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Polarization constant $\mathcal{K}(n, X) = 1$ for entire functions of exponential type

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Abstract

In this paper we will prove that if L is a continuous symmetric n-linear form on a Hilbert space and $\widehat{L}$ is the associated continuous n-homogeneous polynomial, then $||L|| = ||\widehat{L}||$. For the proof we are using a classical generalized inequality due to S. Bernstein for entire functions of exponential type. Furthermore we study the case that if X is a Banach space then we have that

$$\|L\| = \|\widehat{L}\|, \; \forall \;\; L \in \mathcal{L}^s(^nX)\;.$$

If the previous relation holds for every $L \in \mathcal{L}^s(^nX)$, then spaces $\mathcal{P}(^nX)$ and $L \in \mathcal{L}^s(^nX)$ are isometric. We can also study the same problem using Fréchet derivative.

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1. Introduction and preliminaries

Many results which hold in linear forms can be expand in multilinear forms and polynomials. We refer without proof the following proposition (see [4]).

Proposition 1.1. *Let $L : X^n \to Y$ a symmetric n-linear form and $P : X \to Y$, with $P = \widehat{L}$, the associated homogeneous polynomial, where X, Y are Banach spaces. The following are equivalent*

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i. $L : X^n \to Y$ is continuous.
ii. $P : X \to Y$ is continuous.
iii. $P : X \to Y$ is continuous to 0.
iv. There is a constant $M_1 > 0$ such that $\|P(x)\| \leq M_1 \|x\|^n$.
v. There is a constant $M_2 > 0$ such that

$$\|L(x_1, x_2, \dots, x_n)\| \leq M_2 \|x_1\| \cdots \|x_n\|, \quad \forall \ (x_1, \dots, x_n) \in X^n .$$

We can easily prove the following result using Banach-Steinhaus Theorem, (see Theorem 3.5 in [4]).

Proposition 1.2. Let $X_1, \dots, X_n$ Banach spaces and Y a normed space. The multilinear form $L : X_1 \times \cdots \times X_n \to Y$ is continuous if and only if L is continuous for each variable.

If X and Y Banach spaces and $L : X^n \to Y$ a continuous, symmetric n-linear form, we define the norm

$$\begin{aligned} \|L\| : &= \inf\{M : \|L(x_1, x_2, \dots, x_n)\| \leq M \|x_1\| \cdots \|x_n\|, \ \forall \, (x_1, \dots, x_n) \in X^n\} \\ &= \sup\{\|L(x_1, \dots, x_n)\| : \|x_1\| \leq 1, \dots, \|x_n\| \leq 1\} . \end{aligned}$$

We express with $\mathcal{L}^s({}^n X, Y)$ the Banach space of the continuous, symmetric n-linear forms equipped with the above norm. Similarly, we express with $\mathcal{P}({}^n X, Y)$ the Banach space of the continuous, homogeneous polynomials $P : X \to Y$ n-th degree, with norm

$$\begin{aligned} \|P\| : &= \inf\{M > 0 : \|P(x)\| \leq M \|x\|^n, \ \forall \, x \in X\} \\ &= \sup\{\|P(x)\| : \|x\| \leq 1\} . \end{aligned}$$

Remark 1.3. Generally, we can estimate the norm of $L \in \mathcal{L}^s({}^n X, Y)$ easier than the norm of the associated homogeneous polynomial $\widehat{L} \in \mathcal{P}({}^n X, Y)$. Obviously we have

$$\|\widehat{L}\| \leq \|L\| .$$

Mazur-Orlicz study the relation between the norm of $L \in \mathcal{L}^s({}^n X, Y)$ and the norm of the associated homogeneous polynomial $\widehat{L} \in \mathcal{P}({}^n X, Y)$.

We refer the following problem of Mazur-Orlicz from the famous “Scottish Book” [10, Problem 73]:

Problem. Let c_n be the smallest number with the property that if $F(x_1, \dots, x_n)$ is an arbitrary symmetric n-linear operator [in a Banach space and with values in such a space], then

$$\sup_{\substack{\|x_i\| \leq 1 \\ 1 \leq i \leq n}} \|F(x_1, \dots, x_n)\| \leq c_n \sup_{\|x\| \leq 1} \|F(x, \dots, x)\| .$$

It is known (Mr. Banach) that c_n exists. One can show that the number c_n satisfies the inequalities

$$\frac{n^n}{n!} \leq c_n \leq \frac{1}{n!} \sum_{k=1}^{n} \tbinom{n}{k} k^n .$$

Is $c_n = \frac{n^n}{n!}$?

