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Existence and uniqueness results for a nonlinear differential equations of arbitrary order

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Abstract

This paper studies a fractional boundary value problem of nonlinear differential equations of arbitrary orders. New existence and uniqueness results are established using Banach contraction principle. Other existence results are obtained using Schaefer and Krasnoselskii fixed point theorems. In order to clarify our results, some illustrative examples are also presented.

Keywords: Caputo derivative; Boundary Value Problem; fixed point theorem; local conditions.
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1. Introduction

Boundary value problems for fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology In consequence, fractional differential equations have been of great interest, see [3, 4, 5, 6, 7, 8, 10, 15, 16, 19, 20, 24] and references therein. Recently, there has been a significant progress in the investigation of these equations, (see [1, 2, 9, 11, 12, 13, 19, 24]). More recently, some basic theory for the initial boundary value problems of fractional differential equations has been discussed in [1, 4, 13, 17, 19, 23]. Moreover, existence and uniqueness of solutions to boundary value problems for fractional differential equations had attracted the attention of many authors, see for example,[1, 2, 3, 9, 11, 12, 19, 21, 25] and the references therein.

In [9, 13, 21, 22], the existence and uniqueness of solutions was investigated for a nonlinear fractional differential equations with integral boundary conditions by using Schauder and Krasnoselskii's fixed point theorem.

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This paper deals with the existence and uniqueness of solutions for the following problem:

$$\begin{cases} D^{\alpha_0}x(t) = f(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), ..., D^{\alpha_{n-1}}x(t)), t \in [0,1], \\ x(0) = x^*, x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0, J^{\beta}x(1) = \lambda J^{\beta}x(\eta), \end{cases} \tag{1.1}$$

where $D^{\alpha_i}, i = 0, 1, 2, ..., n-1$, denote the Caputo fractional derivatives, with $n - 1 < \alpha_{n-1} < ... < \alpha_1 < \alpha_0 < n$ and $n \in N^*, n \neq 1, J = [0,1], \lambda \neq 0$ is real constant, $x^* \in \mathbb{R}, 0 < \eta < 1$ are real numbers and f is a function which will be specified later.

The rest of this paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to existence of solution of the problem (1.1). In section 4 examples are treated illustrating our results.

2. Preliminaries

The following notations, definitions, and preliminary facts will be used throughout this paper.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0, \infty[$ is defined as:

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \alpha > 0, \tag{2.1}$$

$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du$.

Definition 2.2. The fractional derivative of $f \in C^n([0,\infty[)$ in the Caputo's sense is defined as:

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, n-1 < \alpha, n \in N^*. \tag{2.2}$$

For more details about fractional calculus, we refer the reader to [18].

Let us now introduce the Banach space

$$X := \{x : x \in C([0,1], \mathbb{R}); D^{\alpha_1}x, D^{\alpha_2}x, ..., D^{\alpha_{n-1}}x \in C([0,1], \mathbb{R})\},$$

endowed with the norm

$$\|x\|_X = \|x\| + \|D^{\alpha_1}x\| + \|D^{\alpha_2}x\| + ... + \|D^{\alpha_{n-1}}x\|;$$

$$\|x\| = \sup_{t\in J}|x(t)|, \ \|D^{\alpha_1}x\| = \sup_{t\in J}|D^{\alpha_1}x(t)|,$$
$$\|D^{\alpha_2}x\| = \sup_{t\in J}|D^{\alpha_2}x(t)|, ..., \ \|D^{\alpha_{n-1}}x\| = \sup_{t\in J}|D^{\alpha_{n-1}}x(t)|.$$

We give the following lemmas [14, 15]:

Lemma 2.3. Let $r, s > 0, f \in L_1([a,b])$. Then $I^rI^sf(t) = I^{r+s}f(t), D^sI^sf(t) = f(t), t \in [a,b]$.

Lemma 2.4. Let $s > r > 0, f \in L_1([a,b])$. Then $D^r I^s f(t) = I^{s-r} f(t), t \in [a,b]$.

To study the problem (1), we need the following two lemmas [14]:

Lemma 2.5. For $\alpha > 0$, the general solution of the fractional differential equation $D^\alpha x(t) = 0$ is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.3}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1, n = [\alpha] + 1$.

Lemma 2.6. Let $\alpha > 0$. Then

$$J^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.4}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

We need also the following auxiliary result:

Lemma 2.7. *Let $g \in C([0,1])$, the solution of the equation*

$$D^{\alpha_0} x(t) = g(t), t \in J, n-1 < \alpha_0 < n, \tag{2.5}$$

subject to the boundary condition

$$x(0) = x^*, x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0,$$

and

$$J^\beta x(1) = \lambda J^\beta x(\eta),$$

is given by:

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0 - 1} g(s)\, ds + x^* \\ &- \frac{\Gamma(\beta+n) t^{n-1}}{(1-\lambda\eta^{\beta+n-1}) \Gamma(n) \Gamma(\alpha_0+\beta)} \int_0^1 (1-s)^{\beta+\alpha_0-1} g(s)\, ds \\ &+ \frac{\lambda \Gamma(\beta+n) t^{n-1}}{(1-\lambda\eta^{\beta+n-1}) \Gamma(n) \Gamma(\alpha_0+\beta)} \int_0^\eta (\eta-s)^{\beta+\alpha_0-1} g(s)\, ds \\ &+ \frac{x^* \left(\lambda\eta^\beta - 1\right) \Gamma(\beta+n) t^{n-1}}{(1-\lambda\eta^{\beta+n-1}) \Gamma(n) \Gamma(\beta+1)}. \end{aligned} \tag{2.6}$$

Proof . By lemmas 2.5 and 2.6, the general solution of (6) is given by

$$x(t) = \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} g(s)\, ds - c_0 - c_1 t - c_2 t^2 - .... - c_{n-1} t^{n-1}. \tag{2.7}$$

By $x(0) = x^*$, and $x'(0) = ... = x^{(n-2)}(0) = 0$, we can obtain $c_0 = -x^*$, and $c_1 = c_2 = ... = c_{n-2} = 0$.

