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An inequality related to η-convex functions (II)

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Abstract

Using the notion of η-convex functions as generalization of convex functions, we estimate the difference between the middle and right terms in Hermite-Hadamard-Fejer inequality for differentiable mappings. Also as an application we give an error estimate for midpoint formula.

Keywords: η-convex function; Hermite-Hadamard-Fejer inequality.
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1. Introduction and Preliminaries

The elegance in shape and properties of convex functions makes it attractive to study this kind of functions in mathematical analysis. It should be noticed that in new problems related to convexity, generalized notions about convex functions are required to obtain applicable results. During recently years many efforts have gone on generalization of notion of convex functions. Most important generalizations can be found in works that change the form of defining of functions to a generalized form such as quasi-convex [1], pseudo-convex [7], strongly convex [9], logarithmically convex [8], approximately convex [5], midconvex [6] functions etc.

On the other hand Hermite-Hadamard-Fejer inequality, an interesting result related to convex functions has been proved in [4] as the following:

Theorem 1.1. Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then

$$f(\frac{a+b}{2})\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1.1}$$

where $g : [a, b] \to \mathbb{R}^+ = [0, +\infty)$ is integrable and symmetric about $x = \frac{a+b}{2}\big(g(x) = g(a+b-x), \forall x \in [a, b]\big)$.

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If in (1.1) we consider $g \equiv 1$ then we obtain Hermite-Hadamard inequality:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

An interesting question in (1.2), was estimating the difference between left and middle terms and between right and middle terms. In [2], the difference between middle and right terms in (1.2) has been estimated as the following:

Theorem 1.2. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:

$$\Big|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\Big| \leq \frac{b-a}{8}\big(|f'(a)| + |f'(b)|\big).$$

Motivated by these works we introduce the notion of η-convex functions as generalization of convex functions and estimate the difference between middle and left terms in (1.1), when $|f'|$ is an η-convex function. Also as an application we give an error estimate for midpoint formula.

Definition 1.3. [3] Let I be an interval in real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called *convex with respect to bifunction* $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (briefly η-convex), if

$$f(tx + (1-t)y) \leq f(y) + t\eta\big(f(x), f(y)\big) \tag{1.3}$$

for all $x, y \in I$ and $t \in [0, 1]$.

In fact above definition geometrically says that if a function is η-convex on I, then it's graph between any $x, y \in I$ is on or under the path starting from $\big(y, f(y)\big)$ and ending at $\big(x, f(y) + \eta(f(x), f(y))\big)$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one. Note that by taking $x = y$ in (1.3) we get $t\eta(f(x), f(x)) \geq 0$ for any $x \in I$ and $t \in [0, 1]$ which implies that

$$\eta(f(x), f(x)) \geq 0$$

for any $x \in I$. Also if we take t = 1 in (1.3) we get

$$f(x) - f(y) \leq \eta\big(f(x), f(y)\big)$$

for any $x, y \in I$. If $f : I \to \mathbb{R}$ is a convex function and $\eta : I \times I \to \mathbb{R}$ is an arbitrary bifunction that satisfies

$$\eta(x, y) \geq x - y$$

for any $x, y \in I$, then

$$f(tx + (1-t)y) \leq f(y) + t[f(x) - f(y)] \leq f(y) + t\eta\big(f(x), f(y)\big)$$

showing that f is η-convex.

There are simple examples about η-convexity of a function([3]).