The answer to this problem is yes for any real or complex Banach space. R. S. Martin [9] proved that $c_n \le n^n/n!$ with the aid of an n- dimensional polarization formula. Indeed, if $L \in \mathcal{L}^s\left({}^nX\right)$ and $x_1, \ldots, x_n$ are unit vectors in X, then we obtain that

$$L\left(x_1, \ldots, x_n\right) = \frac{1}{n!} \int_0^1 r_1\left(t\right) \cdots r_n\left(t\right) P\left(\sum_{i=1}^{n} r_i\left(t\right) x_i\right) dt\,. \tag{1.1}$$

The nth Rademacher function r_n is defined on $[0,1]$ by $r_n\left(t\right) = \operatorname{sign} \sin 2^n \pi t$. The Rademacher functions $\{r_n\}$ form an orthonormal set in $L_2\left([0,1], dt\right)$ where dt denotes Lebesgue measure on $[0,1]$. For $x_1, \ldots, x_n \in X$, we can express the above polarization formula in the following convenient form:

$$\begin{aligned}
\|L(x_1,\ldots,x_n)\| &= \tfrac{1}{n!}\left\| \int_0^1 r_1(t)\cdots r_n(t)\widehat{L}\left(\sum_{k=1}^{n} r_k(t)x_k\right) dt \right\| \\
&\le \tfrac{1}{n!} \int_0^1 \left\| \widehat{L}\left(\sum_{k=1}^{n} r_k(t)x_k\right)\right\| dt \\
&\le \tfrac{\|\widehat{L}\|}{n!} \int_0^1 \left\| \sum_{k=1}^{n} r_k(t)x_k \right\|^n dt \\
&\le \tfrac{\|\widehat{L}\|}{n!} \left(\sum_{k=1}^{n} \|x_k\|\right)^n \\
&= \tfrac{n^n}{n!}\|\widehat{L}\|\,.
\end{aligned} \tag{1.2}$$

Thus, $\|L\| \le (n^n/n!)\|\widehat{L}\|$. So for every $L \in \mathcal{L}^s({}^nX, Y)$, we get

$$\|\widehat{L}\| \le \|L\| \le \frac{n^n}{n!}\|\widehat{L}\| \quad . \tag{1.3}$$

The map

$$\begin{array}{rcl}
\widehat{\ }\ : \mathcal{L}^s({}^nX,Y) & \to & \mathcal{P}({}^nX,Y) \\
L & \mapsto & \widehat{L}
\end{array}$$

is obviously linear and onto and because of the polarization formula Eq. (1.1), is one to one. Therefore, from (1.3) we infer that this map is a linear isomorphism.

Proposition 1.4. *If X and Y are Banach spaces, then the map*

$$\widehat{\ }\ : \mathcal{L}^s({}^nX,Y) \to \mathcal{P}({}^nX,Y)$$

is a linear isomorphism.

We will express the reverse form of this map with "$\check{\ }$". That is

$$\begin{array}{rcl}
\check{\ }\ : \mathcal{P}({}^nX,Y) & \to & \mathcal{L}^s({}^nX,Y) \\
P & \mapsto & \check{\mathrm{P}}
\end{array}$$

where $\check{\mathrm{P}}(x,\ldots,x) = P(x)$.

For spesific Banach spaces we can tighten the costant "$n^n/n!$" in (1.3). For that case we need the following definition

Definition 1.5. For X, Y Banach spaces and $n \in \mathbb{N}$, we define

$$\mathbb{K}\left(n, X\right) := \inf\left\{M : \|L\| \leq M\|\widehat{L}\|, \quad \forall\, L \in \mathcal{L}^s\left({}^nX, Y\right)\right\} .$$

$\mathbb{K}(n, X)$ is the n-th polarization constant, of the Banach space X.