Thanks to lemma 2.3, we get

$$J^\beta x(t) = \frac{1}{\Gamma(\beta+\alpha_0)} \int_0^t (t-s)^{\beta+\alpha_0-1} g(s)\, ds + \frac{x^* t^\beta}{\Gamma(\beta+1)} - c_{n-1} \frac{\Gamma(n)}{\Gamma(\beta+n)} t^{\beta+n-1}. \tag{2.8}$$

Using the boundary condition

$$J^{\beta}x(1) = \lambda J^{\beta}x(\eta),$$

we get

$$\begin{aligned} c_{n-1} = {} & \frac{\Gamma(\beta+n)}{(1-\lambda\eta^{\beta+n-1})\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^1 (1-s)^{\beta+\alpha_0-1} g(s)\,ds \\ & - \frac{\lambda\Gamma(\beta+n)}{(1-\lambda\eta^{\beta+n-1})\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^{\eta} (\eta-s)^{\beta+\alpha_0-1} g(s)\,ds \\ & - \frac{x^*\left(\lambda\eta^{\beta}-1\right)\Gamma(\beta+n)}{(1-\lambda\eta^{\beta+n-1})\Gamma(n)\Gamma(\beta+1)}. \end{aligned} \tag{2.9}$$

Substituting the values of $c_0, c_1, c_2, ..., c_{n-2}, c_{n-1}$ in (2.7), we obtain the desired quantity in lemma.
□

3. Main Results

For convenience, we set:

$$\begin{aligned} L_0 &:= \tfrac{1}{\Gamma(\alpha_0+1)} + \tfrac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_0}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)}, \\ L_k &:= \tfrac{1}{\Gamma(\alpha_0-\alpha_k+1)} + \tfrac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_0}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}, k = 1, ..., n-1, \\ \omega &:= \omega_0 + \omega_1 + ... + \omega_{n-1}, \\ \theta &:= \tfrac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_k}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)} + \sum_{k=1}^{n-1} \tfrac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_k}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}, \\ \theta' &= \tfrac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)} + \sum_{k=1}^{n-1} \tfrac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}. \end{aligned} \tag{3.1}$$

Now list the following hypotheses for convenience:

$(H1)$: The function $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$, is continuous.

$(H2)$: There exist non negative continuous functions $a_i \in C([0,1])$, $i = 0, 1, 2, ..., n-1$ such that for all $t \in [0,1]$ and $(x_0, x_1, x_2, ..., x_{n-1}), (y_0, y_1, y_2, ..., y_{n-1}) \in \mathbb{R}^n$, we have

$$\begin{aligned} & |f(t, x_0, x_1, x_2, ..., x_{n-1}) - f(t, y_0, y_1, y_2, ..., y_{n-1})| \\ & \leq a_0(t)|x_0-y_0| + a_1(t)|x_1-y_1| + a_2(t)|x_2-y_2| + ... + a_{n-1}(t)|x_{n-1}-y_{n-1}|. \end{aligned}$$

where,

$$\omega_0 = \sup_{t\in J} a_0(t), \omega_1 = \sup_{t\in J} a_1(t), \omega_2 = \sup_{t\in J} a_2(t), ..., \omega_{n-1} = \sup_{t\in J} a_{n-1}(t).$$

$(H3)$: There exist positive constants $m(t)$ such that

$$|f(t, x_0, x_1, x_2, ..., x_{n-1})| \leq m(t) \text{ for each } t \in J \text{ and all } (x_0, x_1, x_2, ..., x_{n-1}) \in \mathbb{R}^n.$$

with

$$M = \sup_{t \in J} m(t).$$

Our first result is based on Banach contraction principle:

Theorem 3.1. *Suppose that* $\eta^{\beta+n-1} \neq \frac{1}{\lambda}$, *and assume that the hypothesis* $(H2)$*holds. If*

$$\left(L_0 + \sum_{k=1}^{n-1} L_k \right) \omega < 1, \tag{3.2}$$

then the fractional problem (1.1)*has a unique solution.*

Proof . Consider the operator $\phi : X \to X$ defined by:

$$\begin{aligned}
\phi x(t) := & \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0 - 1} f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right) ds + x^* \\
& - \frac{\Gamma(\beta+n) t^{n-1}}{\left(1-\lambda\eta^{\beta+n-1}\right)\Gamma(n)\Gamma(\beta+\alpha_0)} \int_0^1 (1-s)^{\beta+\alpha_0-1} f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right) ds \\
& + \frac{\lambda\Gamma(\beta+n) t^{n-1}}{\left(1-\lambda\eta^{\beta+n-1}\right)\Gamma(n)\Gamma(\beta+\alpha_0)} \int_0^\eta (\eta-s)^{\beta+\alpha_0-1} f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right) ds \\
& + \frac{x^*\left(\lambda\eta^{\beta}-1\right)\Gamma(\beta+n) t^{n-1}}{\left(1-\lambda\eta^{\beta+n-1}\right)\Gamma(n)\Gamma(\beta+1)}.
\end{aligned} \tag{3.3}$$

We shall prove that ϕ is contraction mapping :
For $x, y \in X$ and for each $t \in J$, we have:

$$|\phi x(t) - \phi y(t)| \leq \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} \left| \begin{array}{c} f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right) \\ -f\left(s, y(s), D^{\alpha_1} y(s), ..., D^{\alpha_{n-1}} y(s)\right) \end{array} \right| ds \tag{3.4}$$

$$+\frac{\Gamma(\beta+n) t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)} \int_0^1 (1-s)^{\beta+\alpha_0-1} \left| \begin{array}{c} f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right) \\ -f\left(s, y(s), D^{\alpha_1} y(s), ..., D^{\alpha_{n-1}} y(s)\right) \end{array} \right| ds$$

$$+\frac{|\lambda|\Gamma(\beta+n) t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)} \int_0^\eta (\eta-s)^{\beta+\alpha_0-1} \left| \begin{array}{c} f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right) \\ -f\left(s, y(s), D^{\alpha_1} y(s), ..., D^{\alpha_{n-1}} y(s)\right) \end{array} \right| ds.$$

Using the $(H2)$, we can write:

$$\begin{aligned}
|\phi x(t) - \phi y(t)| \leq & \frac{(\omega_0+\omega_1+...+\omega_{n-1})[\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|]}{\Gamma(\alpha_0+1)} \\
& + \frac{\Gamma(\beta+n)(\omega_0+\omega_1+...+\omega_{n-1})[\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|]}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)} \\
& + \frac{|\lambda|\Gamma(\beta+n)\eta^{\beta+\alpha_0}(\omega_0+\omega_1+...+\omega_{n-1})[\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|]}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)}.
\end{aligned} \tag{3.5}$$

