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Probabilistic analysis of the asymmetric digital search trees

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Abstract

In this paper, by applying three functional operators the previous results on the (Poisson) variance of the external profile in digital search trees will be improved. We study the profile built over n binary strings generated by a memoryless source with unequal probabilities of symbols and use a combinatorial approach for studying the Poissonized variance, since the probability distribution of the profile is unknown.

Keywords: Digital Search Tree; Profile; Functional Operators; Poisson Variance.
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1. Introduction

Digital trees like tries or digital search trees are important in many computer science applications like data compression, pattern matching or hashing. For example, the popular Lempel-Ziv compression scheme is strongly related to digital search trees. Digital trees have been widely studied in the literature. The motivation for studying the profiles of such trees is multifold. Of course, digital trees are used in various applications. For example, the profile represents the number of phrases of length k in the Lempel-Ziv'78 built over n phrases. Second, the profile is a fine shape measure closely connected to many other cost measures (height, saturation level, depth, path length, etc.). And finally, the related analytical problems are mathematically challenging and lead to interesting distributional results [3].

Digital search trees are intended for the same kind of problems as binary search trees. However, they are not constructed from the total order structure of the keys for the data stored in the internal nodes of the tree but from digital representations (or binary sequences) which serve as keys.

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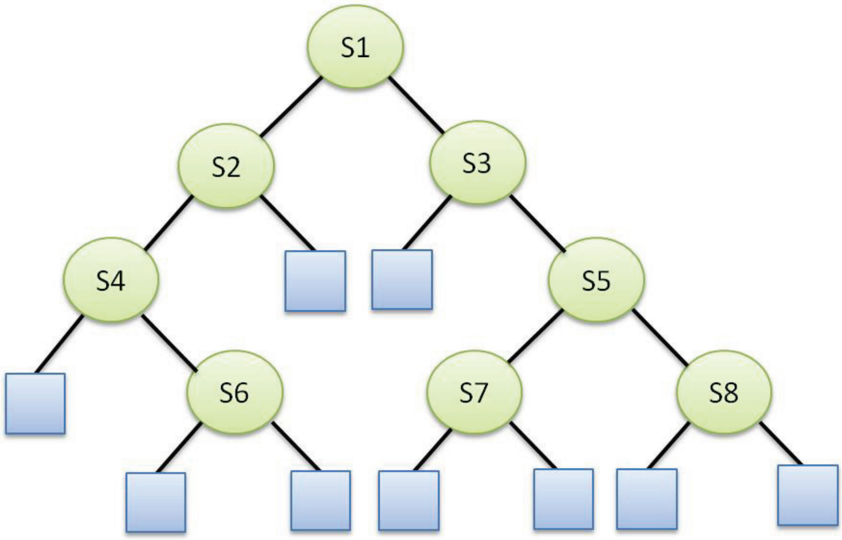

Figure 1: A digital search tree built on eight strings $s_1, \ldots, s_8$.

In a digital search tree strings are directly stored in internal nodes. More precisely, the root contains the first string and the next string occupies the right or the left child of the root depending on whether its first symbol is "0" or "1". The remaining strings are stored in available nodes which are directly attached to nodes already existing in the tree (external nodes). A digital search tree with n internal nodes is completed with $n+1$ external nodes. These external nodes can be seen as those positions where the next item can be stored. The resulting tree is then a complete binary tree with the external nodes as leaves. The search for an available node follows the prefix structure of a new string [2]. Figure 1 shows a digital search tree built on eight strings $s_1, ..., s_8$ (i.e., $s_1 = 0...$, $s_2 = 1...$, $s_3 = 01...$, $s_4 = 11...$, etc.) with internal (ovals) and external (squares) nodes, respectively.

In this paper, we are concerned with probabilistic properties of the external profile defined as the number of external nodes at the same distance from the root. It is a function of the number of strings stored in a tree and the distance from the root. We write $B_{n,k}$ for the external profile. Actually we study the external profile built over n binary strings generated by a *memoryless source* with unequal probabilities of symbols (asymmetric case), that is, we assume each string is a binary independently and identically distributed (i.i.d.) sequence with p being the probability of a "1" and $q = 1-p$ $(0 < p < q < 1)$.

Definition 1.1. Let X_n be a sequence of integer random variable and X_N its corresponding Poisson driven sequence, where N is a Poisson random variable with mean z. Let $G(z,u) = \mathbb{E}(u^{X_N}) = \sum_{n\geq 0} \mathbb{E}(u^{X_n})\frac{z^n}{n!}e^{-z}$ be its Poisson transform. The Poisson variance is introduced as

$$V(z) = G''_u(z,1) + G'_u(z,1) - \Big(G'_u(z,1)\Big)^2,$$

where $G'_u(z,1)$ and $G''_u(z,1)$ denote respectively the first and the second derivative of $G(z,u)$ with respect to u at $u = 1$.

