An assessment of a semi analytical AG method for solving two-dimension nonlinear viscous flow

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Abstract

In this investigation, attempts have been made to solve two-dimension nonlinear viscous flow between slowly expanding or contracting walls with weak permeability by utilizing a semi analytical Akbari Ganji's Method (AGM). As regard to previous papers, solving of nonlinear equations is difficult and the results are not accurate. This new approach is emerged after comparing the achieved solutions with Numerical method and exact solution. Based on the comparison between AGM and numerical methods, AGM can be successfully applied for a broad range of nonlinear equations. Results illustrate, this method is efficient and has enough accuracy in comparison with other semi analytical and numerical methods. Ruge-Kutta numerical method, Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM) and Adomian Decomposition Method (ADM) have been applied to make this comparison. Moreover results demonstrate that AGM could be applicable through other methods in nonlinear problems with high nonlinearity. Furthermore convergence problems for solving nonlinear equations by using AGM appear small.

Keywords: Adomian Decomposition Method (ADM); Akbari-Ganji Method (AGM); Homotopy Perturbation Method (HPM); Variational Iteration Method (VIM).
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1. Introduction

The flow of Newtonian and non-Newtonian fluids in a porous surface channel has attracted the interest of many investigators in view of its applications in engineering practice, particularly in industries. Examples of these are the cases of boundary layer control, transpiration cooling and

gaseous diffusion. Theoretical research on steady flow of this type was initiated by Berman [1] who found a series solution for the two-dimensional laminar flow of a viscous incompressible fluid in a parallel-walled channel for the case of a very low cross-flow Reynolds number. After his work, this problem has been studied by many researchers considering various variations in the problem, e.g., Choi et al. [2] and references cited therein. For the case of a converging or diverging channel with a permeable wall, if the Reynolds number is large and if there is suction or injection at the walls whose magnitude is inversely proportional to the distance along the wall from the origin of the channel, a solution for laminar boundary layer equations can be obtained [3].

Most scientific problems such as two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability and other fluid mechanic problems are inherently nonlinear. Except a limited number of these problems, most of them don't have analytical solution. Therefore, these nonlinear equations should be solved using other methods. In the analytical perturbation method, we should exert the small parameter in the equation. Therefore, finding the small parameter and exerting it into the equation are difficulties of this method. Since there are some limitations with the common perturbation method, and also because the basis of the common perturbation method is upon the existence of a small parameter, developing the method for different applications is very difficult. Therefore, many different methods have recently introduced some ways to eliminate the small parameter. In this study we are comparing the efficiency and accuracy of some of these methods with Akbari-Ganji's Method [4, 5, 6].

The Homotopy perturbation method (HPM) [7, 8, 9] is well-known method to solve the nonlinear equations. This method is introduced by He [10, 11] for the first time. This method has been used by many authors such as Ganji in [12, 13] and the references therein to handle a wide variety of scientific and engineering applications such as linear and nonlinear, homogeneous and inhomogeneous as well, because these methods continuously deform a difficult problem into a simple one, which is easy to solve.

The other method which could be mentioned is variational Iteration Method (VIM) [14, 15, 16] by J.H.He, this method also has been used by many scientists and recently some parts of it has been developed because of introducing some new methods for identification of the Lagrangian multiplier which is used in VIM formula [17].

And the last method which has compare to AGM in this paper is Adomian Decomposition Method (ADM) [18, 19, 20], At the beginning of the 80s, a new method later called ADM for solving various kinds of nonlinear equations had been proposed by Adomian [21, 22]. The convergence of Adomian's method has been investigated by several authors K. Abbaoui, Y. Cherruault [23, 24]).

The main purpose of this paper is introducing AGM as new method and by comparing it with other method we can precisely conclude that AGM has high efficiency and accuracy for solving nonlinear problems with high nonlinearity. It is necessary to mention that a summary of the excellence of this method in comparison with the other approaches can be considered as follows: Boundary conditions are needed in accordance with the order of differential equations in the solution procedure but when the number of boundary conditions is less than the order of the differential equation, this approach can create additional new boundary conditions in regard to the own differential equation and its derivatives. Therefore, it is logical to mention that AGM is operational for miscellaneous nonlinear differential equations in comparison with the other methods.

Description	Symbol
Homotopy Perturbation Method	HPM
Variational Iteration Method	VIM
Adomian Decomposition Method	ADM
Akbari Ganji's Method	AGM
Velocity component in x direction	$\widehat{u}$
Velocity component in y direction	$\widehat{v}$
Dimensionless pressure	$\widehat{p}$
Non-dimensional wall dilation rate which is dened positive for expansion	α
Permeation Reynolds number which is considered positive for injection	Re
Axial coordinate	$\widehat{x}$
Axial coordinate	$\widehat{y}$
Time	t
Kinematic viscosity	ν
Density	ρ

2. Mathematical Fourmulation

Consider the laminar, isothermal, and incompressible flow in a rectangular domain bounded by two permeable surfaces that enable the fluid to enter or exit during successive expansions or contractions [25]. One side of the cross section, representing the distance $(2a)$ between the walls is taken to be smaller than the other two (W and L). Both walls are assumed to have equal permeability and to expand uniformly at a time dependent rate a. Furthermore, the origin $x = 0$ is assumed to be the center of the classic squeeze film problem. This enables us to assume flow symmetry about $x = 0$. Under these assumptions, the equations for continuity and motion become:

$$\frac{\partial \widehat{u}}{\partial \widehat{x}} + \frac{\partial \widehat{v}}{\partial \widehat{y}} \tag{2.1}$$

$$\frac{\partial \widehat{u}}{\partial t} + \widehat{u} \cdot \left(\frac{\partial \widehat{u}}{\partial \widehat{x}}\right) + \widehat{v} \cdot \frac{\partial \widehat{v}}{\partial \widehat{y}} = \left(\frac{-1}{\rho}\right) \cdot \left(\frac{\partial \widehat{p}}{\partial \widehat{x}}\right) + v\nabla^2\widehat{u} \tag{2.2}$$

$$\frac{\partial \widehat{v}}{\partial t} + \widehat{u} \cdot \left(\frac{\partial \widehat{v}}{\partial \widehat{x}}\right) + \widehat{v} \cdot \frac{\partial \widehat{v}}{\partial \widehat{y}} = \left(\frac{-1}{\rho}\right) \cdot \left(\frac{\partial \widehat{p}}{\partial \widehat{y}}\right) + v\nabla^2\widehat{v} \tag{2.3}$$

