0\right),$$

then there exists a tripled coincidence point $(x, y, z) \in \mathcal{X}^3$ of F and g.

Again, the condition $gx \preceq gu$, $gy \preceq gv$ and $gz \preceq gw$ or $gx \succeq gu$, $gy \succeq gv$ and $gz \succeq gw$ was wrongly stated in [4] as $gx \preceq gu, gy \succeq gv$ and $gz \preceq gw$.

Notice that Theorem 1.4 follows from Theorem 1.5 by taking $g = i_{\mathcal{X}}$ and $\varphi(t) = \lambda t$, where $\lambda = j + k + l \in [0, 1)$.

In this paper, we show that Theorems 1.4 and 1.5, as well as several similar results, can be proved in a much easier way, even in an improved form, by reducing them to the known results for mappings with one variable. As a sample, the same is done for some results of Luong and Thuan [6]. Examples are given to show that these improvements are proper. It is also noted that this procedure can be applied for related problems with mixed monotone mappings only in some special situations.

For similar approach to some related problems, see [5], [8] and [9]. Another (more involved) approach can be found in [12].

2. Auxiliary results

The following lemma is easy to prove.

Lemma 2.1. (i) If a relation $\sqsubseteq$ is defined on $\mathcal{X}^3$ by

$$Y \sqsubseteq V \Leftrightarrow x \preceq u \wedge y \preceq v \wedge z \preceq w,\ Y = (x,y,z), V = (u,v,w) \in \mathcal{X}^3,$$

and $d_{\max} : \mathcal{X}^3 \times \mathcal{X}^3 \to \mathbb{R}^+$ is given by

$$d_{\max}(Y,V) = \max\{d(x,u), d(y,v), d(z,w)\}, \quad Y = (x,y,z), V = (u,v,w) \in \mathcal{X}^3,$$

then $(\mathcal{X}^3, d_{\max}, \sqsubseteq)$ is an ordered metric space. The space $(\mathcal{X}^3, d_{\max})$ is complete if and only if $(\mathcal{X}, d)$ is complete. Moreover, the space $(\mathcal{X}^3, d_{\max}, \sqsubseteq)$ is regular if and only if $(\mathcal{X}, d, \preceq)$ is such.

(ii) If $F : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ and if F is g-monotone, then the mapping $T_F : \mathcal{X}^3 \to \mathcal{X}^3$ given by

$$T_F Y = (F(x,y,z), F(y,x,z), F(z,y,x)),\ Y = (x,y,z) \in \mathcal{X}^3$$

is T_g-non-decreasing with respect to $\sqsubseteq$, that is,

$$T_g Y \sqsubseteq T_g V \Rightarrow T_F Y \sqsubseteq T_F V,$$

where $T_g Y = T_g(x,y,z) = (gx, gy, gz)$.

(iii) The mappings T_F and T_g are continuous (resp. compatible) if and only if F and g are continuous (resp. compatible).

(iv) $F(\mathcal{X}^3)$ (resp. $g(\mathcal{X})$) is complete in the metric space $(\mathcal{X}, d)$ if and only if $T_F(\mathcal{X}^3)$ (resp. $T_g(\mathcal{X}^3)$) is complete in the space $(\mathcal{X}^3, d_{\max})$.

(v) The mappings F and g have a tripled coincidence point if and only if the mappings T_F and T_g have a coincidence point in $\mathcal{X}^3$.

(vi) The mappings F and g have a tripled common fixed point if and only if the mappings T_F and T_g have a common fixed point in $\mathcal{X}^3$. In particular, F has a tripled fixed point if and only if T_F has a fixed point in $\mathcal{X}^3$.

In what follows, we will use the class Φ of functions defined by

$$\Phi = \left\{\varphi : [0,+\infty) \to [0,+\infty) \mid \varphi(t) < t,\ t > 0 \text{ and } \lim_{r \to t^+} \varphi(r) < t,\ t > 0\right\}.$$

If $\varphi \in \Phi$ then it is easy to show that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$, where φ^n denotes the n-th iteration of φ.

We will make use of the following result from [9] (see also [1, Theorem 2.1]).