Using Hahn-Banach's Theorem we obtain that in order to calculate $\mathbb{K}(n, X)$, we can consider continuous, symmetric n-linear forms, without loss of generality. That is, we can consider that $Y = \mathbb{K}$. In that case Banach spaces $\mathcal{L}^s({}^nX, Y)$ and $\mathcal{P}({}^nX, Y)$ represent as $\mathcal{L}^s({}^nX)$ and $\mathcal{P}({}^nX)$ respectively. From relation (1.3) turns out, that for every Banach space X we get that

$$1 \leq \mathbb{K}(n, X) \leq \frac{n^n}{n!} \; . \tag{1.4}$$

If H is a Hilbert space then

$$\mathbb{K}(n, H) = 1 \; .$$

An equivalent formulation for this fact is that spaces $\mathcal{L}^s({}^nH)$ and $\mathcal{P}({}^nH)$ are isometric.

2. Entire functions of exponential type

An analytical function $f : \mathbb{C} \to \mathbb{C}$ is called **entire** if it is analytical in all $\mathbb{C}$. Thus, if f is an entire function then

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \;\; \forall \;\; z \in \mathbb{C}$$

and from Cauchy-Hadamard's form, we have that $\limsup \sqrt[n]{|c_n|} = 0$.

Definition 2.1. The entire function f is of exponential type if

$$\limsup_{r\to\infty} \frac{\ln M(r)}{r} < \infty, \;\; \textit{where} \;\; M(r) = \max_{|z|=r} |f(z)| \; .$$

If we have that

$$\sigma = \limsup_{r\to\infty} \frac{\ln M(r)}{r} \; ,$$

then f is **exponential of type** σ.

Proposition 2.2. *Let f be an entire exponential function of type σ, $0 \leq \sigma < \infty$, then $\sigma = \inf\{k \geq 0 : M(r) < e^{rk}, \;\; \forall r \geq R_k\}$.*

Proof . Let $\lambda = \inf\{k \geq 0 : M(r) < e^{rk}, \;\; \forall\; r \geq R_k\}$. For every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $M(r) < e^{(\lambda+\varepsilon)r}$, for every $r > R_\varepsilon$ there exists a sequence $\{r_n\}$, with $r_1 < r_2 < \ldots < r_n < \ldots$, such that $M(r_n) > e^{(\lambda-\varepsilon)r_n}$. In other words we get

$$\frac{\ln M(r)}{r} < \lambda + \varepsilon, \;\; \forall \;\; r > R_\varepsilon$$

and

$$\frac{\ln M(r_n)}{r_n} > \lambda - \varepsilon$$

for suitable large r_n. That means that

$$\lambda = \limsup_{r\to\infty} \frac{\ln M(r)}{r} = \sigma \ .$$

□

From the previous we obtain that:

Corollary 2.3. Function f is an entire exponential function of type σ, if and only if for every $\varepsilon > 0$ but for no ε negative, the following holds

$$M(r) = \left(e^{(\sigma+\varepsilon)r}\right), \quad (r \to \infty) \ .$$

Example 2.4. 1. The analytical functions

$$e^z, \ \sin z, \ \cos z, \ \sinh z, \ \cosh z$$

are exponential functions of type 1.

2. The entire function $f(z) = e^z \cdot \cos z$ is an exponential function of type 2, on the other hand the entire function $g(z) = z^2 \cdot e^{2z} - e^{3z}$ is exponential of type 3.
3. If the entire functions f_1 and f_2 are exponential functions of type σ_1 and σ_2 respectively, then $f_1 \cdot f_2$ is an exponential function of type σ, with $\sigma \le \sigma_1 + \sigma_2$ and $f_1 + f_2$ is an exponential function of type σ^*, where $\sigma^* \le \max\{\sigma_1, \sigma_2\}$.
4. If

$$f(\vartheta) = \sum_{k=-n}^{n} c_k \, e^{ik\vartheta}, \quad c_k \in \mathbb{C} \ ,$$

is a trigonometric polynomial of type $\le n$, then from the previous examples we have that the entire function

$$\sum_{k=-n}^{n} c_k \, e^{ikz}$$

is an exponential function of type $\le n$.