2. The Previous Results

In the following, we first review the main equations and results described in [2]. Let $B_{n,k}$ denotes the (random) number of external nodes at level k in a digital search tree built over n strings generated by a memoryless source with parameter $q > p = 1 - q$. We recall the initial conditions

$$B_{n,0} = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \geq 1. \end{cases}$$

The probability generating function of the external profile, $P_{n,k}(u) = \mathbf{E}(u^{B_{n,k}})$, satisfies the following recurrence relation

$$P_{n+1,k}(u) = \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} P_{j,k-1}(u) P_{n-j,k-1}(u), \tag{2.1}$$

with initial conditions $P_{0,k}(u) = 1$ for $k \geq 1$, $P_{0,0}(u) = u$, $P_{n,0}(u) = 1$ for $n \geq 1$. The corresponding exponential generating function

$$G_k(x,u) = \sum_{n\geq 0} P_{n,k}(u) \frac{x^n}{n!}$$

fulfills the following functional recurrence

$$\frac{\partial}{\partial x} G_k(x,u) = G_{k-1}(px,u) G_{k-1}(qx,u), \qquad k \geq 1, \tag{2.2}$$

with initial conditions $G_0(x,u) = u + e^x - 1$ and $G_k(0,u) = 1$ $(k \geq 1)$. By taking derivatives with respect to u and setting $u = 1$ we obtain for the exponential generating function

$$E_k^{(1)}(x) = \sum_{n\geq 0} \mathbf{E}(B_{n,k}) \frac{x^n}{n!}$$

the following functional recurrence

$$E_k^{\prime(1)}(x) = e^{qx} E_{k-1}^{(1)}(px) + e^{px} E_{k-1}^{(1)}(qx), \tag{2.3}$$

with initial conditions $E_0^{(1)}(x) = 1$ and $E_k^{(1)}(0) = 0$ $(k \geq 1)$. The Poisson transform of $E_k^{(1)}(x)$, namely

$$\Delta_k^{(1)}(x) = e^{-x} \sum_{n\geq 0} \mathbf{E}(B_{n,k}) \frac{x^n}{n!} = e^{-x} E_k^{(1)}(x), \qquad k \geq 0$$

translates recurrence (2.3) into

$$\Delta_k^{\prime(1)}(x) + \Delta_k^{(1)}(x) = \Delta_{k-1}^{(1)}(px) + \Delta_{k-1}^{(1)}(qx), \qquad k \geq 1, \tag{2.4}$$

with initial conditions $\Delta_0^{(1)}(x) = e^{-x}$ and $\Delta_k^{(1)}(0) = 0$ $(k \geq 1)$. Similarly, by taking second derivatives with respect to u and setting $u = 1$ we obtain for the exponential generating function

$$E_k^{(2)}(x) = \sum_{n\geq 0} \mathbf{E}(B_{n,k}^2) \frac{x^n}{n!}$$

the following functional recurrence

$$E_k^{\prime(2)}(x) = e^{qx} E_{k-1}^{(2)}(px) + e^{px} E_{k-1}^{(2)}(qx) + 2E_{k-1}^{(1)}(px) E_{k-1}^{(1)}(qx), \tag{2.5}$$

with initial conditions $E_0^{(2)}(x) = 1$ and $E_k^{(2)}(0) = 0$ $(k \geq 1)$. Furthermore, the Poisson transform of $E_k^{(2)}(x)$, namely

$$\Delta_k^{(2)}(x) = e^{-x} \sum_{n \geq 0} \mathbf{E}(B_{n,k}^2) \frac{x^n}{n!} = e^{-x} E_k^{(2)}(x), \qquad k \geq 0$$

translates recurrence (2.5) into

$$\Delta_k^{\prime(2)}(x) + \Delta_k^{(2)}(x) = \Delta_{k-1}^{(2)}(px) + \Delta_{k-1}^{(2)}(qx) + w_k(x), \qquad k \geq 1, \tag{2.6}$$

where $w_k(x) = 2\Delta_{k-1}^{(1)}(px)\Delta_{k-1}^{(1)}(qx)$, $\Delta_0^{(2)}(x) = e^{-x}$ and $\Delta_k^{(2)}(0) = 0$ $(k \geq 1)$.

By induction it is easy to prove that $\Delta_k^{(2)}(x)$ can be represented as finite linear combinations of functions of the form $e^{-p^{\ell_1} q^{\ell_2} x}$ with $\ell_1, \ell_2 \geq 0$, and of products of two of these functions. Hence, the Mellin transform of $\Delta_k^{(2)}(x)$,

$$\Delta_k^{*(2)}(s) = \int_0^\infty \Delta_k^{(2)}(x) x^{s-1} dx.$$

exists for all s with $\Re(s) > 0$ (see [1]). Since $B_{n,k} = 0$ for $k > n$ it follows that $E_k^{(2)}(x) = \mathcal{O}(x^k)$ as $x \to 0$ which ensures that $\Delta_k^{*(2)}(s)$ actually exists for s with $\Re(s) > -k$.

Let us now express $\Delta_k^{*(2)}(s)$ as

$$\Delta_k^{*(2)}(s) = \Gamma(s) F_k^{(2)}(s),$$

where $\Gamma(s)$ is the Euler gamma function. By definition, we know that $F_k^{(2)}(s)$ is the finite linear combinations of functions a^{-s} (with certain values of a). Thus, $F_k^{(2)}(s)$ is an entire function. Furthermore (2.6) translates into

$$F_k^{(2)}(s) - F_k^{(2)}(s-1) = T(s) F_{k-1}^{(2)}(s) + H_k^{(2)}(s), \qquad k \geq 0 \tag{2.7}$$

where

$$H_k^{(2)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty w_k(x) x^{s-1} dx,$$

and $F_0^{(2)}(s) = 1$. Note that (2.7) does not only hold for $\Re(s) > -k$ where the Mellin transform exists. Since $F_k^{(2)}(s)$ continues analytically to an entire function, (2.7) holds for all s, too. The inhomogeneous part in (2.7) is very large compared to the order of magnitude of the homogeneous equation