Example 1.4. (1) For a convex function f, we may find another function η other than the function $\eta(x,y) = x - y$ such that f is η-convex. Consider $f(x) = x^2$ and $\eta(x,y) = 2x + y$. Then we have

$$\begin{gathered} f\big(\lambda x + (1-\lambda)y\big) = \big(\lambda x + (1-\lambda)y\big)^2 \leq \\ y^2 + \lambda x^2 + \lambda(1-\lambda)2xy \leq y^2 + \lambda x^2 + \lambda(1-\lambda)(x^2+y^2) \leq \\ y^2 + \lambda(x^2 + x^2 + y^2) = y^2 + \lambda(2x^2 + y^2) = f(y) + \lambda\eta\big(f(x), f(y)\big) \end{gathered}$$

for all $x, y \in \mathbb{R}$ and $\lambda \in (0,1)$. Also the facts $x^2 \leq y^2 + (2x^2 + y^2)$ and $y^2 \leq y^2$, for all $x, y \in \mathbb{R}$ show the correctness of inequality for $\lambda = 1$ and $\lambda = 0$ respectively which means that f is η-convex. Note that the function $f(x) = x^2$ is η-convex w.r.t all $\eta(x,y) = ax + by$ with $a \geq 1$, $b \geq -1$ and $x, y \in \mathbb{R}$.

(2) Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} -x, & x \geq 0; \\ x, & x < 0. \end{cases}$$

and define a bifunction η as $\eta(x,y) = -x - y$, for all $x, y \in \mathbb{R}^- = (-\infty, 0]$. It is not hard to check that f is an η-convex function but not a convex one.

(3) Define the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ as $f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$ and a bifunction $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as $\eta(x,y) = \begin{cases} x + y, & x \leq y; \\ 2(x+y), & x > y. \end{cases}$

Then f is η-convex but is not convex.

The first result is the fact that any η-convex function with a bounded bifunction η from above, satisfies the Lipschitz condition. Two definitions are required.

Definition 1.5. [10] A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ if corresponding to any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any collection $\{a_i, b_i\}_1^n$ of disjoint open intervals of $[a,b]$ with $\sum_1^n (b_i - a_i) < \delta$, $\sum_1^n |f(b_i) - f(a_i)| < \varepsilon$.

Definition 1.6. [10] A function $f : [a,b] \to \mathbb{R}$ is said to satisfy Lipschitz condition on $[a,b]$ if there is a constant K so that for any two points $x, y \in [a,b]$, $|f(x) - f(y)| \leq K|x - y|$.

Lemma 1.7. *Suppose that $f : I \to \mathbb{R}$ is an η-convex function and η is bounded from above on $f(I) \times f(I)$. Then f satisfies the Lipschitz condition on any closed interval $[a,b]$ contained in I°, the interior of I. Hence, f is absolutely continuous on $[a,b]$ and continuous on I°.*

Proof . Let M_η be the upper bound of η on $f(I) \times f(I)$. Consider closed interval $[a,b]$ in I° and choose $\varepsilon > 0$ such that $[a - \varepsilon, b + \varepsilon]$ belongs to I. Suppose that x, y are distinct points of $[a,b]$. Set $z = y + \frac{\varepsilon}{|y-x|}(y - x)$ and $t = \frac{|y-x|}{\varepsilon + |y-x|}$. So it is not hard to see that $z \in [a - \varepsilon, b + \varepsilon]$ and $y = tz + (1-t)x$. Then

$$f(y) \leq f(x) + t\eta(f(z), f(x)) \leq f(x) + tM_\eta.$$

This implies that

$$f(y) - f(x) \leq tM_\eta = \frac{|y-x|}{\varepsilon + |y-x|} M_\eta \leq \frac{|y-x|}{\varepsilon} M_\eta = K\,|y-x|\,,$$

where $K = \frac{M_\eta}{\varepsilon}$.

Also if we change the place of x, y in above argument we have $f(x) - f(y) \leq K\,|y-x|$. Therefore $|\ f(y) - f(x)\ | \leq K\,|y-x|$.

It follows that if we choose $\delta < \varepsilon/K$, then f is absolutely continuous. Finally since $[a,b]$ is arbitrary on I°, then f is continuous on I°. $\square$

As a consequence of Lemma 1.7, an η-convex function $f : [a,b] \to \mathbb{R}$ where η is bounded from above on $f([a,b]) \times f([a,b])$ is integrable.

2. Main Result

The first result of this section is a lemma that is generalization of Lemma 2.1 in [2].