The proof is very easy and without using the previous examples, we have

$$|f(z)| = |f(x+iy)| \le C \cdot e^{n|y|}, \quad where \ \ C = \sum_{k=-n}^{n} |c_k| \ ,$$

thus

$$M(r) = \max_{|z|=r} |f(z)| \le C \cdot e^{n \cdot r} \ .$$

Proposition 2.5. We assume that the entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is an exponential function of type σ. If $\lambda = \limsup n|c_n|^{1/n}$, with $0 \le \lambda < \infty$, then

$$\sigma = \frac{\lambda}{e} = \limsup \frac{n}{e} |c_n|^{\frac{1}{n}} \ . \tag{2.1}$$

For the proof of relation (2.1) we refer to R. P. Boas [1] and B. Ya. Levin [8].

Theorem 2.6. (**Bernstein's Inequality**) We assume that $f : \mathbb{C} \to \mathbb{C}$ is an entire exponential function of type $\leq \sigma$. If we have that $\sup_{x\in\mathbb{R}} |f(x)| < \infty$, then $\forall\, \omega \in \mathbb{R}$ we obtain

$$\sup_{x\in\mathbb{R}} |f'(x)\cos\omega + \sigma \cdot f(x)\sin\omega| \leq \sigma \cdot \sup_{x\in\mathbb{R}} |f(x)| \ . \tag{2.2}$$

Equality holds if and only if

$$f(z) = ae^{i\sigma z} + be^{-i\sigma z} \ , \ \ where \ \ a, b \in \mathbb{C} \ .$$

Using the same hypothesis like in Theorem 2.6, from (2.2) for $\omega = 0$, we have that

$$\sup_{x\in\mathbb{R}} |f'(x)| \leq \sigma \cdot \sup_{x\in\mathbb{R}} |f(x)| \ . \tag{2.3}$$

Relation (2.3) is the classical **Bernstein's inequality**. In particular, if f takes real values, then (2.2) implies the **Szegö's inequality**, that is

$$\sqrt{n^2 f\,(x)^2 + f'\,(x)^2} \leq n \sup_{x\in\mathbb{R}} |f\,(x)| \ , \tag{2.4}$$

for every $x \in \mathbb{R}$. Obviously inequalities (2.2), (2.3), and (2.4) hold in the special case where f is a trigonometric polynomial of type $\leq \sigma$.

Remark 2.7. Y. Katznelson [7], discover the relation witch connects Bernstein's inequality (2.3) with **Banach's theory of algebra**. An element a of *a Banach's complex algebra* A *with unit component*, is called *hermitian*, if $\|\exp(ita)\| = 1, \ \ \forall\, t \in \mathbb{R}$. For example, the hermitian elements in algebra of bounded operators in a Hilbert space are the self adjoint operators. It is known that the norm of a self adjoint operator in a Hilbert space is equal to the spectral radius of the operator. Using Bernstein's inequality (2.3), Y. Katznelson proved that : The norm of an hermitian element a of a Banach's algebra A, is equal to the spectral radius $\varrho(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. The above proposition is equivalent to Bernstein's inequality (2.3). Independently and almost simultaneously, Bonsall-Crabb [2], A. M. Sinclair [12] and A. Browder [3] proved the same result.

3. Polarization constant $\mathcal{K}(n, X) = 1$

We study the case that if X is a Banach space then we have that

$$\|L\| = \|\widehat{L}\|, \ \forall \ \ L \in \mathcal{L}^s(^n X) \ .$$

If the previous relation holds for every $L \in \mathcal{L}^s\,(^n X)$, then spaces $\mathcal{P}\,(^n X)$ and $L \in \mathcal{L}^s(^n X)$ are isometric. We can also study the same problem using Fréchet derivative.

Definition 3.1. Let X, Y two normed spaces and U is a non empty open subset of X. A function $f : U \to Y$ is called Fréchet differentiable in $\mathbf{x} \in \mathbf{U}$, if there exists a linear operator $F : X \to Y$ such that :

$$\lim_{y\to 0} \frac{\|f(x + y) - f(x) - F(y)\|}{\|y\|} = 0 \ .$$

F is the Fréchet derivative of f in $\mathbf{x} \in \mathbf{U}$, and expressed by $Df(x)$. That is

$$F = Df(x) \ .$$

If $f : U \to Y$ is Fréchet differentiable in every component of U, then we say that f is Fréchet differentiable in all of U. In that case the map

$$x \in U \mapsto Df(x) \in \mathcal{L}(X, Y)$$

is the Fréchet derivative of f in U and expressed by Df.

