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Existence of mild solutions to a Cauchy problem presented by fractional evolution equation with an integral initial condition

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Abstract

In this article, we apply two new fixed point theorems to investigate the existence of mild solutions for a nonlocal fractional Cauchy problem with an integral initial condition in Banach spaces.

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1. Introduction

The study of abstract nonlocal semi linear initial value problems was originated by Byszewsk [4, 5, 6] and subsequently, many authors have tracked his work. Although, fractional calculus is a topic being more than 300 years old, yet it can be considered as a novel topic, for example see [1, 7, 8]. In particular, the theory of fractional evolution equations is one of the developing branches of this study. Moreover, fractional evolution equations, in some cases, have better effects in applications than traditional ones [14]. Therefore, there have been many published papers in this topic. For example, Jardat et al. [15] investigated the existence and uniqueness of mild solution for the semilinear initial value problem of non-integer order

$$\begin{cases} D^{\alpha}u(t) = Au(t) + f(t, u(t), Gu(t), Su(t)), & t \in (0, T], \\ u(0) = u_0, \end{cases}$$

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where A is the generator of a strongly continuous semigroup and $D^{\alpha}u(t)$ denotes the Caputo's fractional derivative of $u(t)$. Muslim [21] studied the existence and approximation of solutions to following fractional evolution equations in a Banach space.

$$\begin{cases} \frac{d^{\beta}u(t)}{dt^{\beta}} + Au(t) = f(t, u(t)), \quad t > t_0, \alpha \in (0, 1], \\ u(t_0) = u_0, \end{cases}$$

where A is a closed linear operator defined on a dense set and $\frac{d^{\beta}u(t)}{dt^{\beta}}$ denotes the derivative of u, in the Caputo sense and $0 < \beta \leq 1$. Chen et al. [7] investigated the existence of saturated mild solutions and global mild solutions for the initial value problem

$$\begin{cases} D_*^{\alpha}u(t) + Au(t) = f(t, u(t)), \quad t \geq 0, \alpha \in [0, 1], \\ u(0) = u_0, \end{cases}$$

where A is a closed linear operator and $D_*^{\alpha}u(t)$ is the Caputo's fractional derivative of order α. Z.W. Lv et al. [18] studied the existence of solutions to the nonlocal Cauchy problem for the following fractional differential equation

$$\begin{cases} D_*^{\alpha}x(t) = f(t, x(t)), \qquad t \in [0, 1], \\ x(0) = \int_0^1 g(s)x(s)ds, \end{cases}$$

where D_*^{α} is the standard Caputo's derivative of order $0 < \alpha \leq 1$. For more study on existence and uniqueness of different types of mild solutions to the fractional evolution equations, we may refer to [8, 12, 20, 22, 25] and the references therein.

Moreover, nonlocal initial conditions are more realistic than usual ones in treating physical problems [4] and nonlocal Cauchy problem with integral initial condition is rarely considered in the literature. In addition, in most of the existed articles, Schauder's, Krasnoselskii's or Darbo's fixed point theorems have been employed to obtain the solution of Cauchy problems under some restrictive conditions. While in this article, we apply two new fixed point theorems to investigate the existence of mild solutions to the following nonlocal fractional Cauchy problem with integral initial condition in Banach spaces.

$$\begin{cases} D_*^{\alpha}u(t) = Au(t) + f(t, u(t)), \quad t \in J = [0, 1], \\ u(0) = \int_0^1 g(s, u(s))ds, \end{cases} \tag{1.1}$$

where A is generator of a strongly continuous semigroup $\{T(t);\ t \geq 0\}$ in Banach space E and $f, g :\ J \times E \to E$ are given functions satisfying some assumptions that will be specified later.

This paper is organized as follows. In section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. Then we study the existence of mild solutions for evolution equation (1.1), in section 3.

2. Preliminaries

In this section, we present some definitions and auxiliary results which will be needed in the sequel.

Here, we assume that E is a Banach space with the norm $|\ .\ |$ and $J = [0, 1]$. Denote $C(J, E)$ the Banach space of continuous functions from J into E with the norm $\|u\| = \sup_{t \in J} |u(t)|$, where $u \in C(J, E)$.

