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Global attractor for a nonlocal hyperbolic problem on $\mathcal{R}^N$

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Abstract

We consider the quasilinear Kirchhoff's problem

$$u_{tt} - \phi(x)||\nabla u(t)||^2 \Delta u + f(u) = 0, \;\; x \in \mathcal{R}^N, \;\; t \geq 0,$$

with the initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq 3$, $f(u) = |u|^a u$ and $(\phi(x))^{-1} \in L^{N/2}(\mathcal{R}^N) \cap L^\infty(\mathcal{R}^N)$ is a positive function. The purpose of our work is to study the long time behaviour of the solution of this equation. Here, we prove the existence of a global attractor for this equation in the strong topology of the space $\mathcal{X}_1 =: \mathcal{D}^{1,2}(\mathcal{R}^N) \times L^2_g(\mathcal{R}^N)$. We succeed to extend some of our earlier results concerning the asymptotic behaviour of the solution of the problem.

Keywords: quasilinear hyperbolic equations; Kirchhoff strings; global attractor; generalised Sobolev spaces; weighted L^p Spaces.
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1. Introduction

Our aim in this work is to study the following quasilinear hyperbolic initial value problem

$$\begin{aligned} &u_{tt} - \phi(x)||\nabla u(t)||^2 \Delta u + f(u) = 0, \quad x \in \mathcal{R}^N, \quad t \geq 0, \\ &u(x,0) = u_0(x), u_t(x,0) = u_1(x), \qquad x \in \mathcal{R}^N, \end{aligned} \tag{1.1}$$

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with initial conditions $u_0,\ u_1$ in appropriate function spaces, $N \geq 3$. The case of $N = 1$, the first equation of (1.1) describes the nonlinear vibrations of an elastic string. Throughout the paper we assume that the functions $\phi, g : \mathcal{R}^N \longrightarrow \mathcal{R}$ satisfy the following condition:

($\mathcal{G}$) $\phi(x) > 0$, for all $x \in \mathcal{R}^N$ and $(\phi(x))^{-1} =: g(x) \in L^{N/2}(\mathcal{R}^N) \cap L^{\infty}(\mathcal{R}^N)$.

This class will include functions of the form

$$\phi(x) \approx c_0 + \epsilon |x|^a, \ \ \epsilon > 0, \ a > 0,$$

resembling phenomena of slowly varying wave speed around the constant speed c_0. Many results treat the case of $\phi(x) =$ constant (in bounded or unbounded domains). It must be noted, that this case is proved to be totally different from the case of $\phi(x) \to c_{\pm} > 0$, as $x \to \pm\infty$.

Kirchhoff in 1883 proposed the so called Kirchhoff string model in the study of oscillations of stretched strings and plates

$$ph\frac{\vartheta^2 u}{\vartheta t^2} + \delta \frac{\vartheta u}{\vartheta t} = \left\{ p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\vartheta u}{\vartheta x}\right)^2 dx \right\} \frac{\vartheta^2 u}{\vartheta x^2} + f, \tag{1.2}$$

where we have $0 < x < L,\ t \geq 0$, and we have to mention that $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modules, p the mass density, h the cross-section area, L the length, p_0 the initial axial tension, δ the resistance modules and f the external force (see [7]). When $p_0 = 0$ the equation is considered to be of degenerate type and the equation models an unstretched string or its higher dimensional generalization. Otherwise it is of nondegenerate type.

In the case treated here the problem becomes complicated because the equation does not give rise to compact operators. The homogeneous Sobolev spaces combined with equivalent weighted L^p spaces, is the appropriate space to overcome these difficulties. In our paper we assume that $f(u) = |u|^a u$, in order to study the behavior of the solutions for this kind of equations. This case is rather interesting in the case of the homogeneous Sobolev spaces.

In bounded domains there is a vast literature concerning the attractors of semilinear waves equations. We refer to the monographs [2], [13]. Also in the paper [3] the existence of global attractor in a weak topology is discussed for a general dissipative wave equation. K. Ono [9], for $\delta \geq 0$, has proved global existence, decay estimates, asymptotic stability and blow up results for a degenerate non-linear wave equation of Kirchhoff type with a strong dissipation .

On the other hand, it seems that very few results are achieved for the unbounded domain case. In our previous work (see [10]), we proved global existence and blow-up results for an equation of Kirchhoff type in all of $\mathcal{R}^N$. Also, in [12] we proved the existence of compact invariant sets for the same equation. Recently, in [11] we studied the stability of the origin for the generalized equation of Kirchhoff strings on $\mathcal{R}^N$, using central manifold theory. Also, Karahalios and Stavrakakis [4]- [6] proved existence of global attractors and estimated their dimension for a semilinear dissipative wave equation on $\mathcal{R}^N$.

The presentation of this paper has as follows: In Section 2, we discuss the space setting of the problem and the necessary embedding for constructing the evolution triple. In Section 3, we prove existence of an absorbing set for our problem in the energy space $\mathcal{X}_0$. Finally in Section 4, we prove that there exists a global *attractor like* invariant set $\mathcal{A}_{X^s_{\delta_1}}$ in the strong topology of the energy space $\mathcal{X}_1 =: \mathcal{D}^{1,2}(\mathcal{R}^N) \times L^2_g(\mathcal{R}^N)$.