Proposition 3.2. *Let $P \in \mathcal{P}(^nX, Y)$ and $P = \widehat{L}$, for $L \in \mathcal{L}^s(^nX, Y)$, then the homogeneous polynomial P is Fréchet differentiable for every x and satisfies the following relation*

$$DP(x)(y) = nL(x^{n-1}, y) \ ,$$

where $L(x^{n-1}, y) = L(\underbrace{x, \ldots, x}_{n-1}, y)$.

Proof . By the definition of P we easily get that:

$$P(x+y) = L(x+y, x+y, \ldots, x+y) = \sum_{k=0}^{n} \tbinom{n}{k} L\left(x^{n-k}, y^k\right)$$

where

$$L(x^{n-k}, y^k) = L(\underbrace{x, x, \ldots, x}_{n-k}, \underbrace{y, y, \ldots, y}_{k}) \ .$$

Thus we have

$$\begin{aligned}
\lim_{y\to 0} \frac{\|P(x+y) - P(x) - nL\left(x^{n-1}, y\right)\|}{\|y\|} &= \lim_{y\to 0} \frac{\left\| \sum_{k=0}^{n} \binom{n}{k} L\left(x^{n-k}, y^k\right) - \widehat{L}(x) - nL\left(x^{n-1}, y\right) \right\|}{\|y\|} \\
&= \lim_{y\to 0} \frac{\left\| \sum_{k=2}^{n} \binom{n}{k} L(x^{n-k}, y^k) \right\|}{\|y\|} \\
&\leq \lim_{y\to 0} \sum_{k=2}^{n} \tbinom{n}{k} \|L\| \cdot \|x\|^{n-k} \|y\|^{k-1} = 0 \ .
\end{aligned}$$

Hence,

$$\lim_{y\to 0} \frac{\|P(x+y) - P(x) - nL\left(x^{n-1}, y\right)\|}{\|y\|} = 0,$$

that is

$$DP(x)(y) = nL\left(x^{n-1}, y\right) \ .$$

□

Remark 3.3. The value of the linear operator $DP(x) : X \to Y$ in y usually expressed by $DP(x)y$ instead of $DP(x)(y)$. So we have that:

$$DP(x)y = nL\left(x^{n-1}, y\right) \ .$$

Example 3.4. Let $(H, \langle\cdot,\cdot\rangle)$ a real Hilbert space and $P(x) = \|x\|^2,\ \forall\ x \in H$. Because for every $x, y \in H$ we have the following identity $\|x+y\|^2 - \|x\|^2 - 2\langle x, y\rangle = \|y\|^2$, we obtain that

$$DP(x)y = 2\langle x, y\rangle \ .$$

We have that $P(x) = \|x\|^2$ is a homogeneous polynomial of type 2, and so the previous result can be proved using Proposition 3.2, too.

Let $f : \mathbb{R}^n \to \mathbb{R}$ a convex function. If all the partial derivatives of f in $a \in \mathbb{R}^n$ exist, then it is known (lemma 19.4 in [5]) that function f is Fréchet differentiable in a. Also, we have (Theorem 19.5 in [5]) that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then f is Fréchet differentiable almost everywhere. Generally previous result doesn't hold in an infinite dimensional space. Indeed, for every $x \in \ell_1$ the convex function $f(x) = \|x\|_1$ isn't Fréchet differentiable (see examples 1.4(b) and 1.14(a) in [11]). Although the following result is well known, we refer it's proof complementary.

Example 3.5. Norm in ℓ_1 space isn't Fréchet differentiable for any component.