Definition 2.1. By a mild solution of the nonlocal initial value problem (1.1), we mean the function $u \in C(J, E)$ which satisfies

$$u(t) = T(t)\int_0^1 g(s, u(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s, u(s))ds, \qquad \forall t \in J. \tag{2.1}$$

As we mentioned above, operator A generates a C_0 semigroup $T(t)$ on E. A C_0 semigroup $T(t)$ is said to be compact if $T(t)$ is compact for any $t > 0$. If the C_0 semigroup is compact, then $t \longrightarrow T(t)u$ are equicontinuous at all $t > 0$ with respect to u in all bounded subsets of E. This means that semigroup $T(t)$ is equicontinuous.

To prove the existence, we need the following concepts and hence, we introduce the Banach space with a norm, recall some basic definitions and properties from the fractional calculus.

Definition 2.2. ([16]) Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I^{\alpha}_{0^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.3. ([16]) Riemann-Liouville fractional derivative of order $\alpha > 0$ denoted by D^{α} and defined by

$$D^{\alpha}f(x) = \frac{d^m}{dt^m}(I^{m-\alpha}f(x)),$$

where $m - 1 < \alpha \le m, m \in \mathbb{N}$. That is m is the smallest integer greater than α.

Definition 2.4. ([16]) Let $f \in C^n([0, T]$, then Caputo's definition of the fractional-order derivative is defined as

$$D^{\alpha}_{*}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}\mathrm{d}t,$$

where $x \le T$ and $\alpha > 0$ is the order of the derivative and $n = \lceil \alpha \rceil$.

Next, we recall some definitions and properties of measure of non-compactness.

Definition 2.5. ([11]) If E is a sufficiently smooth bounded open set in the plane, the convex hull $co(E)$ of E is the bounded connected open set of minimal perimeter containing E.

Definition 2.6. ([2]) Let Y be a metric space and B a bounded subsets of Y. Then Hausdorff measure of non-compactness of B is defined by

$$\gamma(B) = \inf\Big\{\varepsilon > 0 \ : \ B \ \text{ has a finite cover by closed balls of radius } \varepsilon\Big\}. \tag{2.2}$$

Remark 2.7. Let $B_1, B_2 \subseteq X$ be bounded sets. Then Hausdorff measure of non-compactness has the following properties. For more details and the proof of these properties see [2].

(i) If $B_1 \subseteq B_2$, then $\gamma(B_1) \le \gamma(B_2)$.
(ii) $\gamma(B) = \gamma(\bar{B})$.
(iii) $\gamma(B) = 0$ iff B is totally bounded.
(iv) For $\lambda \in \mathbb{R}$, $\gamma(\lambda B) = |\lambda|\gamma(B)$.

(v) $\gamma(B_1+B_2)\leq\gamma(B_1)+\gamma(B_2)$, where $B_1+B_2=\{b_1+b_2:\ b_1\in B_1,\quad b_2\in B_2\}$.

(vi) $\gamma(B)=\gamma(\overline{\text{co}}(B))$.

Definition 2.8. ([3]) A map $f:J\times E\to E$ is said to be Carathéodory if

(i) $t\longmapsto f(t,u)$ is measurable for each $u\in E$,

(ii) $u\longmapsto f(t,u)$ is continuous for almost each $t\in J$.

Lemma 2.9. ([13, 19]) Let E be a Banach space, and let $D=\{u_n\}\subset C(J,E)$ be a bounded and countable set. Then $\gamma(D(t))$ is the Lebesgue integral on J, and

$$\gamma\Big(\Big\{\int_J u_n(t)dt\mid n\in\mathbb{N}\Big\}\Big)\leq 2\int_J\gamma(D(t))dt.$$

Lemma 2.10. ([24]) If W is bounded, then for each $\varepsilon>0$, there is a sequence $\{u_n\}_{n=1}^{\infty}\subseteq W$, such that

$$\gamma(W)\leq 2\gamma(\{u_n\}_{n=1}^{\infty})+\varepsilon.$$

Lemma 2.11. ([2]) Let $W\subseteq C(J,E)$ and that W is bounded and equicontinuous. Then the set $\overline{co}(W)$ is also bounded and equicontinuous.