$$F_k^{(1)}(s) - F_k^{(1)}(s-1) = T(s) F_{k-1}^{(1)}(s), \qquad k \geq 0, \tag{2.8}$$

for the first moment. Since $F_k^{(1)}(s)$ behaves geometrically as $T(s)^k$ it seems that the term $F_{k+1}^{(1)}(s-1)$ is negligible compared to the other two terms in (2.8). This phenomenon will also occur for $F_k^{(2)}(s)$ and $F_k(s)$. We introduce the so-called Poisson variance

$$V_k(x) := \Delta_k^{(2)}(x) - \left(\Delta_k^{(1)}(x)\right)^2,$$

which should be a good approximation for the variance of the profile ([3]). By (2.4) and (2.6), $V_k(x)$ satisfies

$$V_k(x) + V_k'(x) = V_{k-1}(px) + V_{k-1}(qx) + \left(\Delta_k^{\prime(1)}(x)\right)^2, \tag{2.9}$$

with initial condition $V_0(x) = e^{-x}(1 - e^{-x})$ and $V_k(0) = 0$ $(k \geq 1)$. The Mellin transform of $V_k(x)$ is then given as

$$V_k^*(s) = \int_0^\infty V_k(x) x^{s-1} dx,$$

and again, we can use a factorization of the form

$$V_k^*(s) = \Gamma(s) F_k(s),$$

where $V_k^*(s)$ and $F_k(s)$ can be written in terms of $\Delta_k^{*(1)}(s)$, $\Delta_k^{*(2)}(s)$, $F_k^{(1)}(s)$, and $F_k^{(2)}(s)$ respectively. In particular, (2.9) translates into

$$F_k(s) - F_k(s-1) = T(s) F_{k-1}(s) + H_k(s), \qquad k \geq 0, \tag{2.10}$$

where

$$H_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\Delta_k^{'(1)}(x)\right)^2 x^{s-1} dx$$

and also $F_0(s) = 1 - 2^{-s}$. We also observe that $F_k(-r) = 0$ for $k > r$, since $\Gamma(s)F_k(s)$ is the Mellin transform of $V_k(x)$ that exists for $\Re(s) > -k$. We will use this property.

The inhomogeneous part of (2.10) (i.e., $H_k(s)$) does not dependent on p and q in comparison with the inhomogeneous part of the functional-differential equation satisfied by variance of the profile (see Section 2.1 in [2] for details: page 6). Thus asymptotic analysis of the variance done there through Poissonized variance is not a precise analysis. In other words, the analytic function arising there $D(s, w)$ ([2, Lemma 4]) is not completely explicit. Let $\mathbf{I}$ be an identity operator and $\mathbf{A}[f](s) = \sum_{j \geq 0} f(s-j) T(s-j)$ for some function f.

Kazemi and Vahidi-Asl [2] showed the following results:

Lemma 2.1. The function $f(s, w) = \sum_{k \geq 0} F_k(s) w^k$ can be represented as

$$f(s, w) = D(s, w) g(s, w),$$

where the functions $g(s, w) = (\mathbf{I} - w\mathbf{A})^{-1}(s)$ and $D(s, w)$ is analytic for $|w| < 1/T(\Re(s)/2 - 1)^2$.

Theorem 2.2. Let $V_k(x)$ denote the Poisson variance of the profile in unbalanced digital search trees with underlying probabilities $0 < p < q = 1 - p$. Let k and n be positive integers such that $k/\log n$ satisfies $(\log \frac{1}{p})^{-1} + \varepsilon \leq k/\log n \leq (\log \frac{1}{q})^{-1} - \varepsilon$. Then uniformly

$$V_k(n) = L\left(\rho_{n,k}, \log_{p/q} p^k n\right) \frac{(p^{-\rho_{n,k}} + q^{-\rho_{n,k}})^k n^{-\rho_{n,k}}}{\sqrt{2\pi \beta(\rho_{n,k}) k}} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)\right), \tag{2.11}$$

where $\rho_{n,k} = \rho(k/\log n)$ and $L(\rho, x)$ is a non-zero periodic function with period 1 in x.

The function $L(\rho, x)$ can be represented as $L(\rho, x) = \sum_{j \in Z} f(\rho + it_j)\Gamma(\rho + it_j) e^{-2j\pi i x}$, where $f(s)$ has the form

$$f(s) = g(s-1, 1/T(s)) D(s, 1/T(s)).$$

Lemma 2.3. For every real interval $[a,b]$ there exist k_0, $\gamma > 0$ and $\varepsilon > 0$ such that

$$F_k(s) = f(s)T(s)^k \left(1 + \mathcal{O}\left(e^{-\gamma k}\right)\right), \tag{2.12}$$

uniformly for all s with $\Re(s) \in [a,b]$, $|\Im(s) - 2\ell\pi \log(q/p)| \le \varepsilon$ for some integer ℓ and $k \ge k_0$, where

$$f(s) = D(s, 1/T(s))g(s-1, 1/T(s))$$

is an analytic function that satisfies $f(-r) = 0$ for $r = 1, 2, \ldots$ and is bounded in this region.

Furthermore, if $|\Im(s) - 2\ell\pi \log(q/p)| > \varepsilon$ for for all integers ℓ, then we have

$$F_k(s) = \mathcal{O}\left(T(\Re(s))^k \, e^{-\gamma k}\right), \tag{2.13}$$

uniformly for $\Re(s) \in [a,b]$.