Thus,

$$\begin{aligned}
|\phi x(t) - \phi y(t)| \leq & \frac{(\omega_0+\omega_1+...+\omega_{n-1})[\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|]}{\Gamma(\alpha_0+1)} \\
& + \frac{\Gamma(\beta+n)\left[1+|\lambda|\eta^{\beta+\alpha_0}\right](\omega_0+\omega_1+...+\omega_{n-1})[\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|]}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\alpha_0+\beta+1)}
\end{aligned} \tag{3.6}$$

Consequently,we have

$$|\phi x(t)-\phi y(t)| \leq L_0\omega\left(\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|\right), \tag{3.7}$$

which implies that

$$\|\phi(x)-\phi(y)\| \leq L_0\omega\left(\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|\right). \tag{3.8}$$

On the other hand, for all $k=1,...,n-1$ and for each $t\in[0,1]$, we have

$$\begin{aligned}|D^{\alpha_k}\phi x(t)-D^{\alpha_k}\phi y(t)| &\leq \tfrac{1}{\Gamma(\alpha_0-\alpha_k)}\int_0^t (t-s)^{\alpha_0-\alpha_k-1}\left|\begin{array}{c} f(s,x(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}x(s)) \\ -f(s,y(s),D^{\alpha_1}y(s),...,D^{\alpha_{n-1}}y(s))\end{array}\right| ds \\ &+ \tfrac{\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0)}\int_0^1 (1-s)^{\beta+\alpha_0-1}\left|\begin{array}{c} f(s,x(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}x(s)) \\ -f(s,y(s),D^{\alpha_1}y(s),...,D^{\alpha_{n-1}}y(s))\end{array}\right| ds \\ &+ \tfrac{|\lambda|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0)}\int_0^\eta (\eta-s)^{\beta+\alpha_0-1}\left|\begin{array}{c} f(s,x(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}x(s)) \\ -f(s,y(s),D^{\alpha_1}y(s),...,D^{\alpha_{n-1}}y(s))\end{array}\right| ds.\end{aligned} \tag{3.9}$$

By $(H2)$ we obtain

$$\begin{aligned}|D^{\alpha_k}\phi x(t)-D^{\alpha_k}\phi y(t)| \leq & \tfrac{(\omega_0+\omega_1+...+\omega_{n-1})[\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|]}{\Gamma(\alpha_0-\alpha_k+1)} \\ &+ \tfrac{\Gamma(\beta+n)(\omega_0+\omega_1+...+\omega_{n-1})[\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|]}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)} \\ &+ \tfrac{|\lambda|\Gamma(\beta+n)\eta^{\beta+\alpha_0}(\omega_0+\omega_1+...+\omega_{n-1})[\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|]}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}.\end{aligned} \tag{3.10}$$

Hence, we have

$$\begin{aligned}|D^{\alpha_k}\phi_1 y(t)-D^{\alpha_k}\phi_1 y_1(t)| \leq & \tfrac{(\omega_0+\omega_1+...+\omega_{n-1})[\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|]}{\Gamma(\alpha_0-\alpha_k+1)} \\ &+ \tfrac{\Gamma(\beta+n)\left[1+|\lambda|\eta^{\beta+\alpha_0}\right](\omega_0+\omega_1+...+\omega_{n-1})[\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|]}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}.\end{aligned} \tag{3.11}$$

Therefore,

$$|D^{\alpha_k}\phi x(t)-D^{\alpha_k}\phi y(t)| \leq L_k\omega\left(\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|\right), \tag{3.12}$$

which implies that

$$\|D^{\alpha_k}\phi(x)-D^{\alpha_k}\phi(y)\| \leq L_k\omega\left(\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|\right). \tag{3.13}$$

Thanks to (3.8) and (3.13), we obtain

$$\|\phi(x)-\phi(y)\|_X \leq \left(L_0+\sum_{k=1}^{n-1}L_k\right)\omega\left(\|y-y_1\|+\left\|D^{\beta_1}y-D^{\beta_1}y_1\right\|+...+\left\|D^{\beta_{n-1}}y-D^{\beta_{n-1}}y_1\right\|\right). \tag{3.14}$$

Consequently by (3.2), we conclude that ϕ is contraction. As a consequence of Banach contraction, we deduce that ϕ has a fixed point which is a solution of the boundary value problem (1.1). $\square$

The second result is the following.

Theorem 3.2. *Suppose that for all* $\eta^{\beta+n-1} \neq \frac{1}{\lambda}$ *and assume that the hypotheses* $(H1)$ *and* $(H3)$ *are satisfied.*

Then, the problem (1.1) *has at least a solution on* J.

Proof . We shall use Scheafer's fixed point theorem to prove that ϕ has at least a fixed point on X.

Step1: ϕ is continuous on X : By $(H1)$ we conclude that the operator ϕ is continuous.

Step2: The operator ϕ maps bounded sets into bounded sets in X : For $\sigma > 0$, we take $x \in B_\sigma = \{x \in X; \|x\|_X \leq \sigma\}$. For each $t \in J$, we have:

$$\begin{aligned} |\phi x(t)| \leq & \tfrac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0 - 1} \left| f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right)\right| ds + |x^*| \\ & + \tfrac{\Gamma(\beta+n) t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right| \Gamma(n)\Gamma(\beta+\alpha_0)} \int_0^1 (1-s)^{\beta+\alpha_0-1} \left| f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right)\right| ds \\ & + \tfrac{|\lambda|\Gamma(\beta+n) t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right| \Gamma(n)\Gamma(\beta+\alpha_0)} \int_0^\eta (\eta-s)^{\beta+\alpha_0-1} \left| f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right)\right| ds \\ & \tfrac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n) t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}. \end{aligned} \tag{3.15}$$

Using the $(H3)$, we obtain

$$|\phi x(t)| \leq \sup_{t\in J} m(t) \left[\tfrac{1}{\Gamma(\alpha_0+1)} + \tfrac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_0}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)} \right] + |x^*| + \tfrac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}. \tag{3.16}$$

Thus,

$$|\phi x(t)| \leq L_0 \sup_{t\in J} m(t) + |x^*| + \tfrac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}, t \in J, \tag{3.17}$$

which implies that

$$\|\phi(x)\| \leq ML_0 + |x^*| + \tfrac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}, \tag{3.18}$$