Notation: We denote by B_R the open ball of $\mathcal{R}^N$ with center 0 and radius R. Sometimes for simplicity we use the symbols C_0^∞, $\mathcal{D}^{1,2}$, L^p, $1 \leq p \leq \infty$, for the spaces $C_0^\infty(\mathcal{R}^N)$, $\mathcal{D}^{1,2}(\mathcal{R}^N)$, $L^p(\mathcal{R}^N)$, respectively; $\| \cdot \|_p$ for the norm $\| \cdot \|_{L^p(\mathcal{R}^N)}$, where in case of $p = 2$ we may omit the index. These spaces, as we will see later on when we define them, are very useful for our problem. They are playing a very important role for the product space $\mathcal{X}_0$, as we will see in Section 2. Finally, the symbol $=:$ is used for definitions.

2. Space setting; formulation of the problem

As we have already seen in [10], the space setting for the initial conditions and the solutions of our problem is the product space

$$\mathcal{X}_0 =: D(A) \times \mathcal{D}^{1,2}(\mathcal{R}^N), \qquad N \geq 3.$$

We also define the space $\mathcal{X}_1 =: \mathcal{D}^{1,2}(\mathcal{R}^N) \times L^2_g(\mathcal{R}^N)$, with the following associated norm $e_1(u(t)) =: ||u||^2_{\mathcal{D}^{1,2}} + ||u_t||^2_{L^2_g}$. We have that the embedding $\mathcal{X}_0 \subset \mathcal{X}_1$ is compact. The homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathcal{R}^N)$ is defined as the closure of $C_0^\infty(\mathcal{R}^N)$ functions with respect to the following energy norm $||u||^2_{\mathcal{D}^{1,2}} =: \int_{\mathcal{R}^N} |\nabla u|^2 dx$. It is known that

$$\mathcal{D}^{1,2}(\mathcal{R}^N) = \left\{ u \in L^{\frac{2N}{N-2}}(\mathcal{R}^N) : \nabla u \in (L^2(\mathcal{R}^N))^N \right\}$$

and $\mathcal{D}^{1,2}(\mathcal{R}^N)$ is embedded continuously in $L^{\frac{2N}{N-2}}(\mathcal{R}^N)$, that is, there exists $k > 0$ such that

$$||u||_{\frac{2N}{N-2}} \leq k||u||_{\mathcal{D}^{1,2}}. \tag{2.1}$$

The space $D(A)$ is going to be introduced and studied later in this section. We shall frequently use the following generalized version of Poincaré's inequality

$$\int_{\mathcal{R}^N} |\nabla u|^2 dx \geq \alpha \int_{\mathcal{R}^N} g u^2 dx, \tag{2.2}$$

for all $u \in C_0^\infty$ and $g \in L^{N/2}$, where $\alpha =: k^{-2}||g||^{-1}_{N/2}$ (see [1, Lemma 2.1]). It is shown that $\mathcal{D}^{1,2}(\mathcal{R}^N)$ is a separable Hilbert space. The space $L^2_g(\mathcal{R}^N)$ is defined to be the closure of $C_0^\infty(\mathcal{R}^N)$ functions with respect to the inner product

$$(u, v)_{L^2_g(\mathcal{R}^N)} =: \int_{\mathcal{R}^N} g u v dx. \tag{2.3}$$

It is clear that $L^2_g(\mathcal{R}^N)$ is also a separable Hilbert space. Moreover, we have the following compact embedding.

Lemma 2.1. *Let $g \in L^{N/2}(\mathcal{R}^N) \cap L^\infty(\mathcal{R}^N)$. Then the embedding $\mathcal{D}^{1,2} \subset L^2_g$ is compact. Also, let $g \in L^{\frac{2N}{2N-pN+2p}}(\mathcal{R}^N)$. Then the following continuous embedding $\mathcal{D}^{1,2}(\mathcal{R}^N) \subset L^p_g(\mathcal{R}^N)$ is valid, for all $1 \leq p \leq 2N/(N-2)$.*

Proof . For the proof we refer to [5, Lemma 2.1]. □

To study the properties of the operator $-\phi\Delta$, we consider the equation

$$-\phi(x)\Delta u(x) = \eta(x), \qquad x \in \mathcal{R}^N, \tag{2.4}$$

without boundary conditions. Since for every $u,\ v \in C_0^\infty(\mathcal{R}^N)$ we have

$$(-\phi\Delta u,\ v)_{L_g^2} = \int_{\mathcal{R}^N} \nabla u \nabla v\ dx, \tag{2.5}$$

we may consider equation (2.4) as an operator equation of the form

$$A_0 u = \eta, \quad A_0 : D(A_0) \subseteq L_g^2(\mathcal{R}^N) \to L_g^2(\mathcal{R}^N), \quad \eta \in L_g^2(\mathcal{R}^N). \tag{2.6}$$