Theorem 2.12. ([17]) Let F be a closed and convex subset of a real Banach space X, and $A:F\longrightarrow F$ be a continuous operator and $A(F)$ be bounded. Furthermore, for each bounded subset $B\subset F$, set

$$A^1(B)=A(B),\ A^n(B)=A(\overline{\text{co}}A^{n-1}(B))),\quad n=2,3,\cdots.$$

Now, if there exist a constant $0\leq k<1$ and a positive integer n_0 such that for each bounded subset $B\subset F$,

$$\gamma(A^{n_0}(B))\leq k\gamma(B),$$

then A has a fixed point in F.

Theorem 2.13. ([9]) Let E be a Banach space. Assume that $D\subset E$ is a bounded closed and convex set on E and $F:D\longrightarrow D$ is condensing. Then F has at least one fixed point in D.

3. Main results

For the forthcoming analysis, we introduce the following hypotheses.

(H1) The C_0 semigroup $T(t)$ generated by A is equicontinuous. We denote $M=\sup\{\|T(t)\|;\ t\in J\}$.

(H2) $f:J\times E\longrightarrow E$ satisfies the Carathéodory type conditions.

(H3) $g\ :\ C(J,E)\ \longrightarrow\ E$ is continuous and compact, there exists $p(t)\in L^1(J,\mathbb{R}^+)$ such that $\int_0^1\|g(s,u(s))\|ds\leq\int_0^1 p(s)\|u\|ds,\forall u\in C(J,E)$.

(H4) There exists a function $\phi\in L^1(J,\mathbb{R}^+)$ and a nondecreasing continuous function $\psi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ such that $\|f(t,u)\|\leq\phi(t)\psi(\|u\|)$.

(H5) f satisfies the Lipschitz condition in $u(t)$ i.e. there exist constant L such that $|f(t,u)-f(t,v)|\leq L|u-v|$.

Now, under above hypotheses, we can provide the following results.

Theorem 3.1. *If* (H1)-(H5) *are satisfied, then there is at least one mild solution for* (1.1) *provided that there exist a constant* R *with*

$$R\int_0^1 p(s)ds + \psi(R)I^{\alpha}\phi(t) \leq \frac{R}{M}.$$

Proof . Consider the operator $F : C(J,E) \longrightarrow C(J,E)$ defined by

$$(Fu)(t) = T(t)\int_0^1 g(s,u(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,u(s))ds, \tag{3.1}$$

for all $u \in C(J,E)$ and $t \in J$. First, we show that F is continuous by some ordinary techniques. Let $u_n \longrightarrow u$ in $C(J,E)$. Then

$$\begin{aligned} |Fu_n - Fu| &\leq M\int_0^1 |g(s,u_n(s)) - g(s,u(s))|ds \\ &\quad + \frac{M}{\Gamma(\alpha)}\int_0^t \Big|(t-s)^{\alpha-1}\Big(f(s,u_n(s)) - f(s,u(s))\Big)\Big|ds. \end{aligned}$$

So, $Fu_n \longrightarrow Fu$ in $C(J,E)$ by the Lebesgue's convergence theorem.
Next, we denote $W = \{u \in C(J,E),\ \|u(t)\| \leq R, \text{ for all } t \in J\}$, then $W \subseteq C(J,E)$ is bounded and convex. For any $u \in W$, we have

$$\begin{aligned} \|(Fu)(t)\| &\leq \|T(t)\int_0^1 g(s,u(s))ds\| + \|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,u(s))ds\| \\ &\leq MR\int_0^1 p(s)ds + \frac{M\psi(R)}{\Gamma(\alpha)}\|\int_0^t (t-s)^{\alpha-1}\phi(s)ds\| \\ &\leq MR\int_0^1 p(s)ds + M\psi(R)I^{\alpha}\phi(t) \leq R, \end{aligned}$$

which implies $F : W \longrightarrow W$ is a bounded operator.
Let $B_0 = \overline{co}(FW)$. Then, for any $B \subset B_0$, we use Lemma 2.9 and 2.10. That is for any $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subseteq B$, such that