On the other hand, for all $k = 1, 2, ..., n-1$, we have

$$\begin{aligned} |D^{\alpha_k}\phi x(t)| \leq & \tfrac{1}{\Gamma(\alpha_0-\alpha_k)} \int_0^t (t-s)^{\alpha_0-\alpha_k-1} \left| f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right)\right| ds \\ & + \tfrac{\Gamma(\beta+n) t^{n-\alpha_k-1}}{\left|1-\lambda\eta^{\beta+n-1}\right| \Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0)} \int_0^1 (1-s)^{\beta+\alpha_0-1} \left| f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right)\right| ds \\ & + \tfrac{|\lambda|\Gamma(\beta+n) t^{n-\alpha_k-1}}{\left|1-\lambda\eta^{\beta+n-1}\right| \Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0)} \int_0^\eta (\eta-s)^{\beta+\alpha_0-1} \left| f\left(s, x(s), D^{\alpha_1} x(s), ..., D^{\alpha_{n-1}} x(s)\right)\right| ds \\ & \tfrac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n) t^{n-\alpha_k-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}. \end{aligned} \tag{3.19}$$

By $(H3)$ we have,

$$\|D^{\alpha_k}\phi(x)\| \leq M \left[\tfrac{1}{\Gamma(\alpha_0-\alpha_k+1)} + \tfrac{\Gamma(\beta+n)\left(1+L|\lambda|\eta^{\beta+\alpha_0}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)} \right] + \tfrac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}. \tag{3.20}$$

Hence,

$$\|D^{\alpha_k}\phi(x)\| \leq ML_k + \tfrac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}. \tag{3.21}$$

Combining (3.18) and (3.21) , yields

$$\|\phi(x)\|_X \leq M\left(L_0+\sum_{k=1}^{n-1} L_k\right)+|x^*|+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}. \tag{3.22}$$

Consequently

$$\|\phi(x)\|_X < \infty. \tag{3.23}$$

Step3: In the end we show that ϕ is equicontinuous on J:
Let us take $(x,y)\in B_\sigma$, $t_1,t_2\in J$, such that $t_1<t_2$ and Thanks to $(H3)$. We can write:

$$\begin{aligned}|\phi x(t_2)-\phi x(t_1)| \leq{}& \frac{\sup_{t\in J} m(t)}{\Gamma(\alpha_0)}\int_0^{t_1}\left[(t_2-s)^{\alpha_0-1}-(t_1-s)^{\alpha_0-1}\right]ds+\frac{\sup_{t\in J} m(t)}{\Gamma(\alpha_0)}\int_{t_1}^{t_2}(t_2-s)^{\alpha_0-1}ds \\ &+\frac{\sup_{t\in J} m(t)\Gamma(\beta+n)\left(t_1^{n-1}-t_2^{n-1}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^1(1-s)^{\beta+\alpha_0-1}ds \\ &+\frac{\sup_{t\in J} m(t)|\lambda|\Gamma(\beta+n)\left(t_2^{n-1}-t_1^{n-1}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^\eta(\eta-s)^{\beta+\alpha_0-1}ds \\ &\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)\left(t_2^{n-1}-t_1^{n-1}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}.\end{aligned} \tag{3.24}$$

Thus,

$$\begin{aligned}|\phi x(t_2)-\phi x(t_1)| \leq{}& \frac{M}{\Gamma(\alpha_0+1)}\left(t_1^{\alpha_0}-t_2^{\alpha_0}\right)+\frac{2M}{\Gamma(\alpha_0+1)}(t_2-t_1)^{\alpha_0} \\ &+\frac{M\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)}\left(t_1^{n-1}-t_2^{n-1}\right) \\ &+\left[\frac{M|\lambda|\Gamma(\beta+n)\eta^{\beta+\alpha_0}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)}+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}\right]\left(t_2^{n-1}-t_1^{n-1}\right),\end{aligned} \tag{3.25}$$

on the other hand, for all $k=1,2...,n-1$,

$$\begin{aligned}\left|D^k\phi_1 y(t_2)-D^k\phi_1 y(t_1)\right| \leq{}& \frac{M}{\Gamma(\alpha_0-\alpha_k+1)}\left(t_1^{\alpha_0-\alpha_k}-t_2^{\alpha_0-\alpha_k}\right)+\frac{2M}{\Gamma(\alpha_0-\alpha_k+1)}(t_2-t_1)^{\alpha_0-\alpha_k} \\ &+\frac{M\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}\left(t_1^{n-\alpha_k-1}-t_2^{n-\alpha_k-1}\right) \\ &+\frac{M|\lambda|\Gamma(\beta+n)\eta^{\beta+\alpha_0}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}\left(t_2^{n-\alpha_k-1}-t_1^{n-\alpha_k-1}\right) \\ &+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}\left(t_2^{n-\alpha_k-1}-t_1^{n-\alpha_k-1}\right).\end{aligned} \tag{3.26}$$

Thanks to (3.25) and (3.26), we can state that $\|\phi(x)(t_2)-\phi(x)(t_1)\|\to 0$ as $t_2\to t_1$. By Arzela-Ascoli theorem, we conclude that ϕ is completely continuous operator.

Step4: Finally, we discuss a priory bounds on solutions.
We shall show that the set Ω defined by

$$\Omega=\{x\in X, x=\sigma\phi(x), 0<\sigma<1\}, \tag{3.27}$$

is bounded:
Let $x\in\Omega$, then $x=\sigma\phi(x)$, for some $0<\sigma<1$. Thus, for each $t\in J$, we have:

$$x(t)=\sigma\phi(x). \tag{3.28}$$

Then

$$\frac{1}{\sigma}\left|x\left(t\right)\right| \leq \frac{1}{\Gamma(\alpha_0)}\int_0^t \left(t-s\right)^{\alpha_0-1}\left|f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}x\left(s\right)\right)\right|ds+\left|x^*\right| \tag{3.29}$$
$$+\frac{\Gamma(\beta+n)t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^1 \left(1-s\right)^{\beta+\alpha_0-1}\left|f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}x\left(s\right)\right)\right|ds$$
$$+\frac{|\lambda|\Gamma(\beta+n)t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^\eta \left(\eta-s\right)^{\beta+\alpha_0-1}\left|f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}x\left(s\right)\right)\right|ds$$
$$+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}.$$