In Theorem 3.4 below we show that

$$\begin{aligned} f(s) &= \sum_{\ell=0}^{r} F_\ell(-r) w^\ell T(s)^{-\ell} \frac{h_1(s, 1/T(s))}{A(-r, 1/T(s))} . \frac{T(s) - T(-r)}{T(s)} \\ &+ \sum_{\ell=0}^{r} F_\ell(-r) w^\ell T(s)^{-\ell} \frac{h_2(s, 1/T(s))}{B(-r, 1/T(s))} . \frac{T(s) - T(-r)}{T(s) - \eta} \end{aligned} \tag{2.14}$$

where

$$A(-r, w) = h_1(-r, w) + \frac{1 - wT(-r)}{1 - w(T(-r) - \eta)} h_2(-r, w)$$

and

$$B(-r, w) = h_2(-r, w) + \frac{1 - w(T(-r) - \eta)}{1 - wT(-r)} h_1(-r, w).$$

In this function all parts are explicit and are analytic for $|w| < T(\Re(s) - \eta)^{-1}$ (see [2] for $h_1(s,w)$ and Lemma 3.3 for $h_2(s,w)$).

We obtain an explicit solution of (2.10) by introducing two proper functional operators:

$$\begin{aligned} \mathbf{C}[f](s) &= \sum_{j \ge 0} f(s - j), \\ \mathbf{D}[f](s) &= \sum_{j \ge 1} f(s - j, w) \frac{wT(s-j)}{1 - w(T(s-j) - \eta)} \frac{1 - w(T(s) - \eta)}{1 - wT(s)}, \end{aligned}$$

for some function f and $\eta > 0$. Set

$$\begin{aligned} R_k(s) &= \mathbf{A}^k[1](s), \\ \widetilde{H}_\ell(s) &= \mathbf{C}[H_{\ell-1}](s) - \mathbf{C}[H_{\ell-1}](c), \\ T_{k,\ell}(s) &= \mathbf{A}^k[\widetilde{H}_\ell](s), \\ G(s, w) &= \sum_{k \ge 0} \widetilde{H}_k(s) w^k, \end{aligned}$$

for all $k, \ell \ge 1$.

3. The New Results

Lemma 3.1. *Suppose* $f(s,w)=\sum_{k\geq 0}F_k(s)w^k$. *Then for all* $k\geq 1$ *and complex* s,

$$f(s,w)g(c,w)=g(s,w)+n(s,w), \tag{3.1}$$

where

$$\begin{aligned} n(s,w) &= g(c,w)G(s,w) \\ &+ \frac{(\boldsymbol{I}-w\boldsymbol{A})^{-1}[\widetilde{H}_k](s)-(\boldsymbol{I}-w\boldsymbol{A})^{-1}[\widetilde{H}_k](c)}{1-w} \\ &+ g(c,w)(\boldsymbol{I}-w\boldsymbol{A})^{-1}[G](s,w) \\ &+ g(s,w)(\boldsymbol{I}-w\boldsymbol{A})^{-1}[G](c,w), \quad w\neq 1. \end{aligned} \tag{3.2}$$

Proof . See Appendix A. □

Remark 3.2. The proof of (3.1) makes use of the fact that $F_k(c)=0$ for $k\geq 1$. However, we also have $F_k(-r)=0$ for $k>r$ [2]. In particular, if we set $s=-r$ in (3.1) we find that

$$f(-r,w)g(c,w)=g(-r,w)+n(-r,w)$$

and consequently

$$f(s,w)(g(-r,w)+n(-r,w))=(g(s,w)+n(s,w))\sum_{\ell=0}^{r}F_\ell(-r)w^\ell.$$

First of all, we have the following result by convolution of Laplace transform [2]:

$$\begin{aligned} |H_k(s)| &= \left|\frac{1}{\Gamma(s)}\int_0^\infty \left(\Delta_k^{'(1)}(x)\right)^2 x^{s-1}dx\right| \\ &= \left|\frac{1}{\Gamma(s)}\right|\left|\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Delta_k^{'*(1)}(t)\Delta_k^{'*(1)}(s-t)dt\right| \\ &\leq \frac{C''}{|\Gamma(s)|}\int_{c-i\infty}^{c+i\infty}|t||s-t||\Gamma(t)||\Gamma(s-t-1)|\Big(T(\Re(t)-1)T(\Re(s-t)-1)\Big)^k dt \\ &\leq CT(c-1)^{2k} \qquad\qquad \Re(t)=c=\Re(s-t), \\ &= CT\left(\frac{\Re(s)}{2}-1\right)^{2k} \qquad\qquad c=\frac{\Re(s)}{2}, \\ &\leq C\Big(T(\Re(s))-\eta\Big)^k, \end{aligned} \tag{3.3}$$

for constants C'' and C and for some $\eta>0$.

Lemma 3.3. *There exists a function* $h_2(s,w)$ *that is analytic for all* w *and* s *satisfying*

$$w(T(s-n)-\eta)\neq 1, \quad \textit{for all } n\geq 1,$$

such that

$$n(s,w)=\frac{h_2(s,w)}{1-w(T(s)-\eta)}. \tag{3.4}$$

Thus, $n(s,w)$ *has a meromorphic continuation where* $w_0=1/(T(s)-\eta)$ *is a polar singularity.*

Proof . By definition of $\widetilde{H}_k(s)$ and (3.3), $n(s,w)$ converges absolutely and represents an analytic function, since $|T(s-j)| \le |T(s)| \max(pq)^j$ for $j \ge 0$. Also

$$(\mathbf{I} - w\mathbf{A})[n(s,.)](s) = r(s,w),$$

where

$$r(s,w) = G(s,w) + \Big(\widetilde{H}_k(s) - \widetilde{H}_k(c)\Big)(1-w)^{-1} + G(s,w)m(c,w) - G(c,w)g(s,w).$$