Lemma 2.2. [9, Lemma 2.1] Let f and g be two self mappings on $\mathcal{X}$. Assume that there exists $\varphi \in \Phi$ such that

$$d(fx, fy) \le \varphi(d(gx, gy)) \tag{2.1}$$

for all $x, y \in \mathcal{X}$ with $gx \preceq gy$ or $gx \succeq gy$. If the following conditions hold:

(i) f is g-non-decreasing with respect to $\preceq$ and $f(\mathcal{X}) \subseteq g(\mathcal{X})$;

(ii) there exists $x_0 \in \mathcal{X}$ such that $gx_0 \preceq fx_0$;

(iii) f and g are continuous and compatible and $(\mathcal{X}, d)$ is complete, or

(iii') $(\mathcal{X}, d, \preceq)$ is regular and one of $f(\mathcal{X})$ or $g(\mathcal{X})$ is complete,

then f and g have a coincidence point in $\mathcal{X}$.

3. Main results

Our first main result generalizes and improves both Theorems 1.4 and 1.5.

Theorem 3.1. *Let $(\mathcal{X}, d, \preceq)$ be a partially ordered metric space and let $F : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be such that F is g-monotone. Suppose that there is a function $\varphi \in \Phi$, such that*

$$\begin{aligned} &\max\{d(F(x,y,z), F(u,v,w)), d(F(y,x,z), F(v,u,w)), d(F(z,y,x), F(w,v,u))\} \\ &\quad \leq \varphi(\max\{d(gx,gu), d(gy,gv), d(gz,gw)\}) \end{aligned} \tag{3.1}$$

for all $x, y, z, u, v, w \in \mathcal{X}$ with ($gx \preceq gu$, $gy \preceq gv$ and $gz \preceq gw$) or ($gx \succeq gu$, $gy \succeq gv$ and $gz \succeq gw$). Assume $F(\mathcal{X}^3) \subseteq g(\mathcal{X})$, and also that either F and g are continuous and compatible and $(\mathcal{X}, d)$ is complete or $(\mathcal{X}, d, \preceq)$ is regular and one of $F(\mathcal{X}^3)$ or $g(\mathcal{X})$ is complete. If there exist $x_0, y_0, z_0 \in \mathcal{X}$ such that

$$\begin{aligned} &x_0 \preceq F(x_0,y_0,z_0), \quad y_0 \preceq F(y_0,x_0,z_0) \text{ and } z_0 \preceq F(z_0,y_0,x_0) \text{ or} \\ &x_0 \succeq F(x_0,y_0,z_0), \quad y_0 \succeq F(y_0,x_0,z_0) \text{ and } z_0 \succeq F(z_0,y_0,x_0), \end{aligned}$$

then F and g have a tripled coincidence point $(x, y, z) \in \mathcal{X}^3$.

Proof . Consider the partially ordered metric space $(\mathcal{X}^3, d_{\max}, \sqsubseteq)$ and the mappings T_F and T_g introduced in Lemma 2.1. Then: (i) T_F is T_g-non-decreasing w.r.t. $\sqsubseteq$ and $T_F(\mathcal{X}^3) \subset T_g(\mathcal{X}^3)$; (ii) there exists $Y_0 \in \mathcal{X}^3$ such that $T_gY_0 \sqsubseteq T_FY_0$; (iii) T_F and T_g are continuous and compatible and $(\mathcal{X}, d_{\max})$ is complete, or (iii') $(\mathcal{X}^3, d_{\max}, \sqsubseteq)$ is regular and one of $T_F(\mathcal{X}^3)$, $T_g(\mathcal{X}^3)$ is complete. Moreover,

$$d_{\max}(T_FY, T_FV) \leq \varphi(d_{\max}(T_gY, T_gV)),$$

for all $Y, V \in \mathcal{X}^3$ with $T_gY \sqsubseteq T_gV$ or $T_gY \sqsupseteq T_gV$. Hence, the proof follows from Lemma 2.2 and Lemma 2.1.(v). □

Remark 3.2. If we suppose in Theorem 3.1 that $(\mathcal{X}, d)$ is complete, then it is sufficient that one of $F(\mathcal{X}^3)$ or $g(\mathcal{X})$ is closed.