Thanks to $(H3)$, we can write

$$\frac{1}{\sigma}\left|x\left(t\right)\right| \leq \sup_{t\in J} m\left(t\right)\left[\frac{1}{\Gamma(\alpha_0+1)}+\frac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_0}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)}\right]+\left|x^*\right|+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}. \tag{3.30}$$

Therefore,

$$\left|x\left(t\right)\right| \leq \sigma\sup_{t\in J} m\left(t\right)\left[\frac{1}{\Gamma(\alpha_0+1)}+\frac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_0}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)}\right]+\left|x^*\right|+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}. \tag{3.31}$$

Hence,

$$\left|x\left(t\right)\right| \leq \sigma L_0\sup_{t\in J} m\left(t\right)+\left|x^*\right|+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}, t\in J, \tag{3.32}$$

which implies that,

$$\|x\| \leq \sigma M L_0+\left|x^*\right|+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}. \tag{3.33}$$

On the other hand, for all $k, = 1, 2, ..., n-1$, we have

$$\left|D^{\alpha_k}\phi x\left(t\right)\right| \leq \sigma\sup_{t\in J} m\left(t\right)\left[\frac{1}{\Gamma(\alpha_0-\alpha_k+1)}+\frac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_0}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}\right]+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}, t\in J. \tag{3.34}$$

Thus,

$$\left\|D^{\alpha_k}\phi\left(x\right)\right\| \leq \sigma M L_k+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}. \tag{3.35}$$

From (3.33) and (3.35) we get,

$$\|x\|_X \leq \sigma M\left(L_0+\sum_{k=1}^{n-1}L_k\right)+\left|x^*\right|+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)[\Gamma(n-\alpha_k)+\Gamma(n)]}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(n-\alpha_k)\Gamma(\beta+1)}. \tag{3.36}$$

Hence

$$\left\|\phi\left(x\right)\right\|_X < \infty. \tag{3.37}$$

This shows that the set Ω is bounded.

As consequence of Schaefer's fixed point theorem, we deduce that ϕ at least a fixed point, which is a solution of the fractional differential problem (1.1). □

Our third main result is based on Krasnoselskii theorem [14]. We have:

Theorem 3.3. *Let $\eta^{\beta+n-1}\neq\frac{1}{\lambda}$. Suppose that $(H1)$, $(H2)$ and $(H3)$ are satisfied, such that*

$$\left(\frac{1}{\Gamma(\alpha_0+1)}+\sum_{k=1}^{n-1}\frac{1}{\Gamma(\alpha_0-\alpha_k+1)}\right)\omega<1, \tag{3.38}$$

if there exist $\delta\in\mathbb{R}$ such that

$$M\left(L_0+\sum_{k=1}^{n-1}L_k\right)+|x^*|+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)[\Gamma(n-\alpha_k)+\Gamma(n)]}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(n-\alpha_k)\Gamma(\beta+1)}\leq\delta. \tag{3.39}$$

Then, the fractional problem (1.1) *has at least one solution on* $[0,1]$.

Proof . We shall use Krasnoselskii fixed point theorem to prove that ϕ has at least a fixed point on X.

Suppose that $M\left(L_0+\sum_{k=1}^{n-1}L_k\right)+|x^*|+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)[\Gamma(n-\alpha_k)+\Gamma(n)]}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(n-\alpha_k)\Gamma(\beta+1)}\leq\delta$ and let us take

$$\phi\left(x\right)\left(t\right):=T\left(x\right)\left(t\right)+R\left(x\right)\left(t\right), \tag{3.40}$$

where

$$Tx\left(t\right):=\frac{1}{\Gamma(\alpha_0)}\int_0^t\left(t-s\right)^{\alpha_0-1}f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}\left(s\right)\right)ds+x^*, \tag{3.41}$$

and

$$\begin{aligned}Rx\left(t\right):=&-\frac{\Gamma(\beta+n)t^{n-1}}{\left(1-\lambda\eta^{\beta+n-1}\right)\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^1\left(1-s\right)^{\beta+\alpha_0-1}f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}\left(s\right)\right)ds\\&+\frac{\lambda\Gamma(\beta+n)t^{n-1}}{\left(1-\lambda\eta^{\beta+n-1}\right)\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^\eta\left(\eta-s\right)^{\beta+\alpha_0-1}f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}x\left(s\right)\right)ds\\&+\frac{x^*\left(\lambda\eta^{\beta}-1\right)\Gamma(\beta+n)t^{n-1}}{\left(1-\lambda\eta^{\beta+n-1}\right)\Gamma(n)\Gamma(\beta+1)}.\end{aligned} \tag{3.42}$$

The proof will be given in several steps.

$(1^*:)$ We shall prove that for any $x,y\in B_\delta$, then $T\left(x\right)\left(t\right)+R\left(y\right)\left(t\right)\in B_\delta$, such that $B_\delta=\left\{x\in X;\|x\|_X\leq\delta\right\}$.

For any $x,y\in B_\delta$ and for each $t\in J$ we have:

$$\begin{aligned}\left|Tx\left(t\right)+Ry\left(t\right)\right|\leq&\frac{1}{\Gamma(\alpha_0)}\int_0^t\left(t-s\right)^{\alpha_0-1}\left|f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}\left(s\right)\right)\right|ds+|x^*|\\&+\frac{\Gamma(\beta+n)t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^1\left(1-s\right)^{\beta+\alpha_0-1}\left|f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}\left(s\right)\right)\right|ds\\&+\frac{|\lambda|\Gamma(\beta+n)t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^\eta\left(\eta-s\right)^{\beta+\alpha_0-1}\left|f\left(s,x\left(s\right),D^{\alpha_1}x\left(s\right),...,D^{\alpha_{n-1}}\left(s\right)\right)\right|ds\\&\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}.\end{aligned} \tag{3.43}$$

Using the $(H3)$, we obtain

$$\begin{aligned}\left|Tx\left(t\right)+Ry\left(t\right)\right|\leq&\sup_{t\in J}m\left(t\right)\left[\frac{1}{\Gamma(\alpha_0+1)}+\frac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_0}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)}\right]\\&+|x^*|+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}.\end{aligned} \tag{3.44}$$

Consequently,

$$|Tx(t) + Ry(t)| \leq L_0 \sup_{t\in J} m(t) + |x^*| + \frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}, t \in J, \tag{3.45}$$