If we substitute $n(s,w)$ by

$$\frac{h_2(s,w)}{1 - w(T(s)-\eta)},$$

then

$$h_2(s,w) = \mathbf{D}[h_2](s) + \frac{1 - w(T(s)-\eta)}{1 - wT(s)} r(s,w).$$

For convenience, set

$$V(s-j,w) = \frac{wT(s-j)}{1 - w(T(s-j)-\eta)} . \frac{1 - w(T(s)-\eta)}{1 - wT(s)}, \qquad j \ge 1.$$

By induction it follows that

$$\begin{aligned}
\mathbf{D}^k[1](s) &= \sum_{j_1 \ge 0} \sum_{j_2 \ge 0} \cdots \sum_{j_k \ge 0} V(s-j_1,w)V(s-j_1-j_2,w) \\
&\cdots \ V(s-j_1-\cdots-j_k,w) \\
&= \sum_{n_k \ge k} \sum_{n_{k-1}=k-1}^{n_k-1} \sum_{n_{k-2}=k-2}^{n_{k-1}-1} \cdots \sum_{n_1=1}^{n_2-1} V(s-n_1,w)V(s-n_2,w) \\
&\cdots \ V(s-n_k,w).
\end{aligned}$$

Hence,

$$\begin{aligned}
|\mathbf{D}^k[1](s)| &\le \sum_{n_k \ge k} \sum_{n_{k-1} \ge k-1} \sum_{n_{k-2} \ge k-2} \cdots \sum_{n_1 \ge 1} |V(s-n_1,w)V(s-n_2,w) \\
&\cdots \ V(s-n_k,w)| \\
&\le \sum_{n_k \ge k} |V(s-n_k,w)| \sum_{n_{k-1} \ge k-1} |V(s-n_{k-1},w)| \\
&\cdots \sum_{n_1 \ge 1} |V(s-n_1,w)|.
\end{aligned}$$

Using the fact that $T(s-n) = \mathcal{O}(q^n)$, it follows directly that the series

$$S := \sum_{n \ge 1} |V(s-n,w)| = \sum_{n \ge 1} \frac{\Big| wT(s-n)\Big(1 - w(T(s)-\eta)\Big)\Big|}{|1 - w(T(s-n)-\eta)||1 - wT(s)|},$$

converges if $w(T(s-n)-\eta) \ne 1$ for all $n \ge 1$ and $1 \ne wT(s)$. Let now k_0 be any value such that

$$\sum_{n \ge k_0} |V(s-n,w)| \le \frac{1}{2},$$

Then we have for all $k \geq k_0$,

$$|\mathbf{D}^k[1](s)| \leq \left(\frac{1}{2}\right)^k (2S)^{k_0}.$$

Since $|r(s,w)| \leq C(1 - w(T(\Re(s) - \eta))^{-1}$ and

$$h_2(s,w) = \left[\sum_{k\geq 0} \mathbf{D}^k[1](s)\right] \frac{1 - w(T(s) - \eta)}{1 - wT(s)} r(s,w),$$

thus $|h_2(s,w)| \leq 2(2S)^{k_0} + C'$ for constant C'. □

Theorem 3.4. *For every real interval $[a,b]$ there exist k_0, $\gamma > 0$ and $\varepsilon > 0$ such that*

$$F_k(s) = f(s)T(s)^k \left(1 + O\left(e^{-\gamma k}\right)\right) \tag{3.5}$$

uniformly for all s with $\Re(s) \in [a,b]$, $|\Im(s) - 2\ell\pi \log(q/p)| \leq \varepsilon$ for some integer ℓ and $k \geq k_0$, where $f(s)$ is an analytic function that satisfies $f(-r) = 0$ for $r = 1, 2, \ldots$.

Furthermore, if $|\Im(s) - 2\ell\pi \log(q/p)| > \varepsilon$ for for all integers ℓ then we have

$$F_k(s) = O\left(T(s)^k \, e^{-\gamma k}\right). \tag{3.6}$$

uniformly for $\Re(s) \in [a,b]$ where $f(s)$ described in (2.14).

Proof . Let $h_1(s,w) = \prod_{j\geq 1} 1/(1 - wT(s-j))$ [2]. We have

$$\begin{aligned}
f(s,w) &= \sum_{\ell=0}^{r} F_\ell(-r) w^\ell \frac{g(s,w) + n(s,w)}{g(-r,w) + n(-r,w)} \\
&= \sum_{\ell=0}^{r} F_\ell(-r) w^\ell \frac{g(s,w))}{g(-r,w) + n(-r,w)} \\
&+ \sum_{\ell=0}^{r} F_\ell(-r) w^\ell \frac{n(s,w)}{n(-r,w) + n(-r,w)} \\
&= \sum_{\ell=0}^{r} F_\ell(-r) w^\ell \frac{h_1(s,w)}{A(-r,w)} \frac{1 - wT(-r)}{1 - wT(s)} \\
&+ \sum_{\ell=0}^{r} F_\ell(-r) w^\ell \frac{h_2(s,w)}{B(-r,w)} \frac{1 - w(T(-r) - \eta)}{1 - w(T(s) - \eta)} \\
&= f_1(s,w) + f_2(s,w).
\end{aligned} \tag{3.7}$$

Suppose first that $s > -r - 1$ for some integer $r \geq 0$ but s is not a positive integer. The function $h_1(s,w)$ is analytic for $|w| < 1/T(s-1)$. It also follows that $h_1(s,w)$ is non-zero for real $0 < w < 1/T(s-1)$ and that $h_1(-r,w)$ is analytic and non-zero for $0 < w < 1/T(-r-1)$. Hence, $w_0 = 1/T(s)$ is a singular point of $f_1(s,w)$. Since $F_k(s) = V_k^*(s)/\Gamma(s)$ it follows that all values $F_k(s)$ have the same sign. Hence, the radius of convergence of the series $f(s,w)$ equals $w_0 = 1/T(s)$.