Remark 3.3. It is clear that condition (3.1) implies (1.3). The converse is also true. Indeed, since ($gy \preceq gv$, $gx \preceq gu$ and $gz \preceq gw$) or ($gy \succeq gv$, $gx \succeq gu$ and $gz \succeq gw$), it follows from (1.3) that

$$\begin{aligned} d(F(y,x,z), F(v,u,w)) &\leq \varphi(\max\{d(gy,gv), d(gx,gu), d(gz,gw)\}) \\ &= \varphi(\max\{d(gx,gu), d(gy,gv), d(gz,gw)\}). \end{aligned}$$

Similarly,

$$\begin{aligned} d(F(z,y,x), F(w,v,u)) &\leq \varphi(\max\{d(gz,gw), d(gy,gv), d(gx,gu)\}) \\ &= \varphi(\max\{d(gx,gu), d(gy,gv), d(gz,gw)\}). \end{aligned}$$

Hence, taking the maximum, we have that (3.1) holds.

Theorem 3.1 is more general than Theorem 1.5. The following example can be used to show this.

Example 3.4. Let $\mathcal{X} = [0,1]$ be equipped with the usual metric and order. Consider the mappings $F : \mathcal{X}^3 \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ defined by

$$F(x,y,z) = \frac{x^2 + 2y^2 + 3z^2}{7} \quad \text{and} \quad gx = x^2.$$

All conditions of Theorem 3.1 are satisfied. In particular, we will check that F and g are compatible. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be three sequences in $\mathcal{X}$ such that

$$\begin{aligned}&\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} gx_n = \alpha, \qquad \lim_{n\to\infty} F(y_n, x_n, z_n) = \lim_{n\to\infty} gy_n = \beta \\ &\text{and} \lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} gz_n = \gamma.\end{aligned}$$

Then $\alpha^2 = \frac{\alpha^2+2\beta^2+3\gamma^2}{7}$, $\beta^2 = \frac{\beta^2+2\alpha^2+3\gamma^2}{7}$ and $\gamma^2 = \frac{\gamma^2+2\beta^2+3\alpha^2}{7}$, wherefrom it follows that $\alpha = \beta = \gamma = 0$. Further, we have that

$$\begin{aligned}&d\left(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n)\right) \\ &\quad = \left| \left(\frac{x_n^2 + 2y_n^2 + 3z_n^2}{7} \right)^2 - \frac{x_n^4 + 2y_n^4 + 3z_n^4}{7} \right| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Similarly,

$$d\left(gF(y_n, x_n, z_n), F(gy_n, gx_n, gz_n)\right) \to 0,\ d\left(gF(z_n, y_n, x_n), F(gz_n, gy_n, gx_n)\right) \to 0$$

as $n \to \infty$. Also, it follows immediately that the contractive condition (3.1) is satisfied with $\varphi(t) = \frac{6}{7}t$. Hence, F and g have a tripled coincidence point (which is $(0,0,0)$).

The mappings F and g do not commute and therefore the existence of tripled coincidence point of F and g cannot be obtained using Theorem 1.5.

Taking $\varphi(t) = \lambda t$ and $g = i_{\mathcal{X}}$ in Theorem 3.1 we get the following

Corollary 3.5. *Let $(\mathcal{X}, d, \preceq)$ be a partially ordered metric space and let $F : \mathcal{X}^3 \to \mathcal{X}$ has the monotone property. Assume there is $\lambda \in [0,1)$, such that*

$$\begin{aligned}&\max\left\{d\left(F(x,y,z), F(u,v,w)\right), d\left(F(y,x,z), F(v,u,w)\right), d\left(F(z,y,x), F(w,v,u)\right)\right\} \\ &\quad \le \lambda \max\left\{d(x,u), d(y,v), d(z,w)\right\}\end{aligned}$$