Thus,

$$\|T(x) + R(y)\| \leq ML_0 + |x^*| + \frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}. \tag{3.46}$$

On the other hand, for all $k = 1, 2, ..., n-1$, we have

$$|D^{\alpha_k}Tx(t) + D^{\alpha_k}Ry(t)| \leq \sup_{t\in J} m(t) \left[\frac{1}{\Gamma(\alpha_0-\alpha_k+1)} + \frac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_0}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}\right] + \frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}, \tag{3.47}$$

Thus,

$$\|D^{\alpha_k}Tx(t) + D^{\alpha_k}Ry(t)\| \leq ML_k + \frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}. \tag{3.48}$$

Combining (3.58) and (3.60) yields

$$\|T(x) + R(y)\|_X \leq M\left(L_0 + \sum_{k=1}^{n-1} L_k\right) + |x^*| + \frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)[\Gamma(n-\alpha_k)+\Gamma(n)]}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(n-\alpha_k)\Gamma(\beta+1)}. \tag{3.49}$$

$$\|T(x) + R(y)\|_X \in B_\delta. \tag{3.50}$$

(2^* :) We shall prove that R is continuous and compact. Note that R is continuous on X in view of the continuity of f (hypothesis ($H1$)).

(a^*) : Now, we prove that R maps bounded sets into bounded sets of X.

For $x \in B_\delta$ and for each $t \in J$, we have:

$$\begin{aligned}|Rx(t)| \leq{}& \frac{\Gamma(\beta+n)t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)} \int_0^1 (1-s)^{\beta+\alpha_0-1} \left|f\left(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}(s)\right)\right| ds \\ &+ \frac{|\lambda|\Gamma(\beta+n)t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)} \int_0^\eta (\eta-s)^{\beta+\alpha_0-1} \left|f\left(s, x(s), D^{\alpha_1}x(s), ..., D^{\alpha_{n-1}}x(s)\right)\right| ds \\ &+ \frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)t^{n-1}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}.\end{aligned} \tag{3.51}$$

Using the ($H3$), we obtain

$$|Rx(t)| \leq \frac{\sup_{t\in J} m(t)\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_k}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)} + \frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}, t \in J, \tag{3.52}$$

Thus,

$$\|R(x)\| \leq \frac{M\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_k}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)} + \frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}. \tag{3.53}$$

On the other hand, for all $k = 1, 2, ..., n-1$, we have

$$\|D^{\alpha_k}R(x)\| \leq \frac{M\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_k}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)} + \frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}. \tag{3.54}$$

From (3.53) and (3.54) , we have

$$\|R(x)\|_X \leq M\left(\frac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_k}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)}+\sum_{k=1}^{n-1}\frac{\Gamma(\beta+n)\left(1+|\lambda|\eta^{\beta+\alpha_k}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}\right)$$
$$+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}+\sum_{k=1}^{n-1}\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}.$$

Consequently,

$$\|R(x)\|_X \leq M\theta+\theta' < \infty. \tag{3.55}$$

(b^*) : In the end we show that R is equicontinuous on J :
Let $t_1, t_2 \in J$, such that $t_2 < t_1$ and $(x, y) \in B_\delta$, Then, we have:

$$\begin{aligned}|Rx(t_1)-Rx(t_2)| \leq{}& \frac{\Gamma(\beta+n)\left(t_2^{n-1}-t_1^{n-1}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^1(1-s)^{\beta+\alpha_0-1}\left|f\left(s,x(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}(s)\right)\right|ds\\
&+\frac{|\lambda|\Gamma(\beta+n)\left(t_1^{n-1}-t_2^{n-1}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0)}\int_0^\eta(\eta-s)^{\beta+\alpha_0-1}\left|f\left(s,x(s),D^{\alpha_1}x(s),...,D^{\alpha_{n-1}}x(s)\right)\right|ds\\
&+\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)\left(t_1^{n-1}-t_2^{n-1}\right)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}.\end{aligned} \tag{3.56}$$

Using the $(H3)$, we obtain

$$\begin{aligned}|Rx(t_1)-Rx(t_2)| \leq{}& \frac{M\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)}\left(t_2^{n-1}-t_1^{n-1}\right)\\
&+\frac{M|\lambda|\Gamma(\beta+n)\eta^{\beta+\alpha_0}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+\alpha_0+1)}+\left(t_1^{n-1}-t_2^{n-1}\right)\\
&\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n)\Gamma(\beta+1)}\left(t_1^{n-1}-t_2^{n-1}\right).\end{aligned} \tag{3.57}$$

On the other hand, for all $k = 1, 2..., n-1$,

$$\begin{aligned}|D^{\alpha_k}Rx(t_1)-D^{\alpha_k}Rx(t_2)| \leq{}& \frac{M\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}\left(t_2^{n-1}-t_1^{n-1}\right)\\
&+\frac{M|\lambda|\Gamma(\beta+n)\eta^{\beta+\alpha_0}}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+\alpha_0+1)}\left(t_1^{n-\alpha_k-1}-t_2^{n-\alpha_k-1}\right)\\
&\frac{|x^*|\left|\lambda\eta^{\beta}-1\right|\Gamma(\beta+n)}{\left|1-\lambda\eta^{\beta+n-1}\right|\Gamma(n-\alpha_k)\Gamma(\beta+1)}\left(t_1^{n-\alpha_k-1}-t_2^{n-\alpha_k-1}\right).\end{aligned} \tag{3.58}$$

As $t_2 \to t_1$, the right-hand sides of the inequalities (3.57) and (3.58) tend to zero. Then, as a consequence of the steps (a^*) and (b^*) and by Arzela-Ascoli theorem, we
conclude that R is completely continuous.
$(3^* :)$ Finally, we prove that T is contraction mapping
Let $x, y \in X$. Then, for each $t \in J$ and by $(H2)$ we have

$$|Tx(t)-Ty(t)| \leq \frac{(\omega_0+\omega_1+...+\omega_{n-1})}{\Gamma(\alpha_0+1)}\left(\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|\right). \tag{3.59}$$

On the other hand, for all $k = 1, 2, ..., n-1$, we have

$$|D^{\alpha_k}Tx(t)-D^{\alpha_k}Ty(t)| \leq \frac{(\omega_0+\omega_1+...+\omega_{n-1})}{\Gamma(\alpha_0-\alpha_k+1)}\left(\|x-y\|+\|D^{\alpha_1}x-D^{\alpha_1}y\|+...+\|D^{\alpha_{n-1}}x-D^{\alpha_{n-1}}y\|\right). \tag{3.60}$$

By (3.59) and (3.60), we obtain

$$|Tx(t) - Ty(t)| \leq \left[\frac{1}{\Gamma(\alpha_0+1)} + \sum_{k=1}^{n-1} \frac{1}{\Gamma(\alpha_0-\alpha_k+1)}\right] \omega\left(\|x-y\| + \|D^{\alpha_1}x - D^{\alpha_1}y\| + ... + \|D^{\alpha_{n-1}}x - D^{\alpha_{n-1}}y\|\right). \tag{3.61}$$

Using the condition (3.51) we conclude that T is a contraction mapping.