In a next step we show that $f_1(s,w)$ has no other singularities on the radius of convergence $|w| = 1/T(s)$. Since all terms in $f_1(s,w)$, that is, $\sum_{\ell=0}^{r} F_\ell(-r)w^\ell$, $h_1(s,w)$, $A(-r,w)$, $1 - wT(-r)$, and $1 - wT(s)$ are analytic for $|w| < 1/T(s) + \varepsilon$, a singularity of $f_1(s,w)$ can only be induced by a zero of $A(-r,w)$. Suppose first that $A(-r,w)$ has a zero w_1 with $|w_1| < 1/T(s)$. Since $A(-r,w) \neq 0$

for $0 < w < 1/T(-r-1)$ it follows that $w_1 \neq 1/T(-r)$ and $w_1 \neq 1/(T(-r)-\eta)$. If we assume that $\sum_{\ell=0}^{r} F_\ell(-r)w_1^\ell \neq 0$, then $w = w_1$ must be a zero of $h_1(s,w)$. We slightly decrease s to $s-\gamma$ (for some $\gamma > 0$ such that $s-\gamma$ is not a positive integer) such that $h_1(s-\gamma, w_1) \neq 0$. Then the zero $w = w_1$ of $A(-r,w)$ would induce a singularity w_1 of $f_1(s,w)$ with $|w_1| < 1/T(s)$ although its radius of convergence is $1/T(s-\gamma) > 1/T(s) > |w_1|$. This leads to a contradiction. Hence, if $A(-r,w_1) = 0$ for some w_1 with $|w_1| < 1/T(s)$, then we also have $\sum_{\ell=0}^{r} F_\ell(-r)w_1^\ell = 0$. Actually, it also follows that the order of the zeroes are the same. The above considerations also show that if $w = w_1$ is a zero of $A(-r,w)$ with $|w_1| < 1/T(-r-1)$, then w_1 is also a zero of $\sum_{\ell=0}^{r} F_\ell(-r)w_1^\ell = 0$ of the same order. Namely, if $|w_1| < 1/T(-r-1)$ then there exists a non-integral real number $s > -r-1$ with $|w_1| < 1/T(s)$ and we proceed as above.

This property shows that the only singularity of $f_1(s,w)$ is given by $w = 1/T(s)$ if $s > -r-1$ is real (but not an integer). This singularity is a polar singularity of order 1. Hence, by using Cauchy's formula for a contour of integration on the circle $|w| = e^\gamma/T(s)$ and the residue theorem it follows that [1]

$$[w^k]f_1(s,w) = f_1(s)T(s)^k + \mathcal{O}\left(|T(s)e^{-\gamma}|^k\right), \tag{3.8}$$

where

$$f_1(s) = \sum_{\ell=0}^{r} F_\ell(-r)T(s)^{-\ell}\frac{h_1(s,1/T(s))}{A(-r,1/T(s))}\left(1-\frac{T(-r)}{T(s)}\right).$$

These estimates are uniform for s contained in a compact interval $[a,b] \subseteq (-r-1,-r)$ (for some non-negative integer r) or in a compact interval $[a,b]$ contained in the positive real line. Furthermore, we get the same result if s is sufficiently close to the real axis. Thus, if $a \leq \Re(s) \leq b$ and $|\Im(s)| \leq \varepsilon$ for some sufficiently small $\varepsilon > 0$, then we obtain (3.8). Here we have also used the fact that $f_1(s) \neq 0$ in this range.

Next, suppose that s is real (or sufficiently close to the real axis) and close to a negative integer $-r$, say $-r-\gamma \leq s \leq -r+\gamma$ (for some $\gamma > 0$). Here we use the representation

$$\begin{aligned} f_1(s,w) &= \sum_{\ell=0}^{r} F_\ell(-r)w^\ell\,\frac{h_1(s,w)}{A(-r,w)}\,\frac{1-wT(-r)}{1-wT(s)} \\ &= \sum_{\ell=0}^{r} F_\ell(-r)w^k\,\frac{h_1(s,w)-A(-r,w)}{A(-r,w)}\,\frac{1-wT(-r)}{1-wT(s)} \\ &+ \sum_{\ell=0}^{r} F_\ell(-r)w^k + \sum_{\ell=0}^{r} F_\ell(-r)w^{\ell+1}\frac{T(s)-T(-r)}{1-wT(s)}. \end{aligned}$$

Now if we subtract the finite sum $\sum_{\ell=0}^{r} F_\ell(-r)w^k$, then we can safely multiply by $\Gamma(s)$ (that is singular at $s=-r$) and obtain

$$\begin{aligned} \Gamma(s)\sum_{k>r} F_k(s)w^k &= \sum_{k=0}^{r} F_\ell(-r)w^k\,\frac{\Gamma(s)\Big(h_1(s,w)-A(-r,w)\Big)}{A(-r,w)} \\ &\times \frac{1-wT(-r)}{1-wT(s)} \\ &+ \sum_{\ell=0}^{r} F_\ell(-r)w^{\ell+1}\frac{\Gamma(s)\Big(T(s)-T(-r)\Big)}{1-wT(s)}. \end{aligned}$$