for all $x,y,z,u,v,w \in \mathcal{X}$ with ($x \preceq u$, $y \preceq v$ and $z \preceq w$) or ($x \succeq u$, $y \succeq v$ and $z \succeq w$). Suppose that either F is continuous and $(\mathcal{X}, d)$ is complete or $(\mathcal{X}, d, \preceq)$ is regular and $F(\mathcal{X}^3)$ is complete. If there exist $x_0, y_0, z_0 \in \mathcal{X}$ such that

$$\begin{aligned}&x_0 \preceq F(x_0, y_0, z_0), \quad y_0 \preceq F(y_0, x_0, z_0) \text{ and } z_0 \preceq F(z_0, y_0, x_0) \text{ or} \\ &x_0 \succeq F(x_0, y_0, z_0), \quad y_0 \succeq F(y_0, x_0, z_0) \text{ and } z_0 \succeq F(z_0, y_0, x_0),\end{aligned}$$

then F has a tripled fixed point $(x,y,z) \in \mathcal{X}^3$.

Remark 3.6. A sufficient condition for the uniqueness of tripled fixed (or coincidence) point in Theorem 3.1 and Corollary 3.5 is standard (and the same as in Theorem 1.4.(iii)), so we omit the details.

Our next result can be considered as an extension of a result of Luong and Thuan [6, Theorem 2.1].

First of all, note that all the assertions of Lemma 2.1 remain valid if $d_{\max}$ is replaced with the following metric on $\mathcal{X}^3$:

$$D_{1/3}(Y,V)=\frac{1}{3}[d(x,u)+d(y,v)+d(z,w)],\quad Y=(x,y,z),V=(u,v,w)\in\mathcal{X}^3. \tag{3.2}$$

Consider, further, the following classes of functions:

Φ_1 is the class of all functions $\phi:[0,+\infty)\to[0,+\infty)$ satisfying:

(i) ϕ is continuous and non-decreasing;
(ii) $\phi(t)=0$ iff $t=0$;
(iii) $\phi(t+s)\leq\phi(t)+\phi(s)$ for all $t,s\in[0,+\infty)$.

Ψ is the class of all functions $\psi:[0,+\infty)\to[0,+\infty)$ satisfying:

(i) $\lim_{r\to t}\psi(r)>0$ for all $t>0$;
(ii) $\lim_{t\to 0+}\psi(t)=0$.

Theorem 3.7. *Let $(\mathcal{X},d,\preceq)$ be a complete partially ordered metric space and let $F:\mathcal{X}^3\to\mathcal{X}$ be a monotone mapping. Assume that there are functions $\phi\in\Phi_1$ and $\psi\in\Psi$ such that*

$$\begin{aligned}&\phi(d(F(x,y,z),F(u,v,w))\\&\leq\frac{1}{3}\phi(d(x,u)+d(y,v)+d(z,w))-\psi\left(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\right)\end{aligned} \tag{3.3}$$

holds for all $x,y,z,u,v,w\in\mathcal{X}$ with ($x\preceq u$, $y\preceq v$ and $z\preceq w$) or ($x\succeq u$, $y\succeq v$ and $z\succeq w$). Suppose that either F is continuous or the space $(\mathcal{X},d,\preceq)$ is regular. If there exist $x_0,y_0,z_0\in\mathcal{X}$ such that

$$\begin{aligned}&x_0\preceq F(x_0,y_0,z_0),\quad y_0\preceq F(y_0,x_0,z_0)\ \text{and}\ z_0\preceq F(z_0,y_0,x_0)\ \text{or}\\&x_0\succeq F(x_0,y_0,z_0),\quad y_0\succeq F(y_0,x_0,z_0)\ \text{and}\ z_0\succeq F(z_0,y_0,x_0),\end{aligned}$$

then F has a tripled fixed point $(x,y,z)\in\mathcal{X}^3$.