Lemma 2.1. *Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable function, $g : [a,b] \to \mathbb{R}^+$ is a continuous function and f' is an integrable function on $[a,b]$. Then*

$$\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \int_a^b f(x)g(x)dx =$$
$$\frac{1}{2}\int_a^b\int_a^x g(u)f'(x)dudx - \frac{1}{2}\int_a^b\int_x^b g(u)f'(x)dudx.$$

Proof . By Leibniz integral rule and integration by parts we have

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)\Big(\int_a^x g(u)du\Big)'dx = f(b)\int_a^b g(u)du - \int_a^b\int_a^x g(u)f'(x)dudx. \tag{2.1}$$

With the same argument

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)\Big(-\int_x^b g(u)du\Big)'dx = f(a)\int_a^b g(u)du + \int_a^b\int_x^b g(u)f'(x)dudx. \tag{2.2}$$

Adding relations (2.1) and (2.2), gives the result. $\square$

The following lemma is a consequence of lemma 2.1.

Lemma 2.2. *Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable function, $g : [a,b] \to \mathbb{R}^+$ is a continuous function and symmetric about $\frac{a+b}{2}$ and f' is an integrable function on $[a,b]$. Then*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \int_a^b f(x)g(x)dx =\\ &\frac{(b-a)}{4}\Bigg\{\int_0^1\Bigg(\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du\Bigg)f'\Big(\frac{1+t}{2}a+\frac{1-t}{2}b\Big)dt+\\ &\int_0^1\Bigg(\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du\Bigg)f'\Big(\frac{1-t}{2}a+\frac{1+t}{2}b\Big)dt.\Bigg\}\end{aligned} \tag{2.3}$$

Proof . From Lemma 2.1 we can see

$$\begin{aligned}I =&\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \int_a^b f(x)g(x)dx =\\ &\frac{1}{2}\Bigg\{\int_a^{\frac{a+b}{2}}\int_a^x g(u)f'(x)dudx + \int_{\frac{a+b}{2}}^b\int_a^x g(u)f'(x)dudx-\\ &\int_a^{\frac{a+b}{2}}\int_x^b g(u)f'(x)dudx - \int_{\frac{a+b}{2}}^b\int_x^b g(u)f'(x)dudx.\Bigg\}\end{aligned} \tag{2.4}$$

By changing the variable $x = \frac{1+t}{2}a + \frac{1-t}{2}b$ and $x = \frac{1-t}{2}a + \frac{1+t}{2}b$ in (2.4) we have

$$I = \frac{b-a}{4}\Bigg\{ \int_0^1 \int_a^{\frac{1+t}{2}a+\frac{1-t}{2}b} g(u)f'(\frac{1+t}{2}a + \frac{1-t}{2}b)dudt+ \tag{2.5}$$

$$\int_0^1 \int_a^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)f'(\frac{1-t}{2}a + \frac{1+t}{2}b)dudt- \tag{2.6}$$

$$\int_0^1 \int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{b} g(u)f'(\frac{1+t}{2}a + \frac{1-t}{2}b)dudt- \tag{2.7}$$

$$\int_0^1 \int_{\frac{1-t}{2}a+\frac{1+t}{2}b}^{b} g(u)f'(\frac{1-t}{2}a + \frac{1+t}{2}b)dudt.\Bigg\} \tag{2.8}$$

Consider (2.5) with (2.7) and consider (2.6) with (2.8) together. Then

$$I = \frac{b-a}{4}\Bigg\{ \int_0^1 \Big[2\int_a^{\frac{1+t}{2}a+\frac{1-t}{2}b} g(u)du - \int_a^b g(u)du\Big] f'(\frac{1+t}{2}a + \frac{1-t}{2}b)dt+ \tag{2.9}$$
$$\int_0^1 \Big[2\int_a^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du - \int_a^b g(u)du\Big] f'(\frac{1-t}{2}a + \frac{1+t}{2}b)dt.\Bigg\}$$