We again use the fact that the function $\sum_{\ell=0}^{r} F_\ell(-r)w^\ell/A(-r,w)$ is analytic for $|w| < 1/T(-r-1)$ and observe that $w = 1/T(s)$ is a polar singularity. By applying Cauchy's formula we obtain for $k > r$ (similar to the above)

$$\begin{aligned}\Gamma(s)[w^k]f_1(s,w) &= \sum_{\ell=0}^{r} F_\ell(-r)T(s)^{k-\ell}\left(1-\frac{T(-r)}{T(s)}\right)\\ &\times \frac{\Gamma(s)\Big(h_1(s,1/T(s)) - A(-r,1/T(s)\Big)}{A(-r,1/T(s)}\\ &+ \sum_{\ell=0}^{r} F_\ell(-r)T(s)^{k-\ell-1}\Gamma(s)\Big(T(s)-T(-r)\Big)\\ &+ \mathcal{O}(|T(s)e^{-\gamma}|^k).\end{aligned}$$

Thus, we have actually proved (3.8) for $k > r$ and we also observe that $f_1(-r) = 0$.

Since $|T(s+2\pi i\ell/\log(q/p))| = |T(s)|$ for integer ℓ, it follows that $w = 1/T(s)$ is a polar singularity of $f_1(s,w)$ if $|\Im(s) - 2\pi i\ell/\log(q/p))| < \varepsilon$ for some integer ℓ. Thus, (3.8) follows also for s in this range. Finally, if $|\Im(s) - 2\pi i\ell/\log(q/p))| > \varepsilon$ for some integer ℓ, then there exists $\gamma > 0$ such that $|T(s)| < e^{-2\gamma}|T(\Re(s)|$. Hence it follows that $f_1(s,w)$ is regular for $|w| < e^{2\gamma}/T(\Re(s))$. Thus, for a contour of integration on the circle $|w| = e^{\gamma}/T(\Re(s))$ in Cauchy's formula we obtain

$$[w^k]f_1(s,w) = \mathcal{O}\left(T(\Re(s))^k\, e^{-\gamma k}\right).$$

It should be clear that this estimate is uniform if $\Re(s)$ varies in a finite interval $[a,b]$.

Since $f_2(s,w)$ only has a polar singularity $w = 1/(T(s)-\eta)$ of order 1, thus for a contour of integration on the circle $|w| = e^{\gamma}/(T(s)-\eta)$,

$$[w^k]f_2(s,w) = f_2(s)T(s)^k + \mathcal{O}\left(|(T(s)-\eta)e^{-\gamma}|^k\right),$$

where

$$f_2(s) = \sum_{\ell=0}^{r} F_\ell(-r)T(s)^{-\ell}\frac{h_2(s,1/T(s))}{B(-r,1/T(s))}\left(1-\frac{T(-r)-\eta}{T(s)-\eta}\right).$$

If $|\Im(s) - 2\pi i\ell/\log(q/p))| > \varepsilon$, then

$$[w^k]f_2(s,w) = \mathcal{O}\left(T(\Re(s)-\eta)^k\, e^{-\gamma k}\right).$$

It is obvious that $w_0 = 1/T(s)$ is a dominant singularity of $f(s,w)$. Thus for every real interval $[a,b]$ there exist k_0, $\gamma > 0$ and $\varepsilon > 0$ such that

$$F_k(s) = [w^k]f(s,w) = f(s)T(s)^k\left(1+\mathcal{O}\left(e^{-\gamma k}\right)\right),$$

where

$$\begin{aligned}f(s) &= f_1(s) + f_2(s)\\ &= \sum_{\ell=0}^{r} F_\ell(-r)w^\ell T(s)^{-\ell}\frac{h_1(s,1/T(s))}{A(-r,1/T(s))}.\frac{T(s)-T(-r)}{T(s)}\\ &+ \sum_{\ell=0}^{r} F_\ell(-r)w^\ell T(s)^{-\ell}\frac{h_2(s,1/T(s))}{B(-r,1/T(s))}.\frac{T(s)-T(-r)}{T(s)-\eta}\end{aligned}$$

uniformly for all s with $\Re(s) \in [a,b]$, $|\Im(s) - 2\ell\pi\log(q/p)| \leq \varepsilon$ for some integer ℓ and $k \geq k_0$, where $f(s)$ is an analytic function that satisfies $f(-r) = 0$ for $r = 1, 2, \ldots$. Furthermore, if $|\Im(s) - 2\ell\pi\log(q/p)| > \varepsilon$ for all integers ℓ then we have

$$F_k(s) = \mathcal{O}\left(T(\Re(s))^k\, e^{-\gamma k}\right)$$

uniformly for $\Re(s) \in [a,b]$. $\square$

By the above discussion, we know that $F_k(s)$ and $V_k^*(s) = \Gamma(s)F_k(s)$ behave asymptotically as $T(s)^k$. Thus we are in a situation similar to the analysis of the previous article [2] but here the analytic function $f(s)$ introduced in the above theorem is completely explicit.

References

[1] F. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press. Cambridge, 2008.

[2] R. Kazemi and M.Q. Vahidi-Asl, *The Variance of the Profile in Digita Search Trees*, Discrete Math. Theor. Comput. Sci. 13 (2011) 21–38.