Proof . The subadditivity of function ϕ implies that $\phi(a+b+c)\leq 3\phi(\frac{a+bn+c}{3})$ for $a,b,c\geq 0$. Applying this, it follows from condition (3.3) that

$$\phi(d(F(x,y,z),F(u,v,w)))\leq\phi\left(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\right)-\psi\left(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\right).$$

Similarly, we get that

$$\phi(d(F(y,x,z),F(v,u,w)))\leq\phi\left(\frac{d(y,v)+d(x,u)+d(z,w)}{3}\right)-\psi\left(\frac{d(y,v)+d(x,u)+d(z,w)}{3}\right),$$

and

$$\phi(d(F(z,y,x),F(w,v,u)))\leq\phi\left(\frac{d(z,w)+d(y,v)+d(x,u)}{3}\right)-\psi\left(\frac{d(z,w)+d(y,v)+d(x,u)}{3}\right).$$

Adding up the last three inequalities, we obtain that

$$\frac{\phi(d(F(x,y,z),F(u,v,w))+\phi(d(F(y,x,z),F(v,u,w))+\phi(d(F(z,y,x),F(w,v,u))}{3}$$
$$\leq \phi\left(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\right)-\psi\left(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\right).$$

It follows (using the metric (3.2) and the mapping T_F as introduced in Lemma 2.1) that

$$\begin{aligned}&\phi(D_{1/3}(T_FY,T_FV))\\ &=\phi\left(\frac{d(F(x,y,z),F(u,v,w))+d(F(y,x,z),F(v,u,w))+d(F(z,y,x),F(w,v,u))}{3}\right)\\ &\leq \frac{\phi(d(F(x,y,z),F(u,v,w))+\phi(d(F(y,x,z),F(v,u,w))+\phi(d(F(z,y,x),F(w,v,u))}{3}\\ &\leq \phi(D_{1/3}(Y,V))-\psi(D_{1/3}(Y,V)),\end{aligned}$$

for all $Y,V\in\mathcal{X}^3$, comparable w.r.t. $\sqsubseteq$.

Thus, by a well-known result (see, e.g., [8, Lemma 2.4] for $g=i_{\mathcal{X}}$), and it follows that T_F has a fixed point $Y^*=(x^*,y^*,z^*)\in\mathcal{X}^3$. By Lemma 2.1, this is a tripled fixed point of F. □

Remark 3.8. If the condition (3.3) is replaced by

$$\begin{aligned}&\phi\left(\frac{d(F(x,y,z),F(u,v,w))+d(F(y,x,z),F(v,u,w))+d(F(z,y,x),F(w,v,u))}{3}\right)\\ &\qquad\leq \phi\left(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\right)-\psi\left(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\right),\end{aligned}\tag{3.4}$$

then the conclusion of Theorem 3.7 remains valid without the subadditivity of function ϕ.

Moreover, similarly as in Remark 1.2, Theorem 3.7 remains true with condition (3.4) replaced, e.g., with its cyclic variant. Also, it would be easy to reformulate it for mappings with arbitrary number of variables (see, e.g., [11, Corollary 29.(F)]).

And again, this is not true for the "mixed-monotone" case.

We illustrate the use of previous results by the following

Example 3.9. Consider the set $\mathcal{X}=[0,1]$ equipped with the standard metric and order $\preceq$. Let $F:\mathcal{X}^3\to\mathcal{X}$ and $\phi\in\Phi_1$, $\psi\in\Psi$ be given by

$$F(x,y,z)=\frac{x+y+z}{3}-\frac{1}{2}\left(\frac{x+y+z}{3}\right)^2,\quad \phi(t)=t,\quad \psi(t)=\frac{1}{2}t^2.$$

Let $x\geq u$, $y\geq v$ and $z\geq w$ and denote $\frac{x+y+z}{3}=p$, $\frac{u+v+w}{3}=q$ (hence, $p\geq q$). Then

$$\begin{aligned}&\phi\left(\frac{d(F(x,y,z),F(u,v,w))+d(F(y,x,z),F(v,u,w))+d(F(z,y,x),F(w,v,u))}{3}\right)\\ &=\left(p-\frac{p^2}{2}\right)-\left(q-\frac{q^2}{2}\right)=(p-q)-\frac{p^2-q^2}{2}\\ &\leq (p-q)-\frac{1}{2}(p-q)^2\\ &=\phi\left(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\right)-\psi\left(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\right).\end{aligned}$$

Thus, condition (3.4) is satisfied, as well as other conditions of Theorem 3.7. We conclude that F has a tripled fixed point (which is $(0,0,0)$).

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