Where $\widehat{p}, \rho, \nu$ and t are the dimensional pressure, density, kinematic viscosity, and time. Auxiliary conditions can be specified such as:

$$\widehat{u}(\widehat{x}, 0) = 0, \widehat{v}(a) = -\widehat{v}(w) = -\frac{a}{c} \tag{2.4}$$

$$\frac{\partial \widehat{u}}{\partial \widehat{y}}(\widehat{x}, 0) = 0, \widehat{v}(0) = 0, \widehat{u}(0, \widehat{y}) = 0 \tag{2.5}$$

After some modification and special variable [10] and then we have:

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}F(x) + \alpha\left(y\frac{\mathrm{d}^3}{\mathrm{d}x^3}F(x) + 3\frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x)\right) + Re\left(\frac{\mathrm{d}}{\mathrm{d}x}F(x)\right)\frac{\mathrm{d}^3}{\mathrm{d}x^3}F(x)$$
$$-Re\left(\frac{\mathrm{d}}{\mathrm{d}x}F(x)\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x) = 0 \tag{2.6}$$

With the boundary conditions:

$$F(0) = 0, F(1) = 1, F^{'}(1) = 0, F^{''}(0) = 0 \tag{2.7}$$

Where a prime denotes differentiation with respect to y: Note that Bermans [1] well-known ODE can be viewed as a special case of Eq. (2.6) with $\alpha = 0$, so the above equation could be rewrite as follow:

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}F(x) + Re\left(\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}F(x)\right)F(x) - \left(\frac{\mathrm{d}}{\mathrm{d}x}F(x)\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x)\right) = 0 \tag{2.8}$$

3. Analysis of He's Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following equation:

$$A(u) - f(r) = 0, r \in \Omega \tag{3.1}$$

With the boundary condition of:

$$B(u, \frac{\partial u}{\partial n}) = 0, r \in \Gamma \tag{3.2}$$

Where A is general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω .A can be divided into two parts, which are L and N, where L is linear and N is nonlinear. Eq. (3.1) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, r \in \Omega \tag{3.3}$$

Homotopy perturbation structure is shown as follows:

$$L(v) - L(u_0) + p \cdot L(u_0) + p \cdot [N(v) - f(r)] \tag{3.4}$$

Or:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{3.5}$$

Where:

$$v(r, p) : \Omega \times [0, 1] \to R \tag{3.6}$$

In Eq. (3.4) or Eq. (3.5),$p \in [0.1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (3.1) can be written as a power series in p, as following:

$$v = v_0 + p \cdot v_1 + p^2 \cdot v_2 + \ldots = \sum_{i=0}^{n} p^i \cdot v_i \tag{3.7}$$

And the best approximation for solution is:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 \tag{3.8}$$

3.1. The Application of HPM

In this section, we will apply the HPM to nonlinear ordinary differential Eq. (2.8) with a boundary condition Eq. (2.7). According to the HPM, we can construct homotopy of Eq. (2.8) as follows:

$$H(x,p) = (1-p)\frac{\mathrm{d}^4}{\mathrm{d}x^4}F(x)$$
$$+p\left(\frac{\mathrm{d}^4}{\mathrm{d}x^4}F(x) + R\left(\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}F(x)\right)F(x) - \left(\frac{\mathrm{d}}{\mathrm{d}x}F(x)\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x)\right)\right) \tag{3.9}$$

We consider $F(x)$ as follows:

$$F(x) = F_0(x) + pF_1(x) + p^2F_2(x) \tag{3.10}$$

From Eq. (3.7), if the three terms approximations are sufficient, we will obtain with substituting u from Eq. (3.10) into Eq. (3.9) and some simplification and rearranging based on powers of p-terms, we have :

$$p^0: \frac{\mathrm{d}^4}{\mathrm{d}x^4}F_0(x) = 0 \tag{3.11}$$
$$F_0(0) = 0, F_0(1) = 1, F_0'(1) = 0, F_0''(0) = 0$$

$$p^1: \left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}F_0(x)\right)F_0(x)R - \left(\frac{\mathrm{d}}{\mathrm{d}x}F_0(x)\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_0(x)\right)R + \frac{\mathrm{d}^4}{\mathrm{d}x^4}F_1(x) \tag{3.12}$$
$$F_1(0) = 0, F_1(1) = 0, F_1'(1) = 0, F_1''(0) = 0$$

$$p^2: \left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}F_0(x)\right)F_1(x)R + \left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}F_1(x)\right)F_0(x)R$$
$$- \left(\frac{\mathrm{d}}{\mathrm{d}x}F_0(x)\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_1(x)\right)R - \left(\frac{\mathrm{d}}{\mathrm{d}x}F_1(x)\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_0(x)\right)R + \frac{\mathrm{d}^4}{\mathrm{d}x^4}F_2(x) \tag{3.13}$$
$$F_1(0) = 0, F_1(1) = 0, F_1'(1) = 0, F_1''(0) = 0$$

Solving Eq.s (3.11), (3.12) and (3.13) with boundary conditions, we have:

$$F_0(x) = \frac{-1}{2}x^3 + \frac{3}{2}x \tag{3.14}$$

$$F_1(x) = \frac{Rx^7}{280} - \frac{3\,Rx^3}{280} + \frac{Rx}{140} \tag{3.15}$$

$$F_2(x) = \frac{R^2x^{11}}{92400} - \frac{R^2x^9}{3360} + \frac{3\,R^2x^7}{19600} + \frac{73\,R^2x^3}{107800} - \frac{703\,R^2x}{1293600} \tag{3.16}$$