[3] W. Szpankowski, *Average Case Analysis of Algorithms on Sequences*. John Wiley, New York, 2001.

Appendix A

Proof .[Proof of Lemma 3.1] It is obvious that

$$\begin{aligned} n(s,w) &= \sum_{k\geq 0}\sum_{\ell=0}^{k-1} R_\ell(c)\widetilde{H}_{k-\ell}(s)w^k + \sum_{k\geq 0}\sum_{\ell=1}^{k-1}\Big(T_{k-\ell,\ell}(s) - T_{k-\ell,\ell}(c)\Big)w^k \\ &+ \sum_{k\geq 0}\sum_{j=1}^{k-2}\sum_{\ell=1}^{k-j-1}\Big(R_j(c)T_{k-\ell-j,\ell}(s) - R_j(s)T_{k-\ell-j,\ell}(c)\Big)w^k \end{aligned}$$

and

$$f(s,w)g(c,w) = \sum_{k\geq 0}\ \sum_{\ell=1}^{k} F_\ell(s)R_{k-\ell}(c)w^k.$$

Thus it is enough to show that

$$\begin{aligned} F_k(s) &= R_k(s) - \sum_{\ell=1}^{k-1} F_\ell(s)R_{k-\ell}(c) + \sum_{\ell=0}^{k-1} R_\ell(c)\widetilde{H}_{k-\ell}(s) \\ &+ \sum_{\ell=1}^{k-1}\Big(T_{k-\ell,\ell}(s) - T_{k-\ell,\ell}(c)\Big) + \sum_{j=1}^{k-2}\ \sum_{\ell=1}^{k-j-1}\Big(R_j(c)T_{k-\ell-j,\ell}(s) - R_j(s)T_{k-\ell-j,\ell}(c)\Big). \end{aligned}$$

We will prove this relation by induction. Certainly, it is satisfied for $k = 0$. Now suppose that it holds for some $k \geq 0$. Thus

$$\begin{aligned} F_{k+1}(s) &= R_{k+1}(s) - R_{k+1}(c) - \sum_{\ell=0}^{k-1} R_{k-\ell}(c)F_{\ell+1}(s) \\ &+ \sum_{\ell=0}^{k-1} R_{k-\ell}(c)\widetilde{H}_{\ell+1}(s) + \sum_{\ell=0}^{k-1} R_\ell(c)\mathbf{A}[\widetilde{H}_{k-\ell}](s) - \sum_{\ell=0}^{k-1} R_\ell(c)\mathbf{A}[\widetilde{H}_{k-\ell}](c) \\ &+ \sum_{\ell=1}^{k-1}\Big(T_{k-\ell+1,\ell}(s) - T_{k-\ell+1,\ell}(c)\Big) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\ell=1}^{k-1} R_1(s)T_{k-\ell,\ell}(c) + \sum_{\ell=1}^{k-1} R_1(c)T_{k-\ell,\ell}(c)\\
&+ \sum_{j=1}^{k-2}\sum_{\ell=1}^{k-j-1} R_j(c)T_{k-\ell-j+1,\ell}(s) - \sum_{j=1}^{k-2}\sum_{\ell=1}^{k-j-1} R_j(c)T_{k-\ell-j+1,\ell}(c)\\
&- \sum_{j=1}^{k-2}\sum_{\ell=1}^{k-j-1} R_{j+1}(s)T_{k-\ell-j,\ell}(c)\\
&+ \sum_{j=1}^{k-2}\sum_{\ell=1}^{k-j-1} R_{j+1}(c)T_{k-\ell-j,\ell}(c) + \widetilde{H}_{k+1}(s)\\
&= R_{k+1}(s) - \sum_{\ell=0}^{k} R_{k+1-\ell}(c)F_\ell(s) + \sum_{\ell=0}^{k} R_\ell(c)\widetilde{H}_{k+1-\ell}(s)\\
&+ \sum_{\ell=0}^{k-1} R_\ell(c)\mathbf{A}[\widetilde{H}_{k-\ell}](s) - \sum_{\ell=0}^{k-1} R_\ell(c)\mathbf{A}[\widetilde{H}_{k-\ell}](c)\\
&+ \sum_{\ell=1}^{k}\Big(T_{k-\ell+1,\ell}(s) - T_{k-\ell+1,\ell}(c)\Big) - \Big(\mathbf{A}[\widetilde{H}_k](s) - \mathbf{A}[\widetilde{H}_k](c)\Big)\\
&- \sum_{\ell=1}^{k-1} R_1(s)T_{k-\ell,\ell}(c) + \sum_{\ell=1}^{k-1} R_1(c)T_{k-\ell,\ell}(c)\\
&\\
&+ \sum_{j=1}^{k-1}\sum_{\ell=1}^{k-j} R_j(c)T_{k-\ell-j+1,\ell}(s) - \sum_{\ell=1}^{k-1} R_\ell(c)\mathbf{A}[\widetilde{H}_{k-\ell}](s)\\
&- \sum_{\ell=1}^{k-2} R_1(c)T_{k-\ell,\ell}(c) + \sum_{\ell=2}^{k-1} R_\ell(c)\mathbf{A}[\widetilde{H}_{k-\ell}](c)\\
&- \sum_{j=1}^{k-1}\sum_{\ell=1}^{k-j} R_j(s)T_{k+1-\ell-j,\ell}(c) + \sum_{\ell=1}^{k-1} R_1(s)T_{k-\ell,\ell}(c).
\end{aligned}
$$

Now by removing the same expressions the proof is completed. □