$$\begin{aligned} \gamma(F^1B(t)) &= \gamma(FB(t)) \\ &\leq 2\gamma\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,\{u_n(s)\}_{n=1}^{\infty})ds\Big) + \varepsilon \\ &\leq \frac{4M}{\Gamma(\alpha)}\int_0^t \gamma\Big((t-s)^{\alpha-1}f(s,\{u_n(s)\}_{n=1}^{\infty})\Big)ds + \varepsilon \\ &\leq \frac{4M}{\Gamma(\alpha)}\int_0^t \gamma\Big((t-s)^{\alpha-1}\big|f(s,\{u_n(s)\}_{n=1}^{\infty}) - f(s,0) + f(s,0)\big|\Big)ds + \varepsilon \\ &\leq \frac{4ML}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma\Big(\{u_n(s)\}_{n=1}^{\infty})\Big)ds + \varepsilon \\ &\leq \frac{4ML}{\Gamma(\alpha+1)}\gamma(B)t^{\alpha} + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows from the above inequality that

$$\gamma(F^1B(t)) \leq \frac{4ML}{\Gamma(\alpha+1)}\gamma(B)t^{\alpha}. \tag{3.2}$$

Using Lemma 2.10 one more time, we see that for any $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subseteq \overline{co}(F^1B(t))$, such that

$$\begin{aligned}
\gamma(F^2B(t)) &= \gamma\Big(F\big(\overline{co}(FB(t))\big)\Big) \\
&\leq 2\gamma\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,\{u_n(s)\}_{n=1}^{\infty})ds\Big) + \varepsilon \\
&\leq \frac{4M}{\Gamma(\alpha)}\int_0^t \gamma\Big((t-s)^{\alpha-1}f(s,\{u_n(s)\}_{n=1}^{\infty})\Big)ds + \varepsilon \\
&\leq \frac{4M}{\Gamma(\alpha)}\int_0^t \gamma\Big((t-s)^{\alpha-1}\big|f(s,\{u_n(s)\}_{n=1}^{\infty}) - f(s,0) + f(s,0)\big|\Big)ds + \varepsilon \\
&\leq \frac{4ML}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma\Big(\{u_n(s)\}_{n=1}^{\infty}\Big)ds + \varepsilon \\
&\leq \frac{4ML^2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma(F^1B(s))ds + \varepsilon \\
&\leq \frac{(4ML)^2}{\Gamma(\alpha+1)}\gamma(B)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha}ds + \varepsilon \\
&\leq \frac{(4ML)^2}{\Gamma(2\alpha+1)}\gamma(B)t^{2\alpha} + \varepsilon.
\end{aligned}$$

Hence, by mathematical induction, for any positive integer n and $t \in J$, we obtain

$$\gamma(F^nB(t)) \leq \frac{(4ML)^n}{\Gamma(n\alpha+1)}\gamma(B)t^{n\alpha}.$$

Since $\frac{(4ML)^n}{\Gamma(n\alpha+1)}\gamma(B)t^{n\alpha} \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{(4ML)^{n_0}}{\Gamma(n_0\alpha+1)}\gamma(B)t^{n_0\alpha} = k < 1.$$

Finally, applying Theorem 2.12 it follows that F has a fixed point in W. This fixed point is a mild solution of problem (1.1). □

Similar to idea in [23], we present following theorem.

Theorem 3.2. *If* (H1)-(H5) *are satisfied, then there is at least one mild solution for* (1.1) *provided that*

$$I^{\alpha}\phi(t) \leq \liminf_{R\to\infty} \frac{R - MR\int_0^1 p(s)ds}{M\psi(R)}.$$

Proof . From Theorem 3.1, we know that there exists a constant $R > 0$ such that

$$I^{\alpha}\phi(t) \leq \frac{R - MR\int_0^1 p(s)ds}{M\psi(R)}.$$

The detail proof of this Theorem is similar to the proof of Theorem 3.1 and therefore, we omit it here. □

Our second existence result is based on a generalization of Schauder's fixed point theorem (i.e. Theorem 2.13), presented by Deimling [9].