The solution of this equation, when $p \to 1$, will be as follow:

$$F(x) = F_0(x) + F_1(x) + F_2(x) \tag{3.17}$$

With assumption of $Re = 1$, $F(x)$ would be as follow:

$$F(x) \cong -0.51\,x^3 + 1.506\,x + 0.0037\,x^7 - 0.00029\,x^9 + 0.00001\,x^{11} \tag{3.18}$$

4. Variational Iteration Method

To illustrate the basic idea of variational iteration method, we consider the following general nonlinear system:

$$Lu + Nu = g(x) \tag{4.1}$$

Where L is a linear operator, N nonlinear operator, $g(x)$ a homogeneous term. According to the variational iteration method, we can construct the following iteration formulation:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left(Lu_n(\tau) + Nu_n(\tau) - g(\tau) \right) d\tau \tag{4.2}$$

Where λ is a general Lagrangian multiplier [26, 27], which can be identified optimally via the variational theory [28, 29]. The subscript n indicates the nth approximation and $\widetilde{u}_n$ is considered as a restricted vaeiation [30, 31] , i.e.,$\delta\widetilde{u}_n = 0$, where λ is general Lagrange multiplier. For calculating Lagrange multiplier we are doing the following steps:

1. Linear part of differential equation is solved by Laplace method and will be put equal $(-1)^n$ (where n is the highest order derivative of the linear part)
2. We assume zero the boundary conditions of the problem
3. Wherever the result of Laplace is obtained parametrically, we change the variable, for example, if the result of the Laplace obtained is x, we replace it with $(-1)^n(-x)$

4.1. The application of Variational iteration method (VIM)
First we construct a correction functional which reads:

$$F_{n+1}(x) = F_n(x) + \int_0^x \lambda \left(\frac{d^4}{dx^4} F_n(\tau) + R((\frac{d^3}{dx^3} F_n(\tau)) F_n(\tau) \right.$$
$$\left. -(\frac{d}{dx} F_n(\tau)) \frac{d^2}{dx^2} F_n(\tau))) \right) d\tau \tag{4.3}$$

Linear part of Eq. (4.2) is solved by Laplace method and will be put equal $(-1)^n$ (according to above description):

$$s^4 L[F] - s^3 F(0) - s^2 D(F)(0) - s(D^{(2)})(F)(0)$$
$$-(D^{(3)})(F)(0) = (-1)^n \longrightarrow n = 4 \longrightarrow s^4 L[F] = 1 \tag{4.4}$$

$$F = L^{-1}[\frac{1}{s^4}] = \frac{x^3}{6} \tag{4.5}$$

$$x = (-1)^n \cdot (\tau - x) \tag{4.6}$$

Substituting Eq. (4.6) into Eq. (4.1), so the Lagrange multiplier can be obtaining in that form:

$$\lambda = 1/6 \ (\tau - x)^3 \tag{4.7}$$

As a result, we obtain the following iteration formula:

$$F_{n+1}(x) = F_n(x) + \int_0^x \frac{(\tau - x)^3}{6} (\frac{d^4}{dx^4} F_n(\tau) + R((\frac{d^3}{dx^3} F_n(\tau)) F_n(\tau)$$
$$-(\frac{d}{dx} F_n(\tau)) \frac{d^2}{dx^2} F_n(\tau))) d\tau \tag{4.8}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$F_0(x) = -1/2\,x^3 + 3/2\,x \tag{4.9}$$

Using the above variational formula (4.1) for $n = 0$, substituting Eq. (4.6) into Eq. (4.1) and after some simplifications, we have:

$$F_1(x) = -1/2\,x^3 + 3/2\,x + \frac{Rx^7}{280} \tag{4.10}$$

$$F_2(x) = -1/2\,x^3 + 3/2\,x + \frac{Rx^7}{280} + \frac{R^3x^{15}}{30576000} + \frac{R^2x^{11}}{92400} - \frac{x^9R^2}{3360} \tag{4.11}$$

In the same way, the rest of the components of the iteration formula can be obtained. With assumption of $Re = 1$, $F(x)$ would be as follow:

$$F(x) \cong -0.5\,x^3 + 1.5\,x + 0.0035\,x^7 + 0.00000003\,x^{15} - 0.00029\,x^9$$
$$+0.0000108\,x^{11} \tag{4.12}$$

5. Basic Idea of Adomian's Decomposition Method

We consider a general nonlinear equation in the form:

$$Lu + R(u) + F(u) = g \tag{5.1}$$

Where L is the operator of the highest-ordered derivatives which is assumed to be easily invertible, R the linear differential operator of less order than L, $F(u)$ presents the nonlinear terms and g is the source term. Thus we get:

$$Lu = g - R(u) - F(u) \tag{5.2}$$

The inverse L^{-1} is assumed to be an integral operator, by applying the inverse operator L^{-1} to the both sides of Eq. (40), we obtain:

$$u = f_0 + L^{-1}(g - R(u) - f(u)) \tag{5.3}$$

Where f_0 is the solution of homogeneous equation:

$$Lu = 0 \tag{5.4}$$

Involving the constants of integration. The integration constants involved in the solution of homogeneous equation (5.4) are to be determined by the initial or boundary condition, since the problem is an initial value or a boundary value problem. The ADM assumes that the unknown function $u(x)$ can be expressed by an infinite series of the form:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{5.5}$$

And the nonlinear operator $F(u)$ can be decomposed by an infinite series of polynomials given by:

$$F(x) = \sum_{n=0}^{\infty} A_n(x) \tag{5.6}$$

Where $u(x)$ will be determined recurrently, and A_n are the so-called polynomials of $u_0, u_1, u_2, ..., u_n$ defined by:

$$A_n = \frac{1}{n!} \cdot \frac{d^n}{d\lambda^n}[F(\sum_{n=0}^{\infty} \lambda^i \cdot u_i)]_{\lambda=0} \tag{5.7}$$