Proof . Let $x_0 = (x_k) \in \ell_1$, with $x_n = 0$ for a $n \in \mathbb{N}$. If $e_n = (\underbrace{0, 0, \ldots, 0, 1}_{n}, 0, \ldots)$ and $t \in \mathbb{R}$, then $\|x_0 + te_n\|_1 - \|x_0\|_1 = |t|$. If $f(x) = \|x\|_1$ and $F \in (\ell_1)^*$, then the limit

$$\lim_{t\to 0} \frac{|f(x_0 + te_n) - f(x_0) - F(te_n)|}{\|te_n\|_1} = \lim_{t\to 0} |\ |t|/t - F(e_n)\ |$$

either doesn't exist (if $F(e_n) \neq 0$) or it is equal to 1 (if $F(e_n) = 0$). Thus we have that function f isn't Fréchet differentiable in x_0. We assume now that $x_0 = (x_k) \in \ell_1$, with $x_k \neq 0\ \forall\ k \in \mathbb{N}$. Let

$$F = (sgnx_k) \in \ell_1^* = \ell_\infty, \ \ where \ \ sgnx_k = \frac{x_k}{|x_k|} \ .$$

We also suppose that

$$y = (y_k) \in \ell_1 \ \ where \ \ \|y\|_1 = \sum_{k=1}^{\infty} |y_k| = 1 \ .$$

Because $\lim\limits_{n\to\infty} \sum\limits_{k=n}^{\infty} |y_k| = 0$, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |y_k| < \frac{\varepsilon}{2} \ .$$

For $\delta > 0$ sufficiently small, if $1 \leq k \leq N$ and $|t| < \delta$ then it holds that

$$sgn(x_k + ty_k) = sgnx_k \ .$$

Thus for $|t| < \delta$ we get

$$\begin{aligned}
\tfrac{|f(x_0+ty)-f(x_0)-F(ty)|}{\|ty\|_1} &= \tfrac{1}{|t|} \cdot \left| \sum_{k=1}^{\infty} |x_k + ty_k| - \sum_{k=1}^{\infty} |x_k| - \sum_{k=1}^{\infty} ty_k sgnx_k \right| \\
&= \tfrac{1}{|t|} \cdot \left| \sum_{k=1}^{N} \{(x_k + ty_k) sgn(x_k + ty_k) - |x_k| - ty_k sgnx_k\} + \right. \\
&+ \left. \sum_{k=N+1}^{\infty} (|x_k + ty_k| - |x_k| - ty_k sgnx_k) \right| \\
&= \tfrac{1}{|t|} \cdot \left| \sum_{k=N+1}^{\infty} (|x_k + ty_k| - |x_k| - ty_k sgnx_k) \right| \\
&= \tfrac{1}{|t|} \cdot \sum_{k=N+1}^{\infty} \{|\, |x_k| + ty_k sgnx_k \,| - (|x_k| + ty_k sgnx_k)\} \\
&\le \tfrac{1}{|t|} \cdot \sum_{k=N+1}^{\infty} (|x_k| + t|y_k| - |x_k| - ty_k sgnx_k) \\
&\le 2 \cdot \sum_{k=N+1}^{\infty} |y_k| < \varepsilon \,.
\end{aligned}$$

So, we can say that if f is Fréchet differentiable in x_0, then for $Df(x_0)$ we must define $F = (sgnx_k)$. In order to prove that f isn't Fréchet differentiable in x_0, we consider the following sequence $y_n = (y_k^{(n)})$ of ℓ_1, where

$$y_k^n = \begin{cases} 0 \;, & k < n \\ -2x_k \;, & k \ge n \,. \end{cases}$$

Then

$$\|y_n\|_1 = 2 \cdot \sum_{k=n}^{\infty} |x_k| \quad and \quad \lim_{n\to\infty} \|y_n\| = 0 \,.$$

On the other hand we have

$$\begin{aligned}
|f(x_0 + y_n) - f(x_0) - F(y_n)| &= \left| \|x_0 + y_n\|_1 - \|x_0\| - \sum_{k=1}^{\infty} y_k^{(n)} sgnx_k \right| \\
&= \left| \sum_{k=1}^{\infty} \left| x_k + y_k^{(n)} \right| - \sum_{k=1}^{\infty} |x_k| - \sum_{k=1}^{\infty} y_k^{(n)} sgnx_k \right| \\
&= \left| \sum_{k=1}^{\infty} |x_k| - \sum_{k=1}^{\infty} |x_k| + 2 \sum_{k=n}^{\infty} |x_k| \right| \\
&= 2 \cdot \sum_{k=n}^{\infty} |x_k| \\
&= \|y_n\|_1 \,.
\end{aligned}$$