As a consequence of Krasnoselskii's fixed point theorem we deduce that ϕ has a fixed point which is a solution of (1.1). □

Corollary 3.4. Assume that $\eta^{\beta+n-1} \neq \frac{1}{\lambda}$ and there exist non negative real numbers $\theta_i, i = 0, 1, ..., n-1$ such that for all $t \in [0,1]$ and $(x_0, x_1, x_2, ..., x_{n-1}), (y_0, y_1, y_2, ..., y_{n-1}) \in \mathbb{R}^n$, we have

$$|f_1(t, x_0, x_1, x_2, ..., x_{n-1}) - f_1(t, y_0, y_1, y_2, ..., y_{n-1})| \leq \theta_0 |x_0 - y_0| + ... + \theta_{n-1}|x_{n-1} - y_{n-1}|,$$

If

$$\left(L_0 + \sum_{k=1}^{n-1} L_k\right)(\theta_0 + ... + \theta_{n-1}) < 1, \tag{3.62}$$

then the fractional problem (1.1) has a unique solution on J.

Corollary 3.5. Assume that $(H1)$ holds and $\eta^{\beta+n-1} \neq \frac{1}{\lambda}$. If There exist positive constants k_1 and k_2 such that

$f_1 \leq k_1, f_2 \leq k_2$ on $J \times \mathbb{R}^n$, then, the problem (1.1) has at least a solution on J.

Example 3.6. Consider the following fractional differential problem:

$$\begin{cases} D^{\frac{7}{2}}x(t) = \frac{|x(t)| + \left|D^{\frac{9}{4}}x(t)\right| + \left|D^{\frac{3}{2}}x(t)\right| + \left|D^{\frac{1}{2}}x(t)\right|}{(t^2+32\pi)\left(e^t + |x(t)| + \left|D^{\frac{9}{4}}x(t)\right| + \left|D^{\frac{3}{2}}x(t)\right| + \left|D^{\frac{1}{2}}x(t)\right|\right)} + \cosh(2+t^2), t \in [0,1], \\ x(0) = \sqrt{2}, x'(0) = x''(0) = 0, J^{\frac{1}{2}}x(1) = \frac{4}{5}J^{\frac{1}{2}}x\left(\frac{3}{4}\right), \end{cases} \tag{3.63}$$

For this example, we have

$$f(t, x_1, x_2, x_3, x_4) = \frac{|x_1| + |x_2| + |x_3| + |x_4|}{(t^2+32\pi)(e^t + |x_1| + |x_2| + |x_3| + |x_4|)} + \cosh\left(2+t^2\right), t \in [0,1], x_1, x_2, x_3, x_4 \in \mathbb{R}.$$

For $t \in [0,1]$ and $(x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, we have:

$$|f(t, x_0, x_1, x_2, x_3) - f(t, y_0, y_1, y_2, y_3)| \leq \frac{1}{(t^2+32\pi)}(|x_0-y_0| + |x_1-y_1| + |x_2-y_2| + |x_3-y_3|),$$

So,we have

$$a_i(t) = \frac{1}{(t^2+32\pi)}, i = 0, ..., 3.$$

Then,

$$\omega_i = \sup_{t \in J} a_i(t) = \frac{1}{32\pi}, i = 0, ..., 3,$$

and then,

$$L_0 = 0,228998, L_1 = 2,203434, L_2 = 1,145589, L_3 = 0,424889, \omega = \frac{1}{8\pi},$$

and

$$\left(L_0+\sum_{k=1}^{n-1}L_k\right)\omega=1,799476.0,039808=0,071633<1.$$

Hence by Theorem 3.1 then the problem (3.63) has a unique solution on $[0,1]$.

Example 3.7. Let us consider the following fractional boundary value problem:

$$\begin{cases} D^{\frac{14}{4}}x\left(t\right)=\frac{\sin x(t)+\cos\left(D^{\frac{5}{2}}x(t)+D^{\frac{4}{3}}x(t)\right)}{\left(5\sqrt{\pi}+e^{t^2}\right)^2}, t\in\left[0,1\right], \\ x\left(0\right)=3, x'\left(0\right)=x''\left(0\right)=0, D^{\frac{4}{5}}x\left(1\right)=\frac{3}{4}D^{\frac{4}{5}}x\left(\frac{1}{3}\right). \end{cases} \tag{3.64}$$

Then, we have

$$f(t,x,y,z)=\frac{\sin\left(x\right)+\cos\left(y+z\right)}{\left(5\sqrt{\pi}+e^{t^2}\right)^2}, t\in\left[0,1\right], (x,y,z)\in\mathbb{R}^3.$$

Let $x,y,z\in\mathbb{R}$ and $t\in[0,1]$. Then

$$|f(t,x,y,z)|\leq\frac{2}{\left(5\sqrt{\pi}+e^{t^2}\right)^2}.$$

So, we can take

$$m\left(t\right)=\frac{2}{\left(5\sqrt{\pi}+e^{t^2}\right)^2}.$$

Then,

$$M=\frac{2}{\left(5\sqrt{\pi}+1\right)^2}.$$

Thanks to Theorem 3.2, the problem (3.64) has at least one solution on $[0,1]$.