5.1. The application of Adomian Decomposition Method

In order to apply ADM to nonlinear equation in fluids problem, we rewrite Eqs. (2.8) In the following operator form, with assumption of $Re = 1$:

$$L_x\left(F\left(x\right)\right) = -\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}F\left(x\right)\right)F\left(x\right) + \left(\frac{\mathrm{d}}{\mathrm{d}x}F\left(x\right)\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}F\left(x\right) \tag{5.8}$$

Where the notation:

$$L_x = \frac{\mathrm{d}^4}{\mathrm{d}x^4} \tag{5.9}$$

Is the linear operator. By using the inverse operator, we can write Eq.s (5.9) in the following form:

$$F(x) = L_x^{-1}\left(-\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}F\left(x\right)\right)F\left(x\right) + \left(\frac{\mathrm{d}}{\mathrm{d}x}F\left(x\right)\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}F\left(x\right)\right) \tag{5.10}$$

Where the inverse operator is defined by:

$$L_x^{-1} = \int_0^x\int_0^x\int_0^x\int_0^x \mathrm{d}x\mathrm{d}x\mathrm{d}x\mathrm{d}x \tag{5.11}$$

Where:

$$N_1\left(F\left(x\right)\right) = \left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}F\left(x\right)\right)F\left(x\right) \tag{5.12}$$

$$N_2\left(F\left(x\right)\right) = -\left(\frac{\mathrm{d}}{\mathrm{d}x}F\left(x\right)\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}F\left(x\right) \tag{5.13}$$

The nonlinear operators are defined by the following infinite series:

$$N_i(F) = \sum_{n=0}^{\infty}A_{in}, i = 1, 2 \tag{5.14}$$

Where is called Adomian polynomials and defined by:

$$A_{i,n} = \frac{-1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}N_i[\sum_{k=0}^{n}\lambda^kF_k]\right]_{\lambda=0} \tag{5.15}$$

Hence we obtain the components series solution by the following recursive relation:

$$F_n(x) = L_x^{-1}\left(-\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}F_n\left(x\right)\right)F_n\left(x\right) + \left(\frac{\mathrm{d}}{\mathrm{d}x}F_n\left(x\right)\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_n\left(x\right)\right) \tag{5.16}$$

Where , Adomians polynomials formula, Eq. (5.15), is easy to set computer cod to get as many polynomials as we need in the calculation. We can give the first few Adomians polynomials of the as:

$$A_{1,0} = N_1\left(\sum_{K=0}^{0}\lambda^k\cdot F_k(x)\right) = \frac{3}{2}x^3 - \frac{9}{2}x \tag{5.17}$$

$$A_{2,0} = N_2 \left(\sum_{K=0}^{0} \lambda^k \cdot F_k(x) \right) = 3 \left(-\frac{3}{2}x^2 + \frac{9}{2} \right) x \tag{5.18}$$

And so on, the rest of the polynomials can be constructed in a similar manner. Using the recursive relation, Eq. (5.16) and Adomians polynomials formula, Eq. (5.15), with the initial conditions, Eq. (2.7), gives:

$$F_0(x) = \frac{3}{2}x - \frac{1}{2}x^3 \tag{5.19}$$

$$F_1(x) = -\frac{3}{40}x^5 + \frac{1}{140}x^7 \tag{5.20}$$

$$F_2(x) = \frac{1}{4480}x^9 - \frac{1}{5600}x^{11} + \frac{3}{400400}x^{13} \tag{5.21}$$

Where:

$$F(x) = \frac{3}{2}x - \frac{1}{2}x^3 - \frac{3}{40}x^5 + \frac{1}{140}x^7 + \frac{1}{4480}x^9 - \frac{1}{5600}x^{11} + \frac{3}{400400}x^{13} \tag{5.22}$$

And so on. In the same manner the rest of the components of the iteration formula can be obtained.

6. Ajbari-Ganki's Method (AGM)

Boundary conditions and initial conditions are required for analytical methods of each linear and nonlinear differential equation according to the physics of the problem. Therefore, we can solve every differential equation with any degrees. In order to comprehend the given method in this paper, two differential equations governing on engineering processes will be solved in this new manner.

In accordance with the boundary conditions, the general manner of a differential equation is as follows: The nonlinear differential equation of p which is a function of u, the parameter u which is a function of x and their derivatives are considered as follows:

$$P_k : f\left(u, u^{'}, u^{''}, ..., u^m \right) = 0 \, ; u = u(x) \tag{6.1}$$

Boundary conditions:

$$\begin{aligned} u(x) &= u_0, u^{'}(x) = u_1, ..., u^{m-1}(x) = u_{m-1} at x = 0 \\ u(x) &= u_{L_0}, u^{'}(x) = u_{L_1}, ..., u^{m-1}(x) = u_{L_{m-1}} at x = L \end{aligned} \tag{6.2}$$

To solve the first differential equation with respect to the boundary conditions in x=L in Eq. (6.2), the series of letters in the n-th order with constant coefficients which is the answer of the first differential equation is considered as follows:

$$u(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x^1 + a_2 x^2 + ... + + a_n x^n \tag{6.3}$$

The more precise answer of Eq. (6.2), the more choice of series sentences from Eq. (6.3). In applied problems, approximately five or six sentences from the series are enough to solve nonlinear differential equations. In the answer of differential Eq. (6.3) regarding the series from degree (n), there are (n+1) unknown coefficients that need (n+1) equations to be specified. The boundary conditions of Eq. (6.2)

are used to solve a set of equations which is consisted of (n+1) ones. The boundary conditions are applied on the functions such as follows:

a) The application of the boundary conditions for the answer of differential Eq. (6.3) is in the form of: When x = 0:

$$u(0) = a_0 = u_0$$
$$u'(0) = a_1 = u_1$$
$$u''(0) = a_2 = u_2$$
$$. = . = .$$
$$. = . = .$$
$$. = . = .$$