Hence f isn't Fréchet differentiable in x_0. $\square$

Now, if $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ (where $a_0, a_1, \ldots, a_n$ are real or complex coefficients) is a polynomial of type n, we have that the following Bernstein's inequality holds true

$$|P'(z)| \le n \|P\|_\infty, \quad |z| \le 1 \,, \tag{3.1}$$

where

$$\|P\|_\infty = \max\{|P(z)| : |z| = 1\} .$$

If P is a real polynomial of real variable then instead of relation (3.1) we have Markov's inequality:

$$|P'(x)| \leq n^2 \|P\|_{[-1,1]}, \quad -1 \leq x \leq 1 , \tag{3.2}$$

where

$$\|P\|_{[-1,1]} = \max\{|P(x)| : \|x\| \leq 1\} .$$

We have to mention here that constants n and n^2 in inequalities (3.1) and (3.2) are the best that is possible. L. Harris [6] proved that generalization of (3.1) holds for every polynomial of type $\leq n$ in a complex Hilbert space. It is noteworthy to mention that inequality (3.2) can be generalized in every real normed space X.

Now, what can we say for homogeneous polynomials in a normed space X ? For this kind of polynomials we give an equivalent form of Bernstein's inequality in a normed space X.

Proposition 3.6. *Let $P \in \mathcal{P}(^n X)$, then Bernstein's inequality*

$$\|DP\| = \sup_{\|\mathrm{x}\| \leq 1} \|\mathrm{DP(x)}\| \leq \mathrm{n}\|\mathrm{P}\|, \quad \|\mathrm{x}\| \leq 1 , \tag{3.3}$$

holds for every $n \in \mathbb{N}$, if and only if $\|L\| = \|P\|$, where $L \in \mathcal{L}^s(^n X)$ and $\widehat{L} = P$.

Proof . For $\|L\| = \|P\|$ and $\|x\| \leq 1$ we have:

$$\|DP(x)\| = \sup_{\|y\| \leq 1} |DP(x)y| = n \sup\{|L\left(x^{n-1}, y\right)| : \|y\| \leq 1\} \leq n\|L\| = n\|P\| .$$

Reverse, we assume that (3.3) holds for every $n \in \mathbb{N}$. Then we obtain

$$\sup\{|L\left(x^{n-1}, y\right)| : \|x\| \leq 1, \|y\| \leq 1\} \leq \|P\| . \tag{3.4}$$

We only have to prove that

$$\|L\| = \|P\| .$$

We can achieve proof by deductive.
For $n = 2$, using the previous inequality we get that $\|L\| \leq \|P\|$ and then

$$\|L\| = \|P\| .$$

We assune now that $\|F\| = \|\widehat{F}\|$, for every $F \in \mathcal{L}^s(^{n-1} X)$. If $L \in \mathcal{L}^s(^n X)$ and $P = \widehat{L}$, then for every $x_n \in X$, with $\|x_n\| \leq 1$, we define $L_{x_n} \in \mathcal{L}^s(^{n-1} X)$ as

$$L_{x_n}(x_1, x_2, \ldots, x_{n-1}) := L(x_1, x_2, \ldots, x_n) .$$

Using deductive hypothesis and relation (3.4) we obtain eventually

$$\begin{aligned} \|L\| &= \sup\{|L(x_1, x_2, \ldots, x_n)| : \|x_i\| \leq 1, 1 \leq i \leq n\} = \\ &= \sup\{|L_{x_n}(x_1, x_2, \ldots, x_{n-1})| : \|x_i\| \leq 1, 1 \leq i \leq n\} \\ &= \sup\{|L_{x_n}(x, x, \ldots, x)| : \|x_n\| \leq 1, \|x\| \leq 1\} \\ &= \sup\{|L\left(x^{n-1}, x_n\right)| : \|x\| \leq 1\} \\ &\leq \|P\| . \end{aligned}$$

Thus we get that

$$\|L\| = \|P\| .$$

□

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