Example 3.8. Our third example is the following:

$$\begin{cases} D^{\frac{17}{5}}x\left(t\right)=\frac{e^{-t^2}\left(e^{-|x(t)|}+\sin D^{\frac{15}{6}}x(t)+\sin^2\left(D^{\frac{11}{7}}x(t)\right)+\cos\left(D^{\frac{2}{3}}x(t)\right)\right)}{20\pi+e^{t^2}}+\ln\left(2+t^2\right), t\in\left[0,1\right], \\ x\left(0\right)=2, x'\left(0\right)=x''\left(0\right)=0, D^{\frac{5}{4}}x\left(1\right)=\frac{5}{6}D^{\frac{5}{4}}x\left(\frac{2}{5}\right), \end{cases} \tag{3.65}$$

where,

$$f\left(x_1,x_2,x_3,x_4\right)=\frac{e^{-t^2}\left(e^{-|x_1|}+\sin\left(x_2\right)+\sin^2\left(x_3\right)+\cos\left(x_4\right)\right)}{20\pi+e^{t^2}}+\ln\left(2+t^2\right), t\in\left[0,1\right], x_1,x_2,x_3,x_4\in\mathbb{R}.$$

Taking $x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3\in\mathbb{R}, t\in[0,1]$,then

$$\left|f\left(t,x_0,x_1,x_2,x_3\right)-f\left(t,y_0,y_1,y_2,y_3\right)\right|\leq\frac{e^{-t^2}\left(|x_0-y_0|+|x_1-y_1|+|x_2-y_2|+|x_3-y_3|\right)}{20\pi+e^{t^2}}.$$

So,we can take:

$$a_i\left(t\right)=\frac{e^{-t^2}}{20\pi+e^{t^2}}, i=0,1,...,3,$$

then,

$$\omega_i = \sup_{t\in[0,1]} a_i(t) = \frac{1}{20\pi+1}, i = 0, ..., 3.$$

For $k = 1, 2, 3$ we have

$$\tfrac{1}{\Gamma(\alpha_0+1)} + \sum_{k=1}^{n-1} \tfrac{1}{\Gamma(\alpha_0-\alpha_k+1)} = 1,951303.$$

We have also

$$\left(\tfrac{1}{\Gamma(\alpha_0+1)} + \sum_{k=1}^{n-1} \tfrac{1}{\Gamma(\alpha_0-\alpha_k+1)}\right)\omega = 0,1223387.$$

By Theorem 3.3, we can state that the problem (3.65) has at least one solution on $[0,1]$.

References

[1] B. Ahmad and S.k. Ntouyas, *A Boundary value problem of fractional differential inclusions with three-point nonlocal fractional boundary conditions*, Elect. J. Math. Anal. Appl. 1 (2013) 67–81.

[2] B. Ahmad, J. Nieto, A. Alsaedi and N. Mohamad, *On a New Class of Antiperiodic Fractional Boundary Value Problems,* Abst. Appl. Anal., Volume 2013, Article ID 606454, 7 pages, 2013.

[3] A. Bashir. N. Juan and J.A. Aalsaedi, *Existence and uniqueness of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions,* Acta Math. Scientia 31 (2011) 2122–2130.

[4] M.E. Bengrine and Z. Dahmani, *Boundary value problems for fractional differential equations*, Int. J. Open problems compt. Math. 5 2012.

[5] G. Chai, *Existence results of positive solutions for boundary value problems of fractional differential equations*, Boundary Value Problems (2013), 109, 2013.

[6] K. Diethelm and N.J. Ford, *Analysis of fractional differential equations,* J. Math. Anal. Appl. 265 (2002) 229–248.

[7] A.M.A. El-Sayed, *Nonlinear functional differential equations of arbitrary orders*, Nonlinear Anal. 33 (1998) 181–186.

[8] V. Gafiychuk, B. Datsko and V. Meleshko, *Mathematical modeling of time fractional reaction-diffusion systems,* J. Comput. Appl. Math. 220 (2008) 215–225.

[9] A. Guezane-Lakoud and R. Khaldi, *Solvability of a fractional boundary value problem with fractional integral condition*, Nonlinear Anal. 75 (2012) 2692–2700.

[10] R. Hilfer, *Applications of fractional calculus in physics,* World Scientific, Publishing Co., Inc., River Edge, NJ. 2000.

[11] M. Houas, Z. Dahmani and M. Benbachir, *New results for a boundary value problem for differential equations of arbitrary order*, Int. J. Modern Math. Sci. 7 (2013) 195–211.

[12] M. Houas and Z. Dahmani, *New results for a Caputo boundary value problem*, Amer. J. Comput. Appl. Math. 3 (2013) 143–161.

[13] M. Houas and Z. Dahmani, *New results for a coupled system of fractional differential equations*, Facta Universitatis, Series Mathematics and Informatics, Accepted paper, 2013.

[14] A.A. Kilbas and S.A. Marzan, *Nonlinear differential equation with the Caputo fraction derivative in the space of continuously differentiable functions,* Differ. Equ. 41 (2005) 84–89.

[15] V. Lakshmikantham and A.S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. 69 (2008) 2677–2682.

[16] Y. Li, S. Sun, D. Yang and Z. Han, *Three-point boundary value problems of fractional functional differential equations with delay,* (2013), 38, 2013.

[17] C. Li, X. Luo and Y. Zhou, *Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations,* Comput. Math. Appl. 59 (2010) 1363–1375.

[18] F. Mainardi, *Fractional calculus*, *Some basic problem in continuum and statistical mechanics*, *Fractals and fractional calculus in continuum mechanics*, Springer, Vienna, 1997.

[19] S.M. Momani and S.B. Hadid, *Some comparison results for integro- fractional differential inequalities*, J. Fract. Calc. 24 (2003) 37–44.

[20] J. Nieto and J. Pimente, *Positive solutions of a fractional thermostat model*, Boundary Value Problems. art. no. 5, 2013.

[21] S.K. Ntouyas, *Existence results for first order boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions*, J. Fract. Calculus Appl. 3 (2012) 1–14.

[22] M.U. Rehman, R.A Khan and N.A. Asif, *Three point boundary value problems for nonlinear fractional differential equations*, Acta Math. Scientia. 31B (2011) 1337–1346.

[23] G. Wang, R.P. Agarwal and A. Cabada, *Existence results and monotone iterative technique for systems of nonlinear fractional differential equations,* Appl. Math. Lett. 25 (2012) 1019–1024.

[24] S. Zhang, *Positive solutions to singular boundary value problem for nonlinear fractional differential equation,* Comput. Math. Appl. 59 (2010) 1300–1309.

[25] Y. Zhag and Z. Bai, *Existence of solutions for a nonlinear fractional boundary value problem at resonance,* J. Appl. Math. Comput. 36 (2011) 417–440.