Theorem 3.3. *If* (H1)-(H5) *are satisfied, then there is at least one mild solution for* (1.1) *provided that*

$$\frac{4LM}{\Gamma(\alpha+1)} \leq 1.$$

Proof . Consider operator F defined by (3.1). In the proof of Theorem 3.1 we proved $F : W \longrightarrow W$ is continuous and bounded operator on the bounded closed and convex set W which is defined by

$$W = \{u \in C(J, E),\ \|u(t)\| \leq R,\ \text{for all } t \in J\}.$$

Now, we prove that the operator $F : W \longrightarrow W$ is equicontinuous.
Let $C = \sup\{\|f(t, u(t)\| \ :\ \|u(t)\| \leq R,\ t \in J\}$. Then for any $u \in W$ and $0 \leq t_1 < t_2 \leq 1$, we obtain that

$$\|(Fu)(t_2) - (Fu)(t_1)\| \leq \sum_{i=1}^{4} \|I_i\|,$$

where

$$\begin{aligned}
I_1 &= \big(T(t_2) - T(t_1)\big) \int_0^1 g(s, u(s))ds,\\
I_2 &= \int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} T(t_2 - s) f(s, u(s))ds,\\
I_3 &= \int_0^{t_1} (t_1 - s)^{\alpha-1} \Big(T(t_2 - s) - T(t_1 - s)\Big) f(s, u(s))ds,\\
I_4 &= \int_0^{t_1} \Big((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\Big) T(t_2 - s) f(s, u(s))ds,
\end{aligned}$$

Therefore, we only need to check $\|I_i\|$, $(i = 1, ..., 4)$ tend to 0 independently of $u \in W$ when $t_2 \longrightarrow t_1$.
For I_1, by assumptions $(H1)$ and $(H3)$, it is obvious that $\|I_1\| \longrightarrow 0$ as $t_2 \longrightarrow t_1$.
For I_2, by assumption $(H1)$ we get that

$$\begin{aligned}
\|I_2\| &\leq M \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, u(s))\| ds\\
&\leq MC \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds\\
&\leq \frac{MC}{\Gamma(\alpha+1)} (t_2 - t_1)^{\alpha}\\
&\longrightarrow 0 \qquad \text{as } t_2 \longrightarrow t_1.
\end{aligned}$$

For I_3, by assumption $(H1)$ and properties of C_0 semigroup we get that

$$\begin{aligned}
\|I_3\| &\leq \int_0^{t_1} (t_1 - s)^{\alpha-1} \big\|\big(T(t_2 - s) - T(t_1 - s)\big) f(s, u(s))\big\| ds\\
&\leq C\big\|T(t_2 - t_1) - I\big\| \int_0^{t_1} (t_1 - s)^{\alpha-1} \big\|T(t_1 - s)\big\| ds\\
&\leq MC\big\|T(t_2 - t_1) - I\big\| \int_0^{t_1} (t_1 - s)^{\alpha-1} ds\\
&\leq \frac{MCt^{\alpha}}{\Gamma(\alpha+1)} \big\|T(t_2 - t_1) - I\big\|\\
&\longrightarrow 0 \qquad \text{as } t_2 \longrightarrow t_1.
\end{aligned}$$

For I_4, by assumption $(H1)$ we get that

$$\begin{aligned}
\|I_4\| &\leq M\int_0^{t_1}\Big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big)\|f(s,u(s))\|ds\\
&\leq MC\int_0^{t_1}\Big((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big)ds\\
&\leq \frac{MC}{\Gamma(\alpha+1)}\big(t_2^\alpha-t_1^\alpha+(t_2-t_1)^\alpha\big)\\
&\leq \frac{2MC}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha\\
&\longrightarrow 0 \quad \text{as } t_2\longrightarrow t_1.
\end{aligned}$$

Hence, $\|(Fu)(t_2)-(Fu)(t_1)\|$ tends to 0 independently of $u\in W$ when $t_2\longrightarrow t_1$. This means that operator $F:W\longrightarrow W$ is equicontinuous.
We defined B_0 in the proof of Theorem 3.1 as $B_0=\overline{co}(FW)$. It is easy to show that F maps B_0 into itself and by Lemma 2.11, we know $B_0\subset C(J,E)$ is equicontinuous. Since $t\in I$, similar to proof of equation (3.2) we have

$$\begin{aligned}
\gamma(FB_0) &\leq \frac{4ML}{\Gamma(\alpha+1)}\gamma(B_)\\
&\leq \gamma(B_0).
\end{aligned}$$