(6.4)

And when $x = L$:

$$u(L) = a_0 + a_1 L + a_2 L^2 + ... + a_n L^n = u_{L_0}$$
$$u'(L) = a_1 + 2a_2 L + 3a_3 L^2 + ... + na_n L^{n-1} = u_{L_1}$$
$$u''(L) = 2a_2 + 6a_3 L + 12a_4 L^2 ... + n(n-1)a_n L^{n-2} = u_{L_{m-1}}$$
$$. \qquad = \quad . \quad = \quad .$$
$$. \qquad = \quad . \quad = \quad .$$
$$. \qquad = \quad . \quad = \quad .$$

(6.5)

b) After substituting Eq. (6.5) into Eq. (6.2), the application of the boundary conditions on differential Eq. (6.2) is done according to the following procedure:

$$p_0 : f\left(u(0), u'(0), u''(0), ..., u^m(0)\right)$$
$$p_1 : f\left(u(L), u'(L), u''(L), ..., u^m(L)\right)$$
$$. : \quad . \quad . \quad . \quad .$$
$$. : \quad . \quad . \quad . \quad .$$

(6.6)

With regard to the choice of n ;$(n \prec m)$ sentences from Eq. (6.4) and in order to make a set of equations which is consisted of (n+1) equations and (n+1) unknowns, we confront with a number of additional unknowns which are indeed the same coefficients of Eq. (6.4). Therefore, to remove this problem, we should derive m times from Eq. (6.4) according to the additional unknowns in the afore-mentioned sets of differential equations and then applying the boundary conditions on them.

$$p'_k : f\left(u', u'', u''', ..., u^{m+1}\right)$$
$$p''_k : f\left(u'', u''', u'''', ..., u^{m+2}\right)$$
$$. : \quad . \quad . \quad . \quad .$$
$$. : \quad . \quad . \quad . \quad .$$

(6.7)

c) Application of the boundary conditions on the derivatives of the differential equation P_k in Eq. (6.8) is done in the form of:

$$p'_k : f\left(u'(0), u''(0), u'''(0), ..., u^{m+1}(0)\right)$$
$$p'_k : f\left(u'(L), u''(L), u'''(L), ..., u^{m+1}(L)\right)$$

(6.8)

$$p_k'' : f\left(u''(0), u'''(0), u''''(0), ..., u^{m+2}(0)\right)$$

$$p_k'' : f\left(u''(L), u'''(L), u''''(L), ..., u^{m+2}(L)\right) \tag{6.9}$$

$(n+1)$ Equations can be made from Eq. (6.5) to Eq. (6.9) so that unknown coefficients of Eq. (64) take for example $a_0, a_1, a_2.a_n$ will be computed. The answer of the nonlinear differential Eq. (6.2) will be gained by determining coefficients of Eq. (6.4).

6.1. *Solving the differential equation with AGM*

First of all we rewrite the problem Eq. (2.8) in the following order:

$$g(x) = \frac{\mathrm{d}^4}{\mathrm{d}x^4}F(x) + R\left(\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}F(x)\right)F(x) - \left(\frac{\mathrm{d}}{\mathrm{d}x}F(x)\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x)\right) = 0 \tag{6.10}$$

In AGM, the answer of the differential equation is considered as a finite series of polynomials with constant coefficients as follows:

$$F(x) = \sum_{i=0}^{10} a_i \cdot x^i = a_10x^{10} + a_9x^9 + a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4$$

$$+a_3x^3 + a_2x^2 + a_1x^1 + a_0 \tag{6.11}$$

It is notable that in the afore-mentioned equation, the constant coefficients a_0 to a_{10} are obtained by applying the introduced boundary conditions.

6.2. *Applying boundary conditions*

In AGM, the boundary conditions are applied in order to compute constant coefficients of Eq. (6.11) in two ways as follows:

a) Applying the boundary conditions on Eq. (6.11) is expressed as follows:

$$F = F(BC) \tag{6.12}$$

It is notable that BC is the abbreviation of boundary conditions. According to the above explanations, the boundary conditions are applied on Eq. (6.11) in the following form:

$$F(0) = 0 \qquad as \qquad a_0 = 0 \tag{6.13}$$

$$F''(0) = 0 \qquad therefore \qquad 2a_2 = 0 \tag{6.14}$$

$$F(1) = 1 \qquad then \qquad a_{10} + a_9 + a_8 + a_7 + a_6 + a_5 + a_4 + a_3 + a_2$$
$$+a_1 + a_0 = 1 \tag{6.15}$$

$$F'(1) = 0 \qquad and finally \qquad 10a_{10} + 9a_9 + 8a_8 + 7a_7 + 6a_6 + 5a_5$$
$$+4a_4 + 3a_3 + 2a_2 + a_1 = 0 \tag{6.16}$$

b) Applying the boundary conditions on the main differential equation, which in this case study is Eq. (6.11), and also on its derivatives is done after substituting Eq. (6.12) into the main differential equation as follows:

$$g(F(x)) \quad \rightarrow \quad g(F(BC)) = 0 \quad , \quad g'(F(BC)) \quad , \quad ... \tag{6.17}$$

So, after substituting Eq. (6.12) which has been considered as the answer of the main differential equation into Eq. (6.11), the initial conditions are applied on the obtained equation and also on its derivatives on the basis of Eq. (6.17) as follows:

$$g(F(0)): \quad \rightarrow \quad 24a_4 + Re(6a_0a_3 - 2a_1a_2) = 0 \tag{6.18}$$