Since g is symmetric with respect to $\frac{a+b}{2}$ then

$$2\int_a^{\frac{1+t}{2}a+\frac{1-t}{2}b} g(u)du - \int_a^b g(u)du = \int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du, \tag{2.10}$$

and

$$2\int_a^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du - \int_a^b g(u)du = \int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du. \tag{2.11}$$

Implying (2.10) and (2.11) in (2.9) we have

$$I = \frac{(b-a)}{4}\Bigg\{ \int_0^1 \Big(\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du\Big) f'\big(\frac{1+t}{2}a + \frac{1-t}{2}b\big)dt+$$
$$\int_0^1 \Big(\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du\Big) f'\big(\frac{1-t}{2}a + \frac{1+t}{2}b\big)dt.\Bigg\}$$

□

Remark 2.3. Lemma 2.1 and 2.2 are equivalent to Lemma 2.1 in [2], if we set $g \equiv 1$.

Based on Lemma 2.2, we obtain the main theorem of the paper.

Theorem 2.4. *Suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable function, $g : [a, b] \to \mathbb{R}^+$ is a continuous function and symmetric about $\frac{a+b}{2}$ and $|f'|$ is an η-convex function where η is bounded from above on $[a, b]$. Then*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \int_a^b f(x)g(x)dx\right| \leq$$
$$\frac{(b-a)}{4}\Big[2\big|f'(b)\big| + \big|\eta\big(f'(a), f'(b)\big)\big|\Big] \int_0^1 \int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)dudt. \tag{2.12}$$

Proof . From Lemma 2.2 and the fact that $|f'|$ is η-convex where η is bounded from above we have

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \int_a^b f(x)g(x)dx\right| \leq$$
$$\frac{(b-a)}{4}\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)\Big[\big|f'\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big)\big| + \big|f'\big(\frac{1-t}{2}a+\frac{1+t}{2}b\big)\big|\Big]dudt \leq$$
$$\frac{(b-a)}{4}\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)\Big[|f'(b)| + \frac{1+t}{2}\eta\big(|f'(a)|,|f'(b)|\big) + |f'(b)| + \frac{1-t}{2}\eta\big(|f'(a)|,|f'(b)|\big)\Big]dudt =$$
$$\frac{(b-a)}{4}\Big[2\big|f'(b)\big| + \big|\eta\big(f'(a),f'(b)\big)\big|\Big]\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)dudt.$$

□

Remark 2.5. Theorem 2.4 reduces to Theorem 1.2, if we consider $g \equiv 1$ and $\eta(x,y) = x - y$ for all $x, y \in [a,b]$.

Finally as an application of Theorem 2.4, we give an error estimate for midpoint formula that is generalization of Proposition 4.1 in [2].

Suppose that d is a partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of interval $[a,b]$. Consider formula

$$\int_a^b f(x)g(x)dx = T(f,g,d) + E(f,g,d),$$

where

$$T(f,g,d) = \sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}\int_{x_i}^{x_{i+1}} g(x)dx$$

and $E(f,g,d)$ is the approximation error.

Theorem 2.6. *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable function, $g:[a,b]\to\mathbb{R}^+$ is a continuous function and symmetric with respect to $\frac{a+b}{2}$ and $|f'|$ is an η-convex function where η is bounded from above on $[a,b]$. Then*

$$\Big|E(f,g,d)\Big| \leq \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)}{4}\Big[2\big|f'(x_{i+1})\big| + \big|\eta\big(f'(x_i),f'(x_{i+1})\big)\big|\Big]\int_0^1\int_{\frac{1+t}{2}x_i+\frac{1-t}{2}x_{i+1}}^{\frac{1-t}{2}x_i+\frac{1+t}{2}x_{i+1}} g(x)dxdt.$$

Proof . It is enough to apply Theorem 2.4 on the subinterval $[x_i, x_{i+1}]$ $(i = 0,1,\cdots,n-1)$ of the partition d for interval $[a,b]$, and to sum all achieved inequalities over i and then using triangle inequality. □

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