Therefore, $F:B_0\longrightarrow B_0$ is a condensing operator. It follows from Theorem 2.13 that F has one fixed point, which means (1.1) has one mild solution on J. $\square$

References

[1] M.H. Akrami and G.H. Erjaee, *Examples of Analytical Solutions by Means of Mittag-Leffler Function of Fractional Black-Scholes Option Pricing Equation*, Fract. Calc. Appl. Anal. 18 (2015) 38–47.

[2] J. Banaś and K. Goebel, *Measures of noncompactness in Banach Spaces*, vol. 60 of Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 1980.

[3] M. Benchohra, A. Cabada and D. Seba, *An Existence result for nonlinear fractional differential equations on Banach spaces*, Bound. Value Probl., Volume 2009, Article ID 628916, 11 pages doi:10.1155/2009/628916.

[4] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. 162 (1991) 494–505.

[5] L. Byszewski, *Existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problem*, Zesz. Nauk. Pol. Rzes. Mat. Fiz. 18 (1993) 109–112.

[6] L. Byszewski and V. Lakshmikantham, *Theorem about the existence and uniqueness of solutions of a nonlocal Cauchy problem in a Banach space*, Appl. Anal. 40 (1990) 11–19.

[7] P. Chena, Y. Li, Q. Chenb and B. Fenga, *On the initial value problem of fractional evolution equations with noncompact semigroup*, Comput. Math. Appl. 67 (2014) 1108–1115.

[8] J. Dabaš and A. Chauhan, *Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay*, Math. Compu. Model. 57 (2013) 754–763.

[9] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[10] Z. Fan, Q. Dong, G. Li,*Semilinear differential equations with nonlocal conditions in Banach spaces*, Inter. J. Nonlinear Sci. 2 (2006) 131–139.

[11] A. Ferriero and N. Fusco,*A note on the convex hull of sets of finite Perimeter In The Plane*, Discrete Continuous Dyn. Syst. Series B 11(1) (2009).

[12] X. Fu and K. Ezzinbi,*Existence of solutions for neutral differential evolution equations with nonlocal conditions*, Nonlinear Anal. 54 (2003) 215–227.

[13] H.P. Heinz, *On the behaviour of measure of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. 7 (1983) 1351–1371.

[14] H. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publ. Co., Singapure, 2000.
[15] O.K. Jardat, A. Al-Omari and S. Momani,*Existence of the mild solution for fractional semilinear initial value problems*, Nonlinear Anal. 69 (2008) 3153–3159.
[16] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of North-Holland Mathematics Studies. Elsevier Science Amsterdam, 2006.
[17] L.S. Liu, F. Guo, C.X. Wu and Y.H. Wu,*Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces*, J. Math. Anal. Appl. 309 (2005) 638–649.
[18] Z.W. Lv, J. Liang, T.J. Xiao,*Solutions to fractional differential equations with nonlocal initial condition in Banach Spaces*, Advances Diff. Equ. 2010, 10 pages, Article ID 340349.
[19] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Anal. TMA 4 (1980) 985–999.
[20] G.M. Mophou, *Existence and uniqueness of mild solutions to impulsive fractional differential equations*, Nonlinear Anal. 72 (2010) 1604–1615.
[21] M. Muslim,*Existence and approximation of solutions to fractional differential equations*, Math. Comput. Model. 49 (2009) 1164–1172.
[22] J.R. Wang, Y. Zhou, M. Fečkan, *Abstract Cauchy problem for fractional differential equations*, Nonlinear Dyn. 71 (2013) 685–700.
[23] X. Xue, *Existence of solutions for semilinear nonlocal Cauchy problems in Banach spaces*, Elec. J. Differ. Eq. 64 (2005) 1–7.
[24] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific Publ. Company, 2014.
[25] Y. Zhou and F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Comput. Math. Appl. 59 (2010) 1063–1077.