And then:

$$g'(F(0)): \quad \rightarrow \quad 120\,a_5 + Re\left(24\,a_0a_4 - 4\,a_2{}^2\right) = 0 \tag{6.19}$$

$$g''(F(0)): \quad \rightarrow \quad 720\,a_6 + Re\left(120\,a_0a_5 + 24\,a_1a_4 - 24\,a_2a_3\right) = 0 \tag{6.20}$$

$$g'''(F(0)): \quad \rightarrow \quad 5040\,a_7 + Re\left(720\,a_0a_6 + 240\,a_1a_5 - 48\,a_2a_4 - 72\,a_3{}^2\right)$$
$$= 0 \tag{6.21}$$

$$g^4(F(0)): \quad \rightarrow \quad 40320\,a_8 + Re\Big(5040\,a_0a_7 + 2160\,a_1a_6 + 240\,a_2a_5$$
$$-720\,a_3a_4\Big) = 0 \tag{6.22}$$

$$g^5(F(0)): \quad \rightarrow \quad 362880\,a_9 + Re\Big(40320\,a_0a_8 + 20160\,a_1a_7 + 5760\,a_2a_6$$
$$-2880\,a_3a_5 - 2880\,a_4{}^2\Big) = 0 \tag{6.23}$$

$$g^6(F(0)): \quad \rightarrow \quad 3628800\,a_{10} + R\Big(362880\,a_0a_9 + 201600\,a_1a_8 + 80640\,a_2a_7$$
$$-40320\,a_4a_5\Big) = 0 \tag{6.24}$$

By choosing constant amount below:

$$Re = 1 \tag{6.25}$$

By solving a set of algebraic equations which is consisted of eleven equations with eleven unknowns from Eq.s $(6.14 - (6.17))$ and Eq.s $(6.19 - (6.25))$, the constant coefficients of Eq. (6.12) can easily be gained.

$$a_0 = 0, a_1 = 1.506495911, a_2 = 0, a_3 = -0.5098992971, a_4 = 0, a_5 = 0$$
$$, a_6 = 0, a_7 = 0.003714247046, a_8 = 0, a_9 = -0.0003108609993, a_{10} = 0 \tag{6.26}$$

Eq. (6.10) which is the solution of the proposed problem is rewritten in the form of:

$$F(x) = +1.506495911\,x - 0.5098992971\,x^3 + 0.003714247046\,x^7$$
$$-0.0003108609993\,x^9 \tag{6.27}$$

7. Conclsuions

The current study through two-dimension nonlinear viscous flow and mainly solving the obtained equation with different methods presents that AGM is an effective method for solving nonlinear differential equations and by comparing with numerical method (Ruge-Kutta,), HPM, VIM, ADM. With acceptance of accuracy of mentioned methods for solving the equation by doing the solving process it would be definitely obtained that AGM is so close to exact amount of the equation and have enough accuracy and efficiency through other methods. For revealing the simplicity of AGM, our trial function for assumed solution include eleven constants which according to basic idea of this method set of simple algebraic calculation has been done otherwise the other methods use difficult process for obtaining the solution. It can precisely obtained through figures and table that AGM would be able to solve nonlinear equations with high nonlinearity with different amount for its constant part which in this paper is Reynolds number.

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Figure Caption

Fig. 1. A comparison of the results of the methods.

Fig.2. A comparison between the first derivatives of the methods.

Fig.3. A comparison between the second derivatives of the methods.

Fig.4.Comparing the charts of the achieved solution by AGM for various amounts of Re.

Fig.5. A comparison amongst the obtained charts by AGM for the first derivative of the achieved solution in terms of different values of Re.

Fig.6.A comparison amongst the obtained charts by AGM for the second derivative of the achieved solution in terms of different values of Re.

Fig.7..A comparison between AGM and numeric method

Table Caption

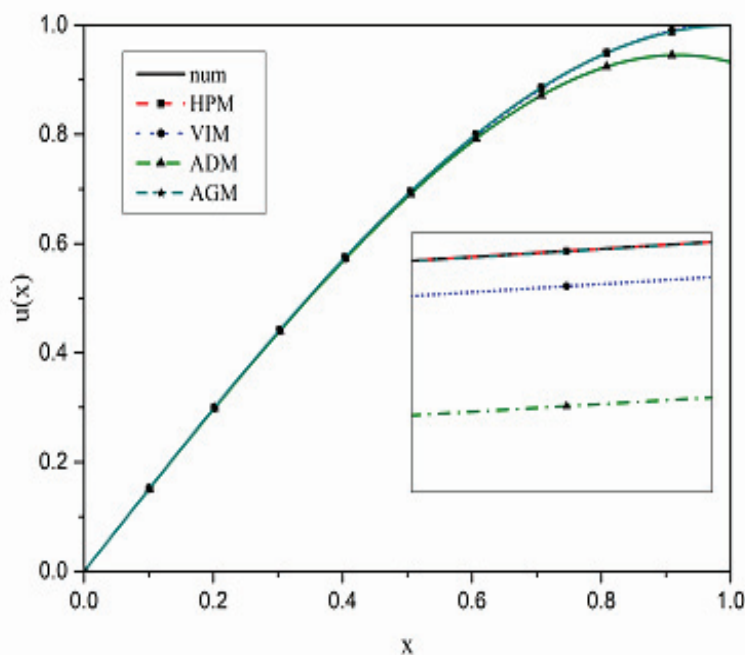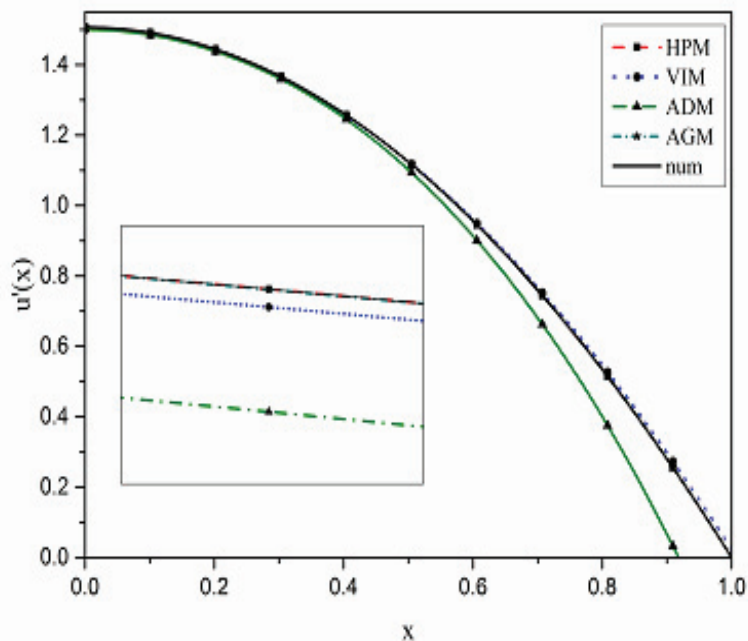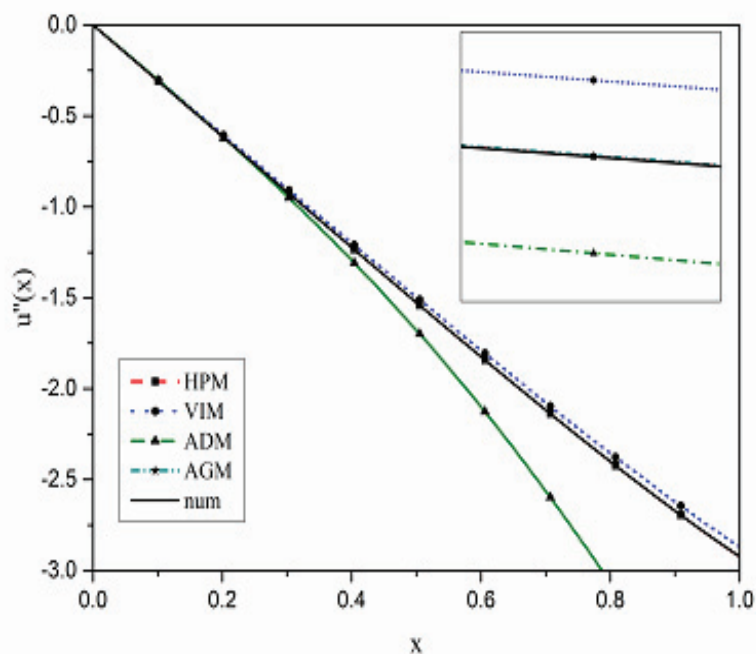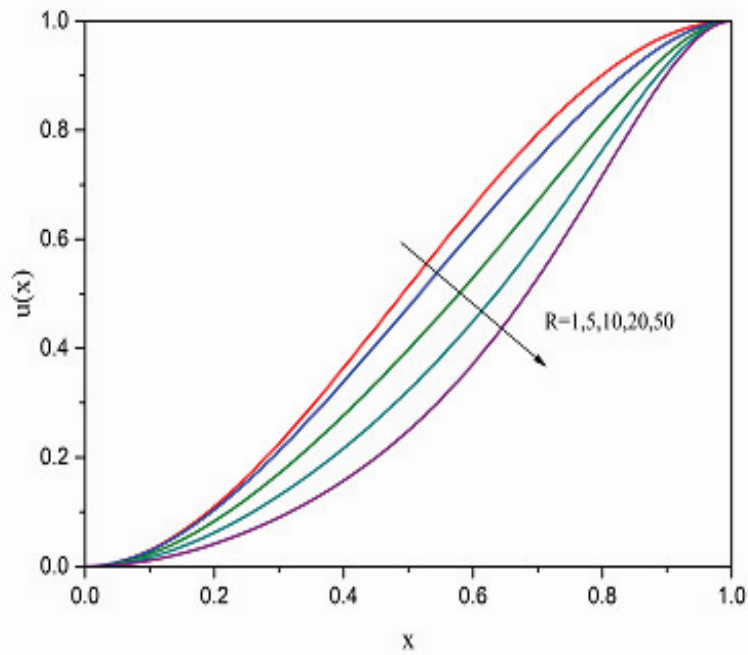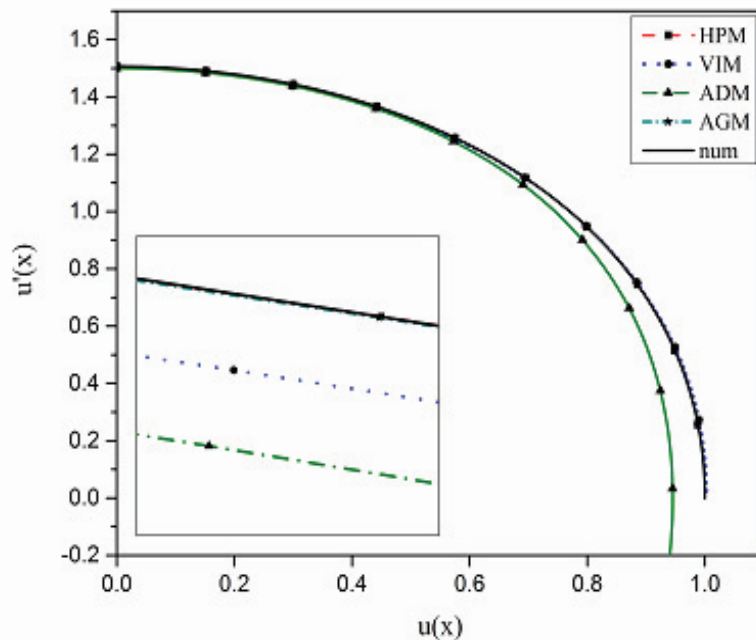**Table.1**..The obtained results in accordance with the Numerical Solution of Eq. (1).

Fig. 1. A comparison of the results of the Methods.

Fig.2. A comparison between the first derivative of the Methods.

Fig.3. A comparisoo between the second derivatives of the methnds.

cig.4. Comparing the chtrts of ahe aFhieved solution by AGM for various amounts of **Re.**

Fig.5.A comparison aaongst the obtained charts by AGM for the first derivative of the mchieved solution in terms of different values of **Re.**
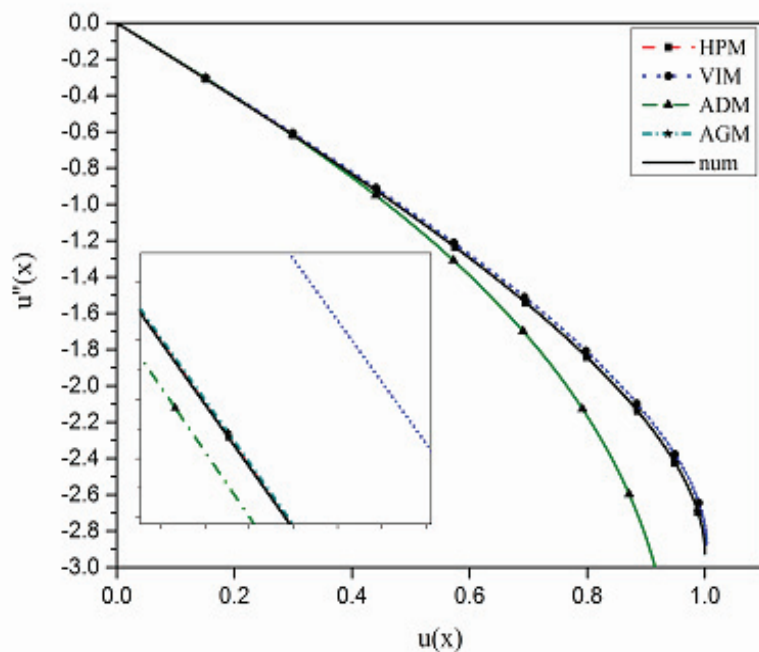
Fig.6. A comparison amongst the obtained chauos by AGM for the second derivative of the achieved selrtion in terms of differont values tf **Re.**
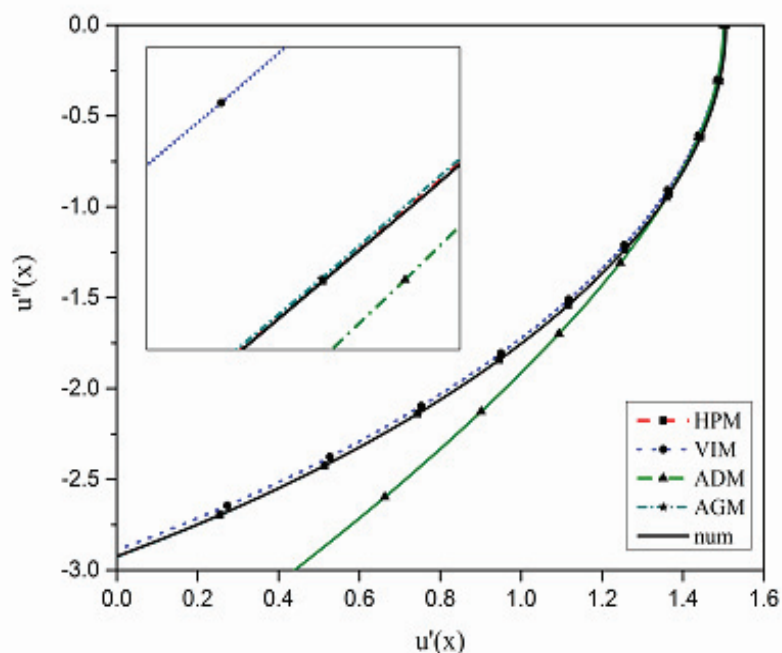

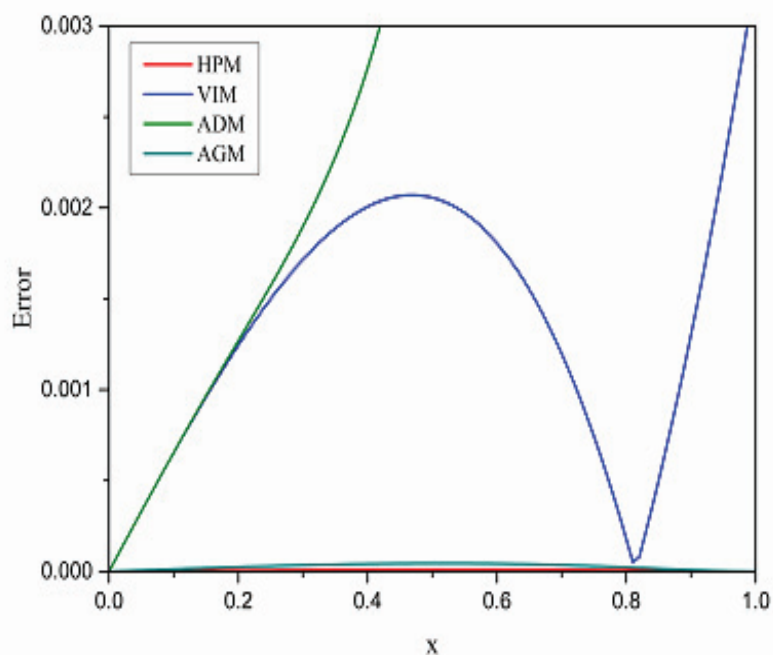**Fig.7.** A comearison betwepn AGM and eumnric method.

Fig.8. A comparison between Errors of methods.

Table.1. The obtained results in accordaece with the bumnrical Solution of Eq. (2.1).

x	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**
$F(x)$	0	0.29724	0.57001	0.79390	0.94488	1
$F'(x)$	1.50662	1.44542	1.26190	0.95692	0.53369	0
$F''(x)$	0	0.61202	1.22259	1.82468	2.40137	2.92327