The operator $A_0 = -\phi\Delta$ is a symmetric, strongly monotone operator on $L_g^2(\mathcal{R}^N)$. Hence, Friedrich's extension theorem is applicable. The energy scalar product given by (2.5) is

$$(u,\ v)_E = \int_{\mathcal{R}^N} \nabla u \nabla v dx$$

and the energy space X_E is the completion of $D(A_0)$ with respect to $(u,\ v)_E$. It is obvious that the energy space is the homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathcal{R}^N)$. The energy extension $A_E = -\phi\Delta$ of A_0,

$$-\phi\Delta : \mathcal{D}^{1,2}(\mathcal{R}^N) \ \to \mathcal{D}^{-1,2}(\mathcal{R}^N), \tag{2.7}$$

is defined to be the duality mapping of $\mathcal{D}^{1,2}(\mathcal{R}^N)$. We define $D(A)$ to be the set of all solutions of equations (2.4), for arbitrary $\eta \in L_g^2(\mathcal{R}^N)$. Friedrich's extension A of A_0 is the restriction of the energy extension A_E to the set $D(A)$. The operator $A = -\phi\Delta$ is self-adjoint and therefore graph-closed. Its domain $D(A)$, is a Hilbert space with respect to the graph scalar product

$$(u,\ v)_{D(A)} = (u,\ v)_{L_g^2} + (Au,\ Av)_{L_g^2}, \ \text{ for all } \ u,\ v \in D(A).$$

The norm induced by the scalar product is

$$||u||_{D(A)} = \left\{ \int_{\mathcal{R}^N} g|u|^2\ dx + \int_{\mathcal{R}^N} \phi|\Delta u|^2\ dx \right\}^{\frac{1}{2}},$$

which is equivalent to the norm

$$||Au||_{L_g^2} = \left\{ \int_{\mathcal{R}^N} \phi|\Delta u|^2\ dx \right\}^{\frac{1}{2}}.$$

So we have established the evolution quartet

$$D(A) \subset \mathcal{D}^{1,2}(\mathcal{R}^N) \subset L_g^2(\mathcal{R}^N) \subset \mathcal{D}^{-1,2}(\mathcal{R}^N), \tag{2.8}$$

where all the embeddings are dense and compact.

Finally, we give the definition of weak solutions for the problem (1.1).

Definition 2.2. A weak solution of the problem (1.1) is a function u such that

(i) $u \in L^2[0,T; D(A)]$, $u_t \in L^2[0,T; \mathcal{D}^{1,2}(\mathcal{R}^N)]$, $u_{tt} \in L^2[0,T; L_g^2(\mathcal{R}^N)]$,

(ii) for all $v \in C_0^\infty([0,T] \times (\mathcal{R}^N))$, satisfies the generalized formula

$$\begin{aligned} \int_0^T (u_{tt}(\tau), v(\tau))_{L_g^2}\, d\tau + \int_0^T \left(||\nabla u(t)||^2 \int_{\mathcal{R}^N} \nabla u(\tau) \nabla v(\tau) dx d\tau \right) \\ + \int_0^T (f(u(\tau)), v(\tau))_{L_g^2}\, d\tau = 0, \end{aligned} \tag{2.9}$$

where $f(s) = |s|^a s$, and

(iii) satisfies the initial conditions

$$u(x,0) = u_0(x), \ \ u_0 \in D(A), \quad u_t(x,0) = u_1(x), \ \ u_1 \in \mathcal{D}^{1,2}(\mathcal{R}^N).$$

3. Existence of an absorbing set

In this section we prove existence of an absorbing set for our problem (1.1) in the energy space $\mathcal{X}_0$. First, we give existence and uniqueness results for the problem (1.1) using the space setting established previously. Let $(m, m_t) \in C\left(0, T; D(A) \times \mathcal{D}^{1,2}\right)$ be given. In order to obtain a local existence result for the problem (1.1), we need information concerning the solvability of the corresponding nonhomogeneous linearized (around the function m) problem restricted to the sphere B_R:

$$\begin{aligned} &u_{tt} - \phi(x)||\nabla m(t)||^2 \Delta u + f(m) = 0, \quad (x,t) \in B_R \times (0,T), \\ &u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in B_R, \\ &u(x,t) = 0, \qquad (x,t) \in \partial B_R \times (0,T). \end{aligned} \tag{3.1}$$

Then, we have the following local existence result:

Theorem 3.1. *Consider that* $(u_0,\ u_1) \in D(A) \times \mathcal{D}^{1,2}$ *and satisfy the nondegenerate condition*

$$||\nabla u_0|| > 0. \tag{3.2}$$

Then there exists $T = T(||u_0||_{D(A)},\ ||\nabla u_1||) > 0$ *such that the problem* (1.1) *admits a unique local weak solution* u *satisfying*

$$u \in C(0,T; D(A)) \ \ and \ \ u_t \in C(0,T; \mathcal{D}^{1,2}).$$

Moreover, at least one of the following statements holds true, either

(i) $T = +\infty$, *or*
(ii) $e(u(t)) =: ||u(t)||^2_{D(A)} + ||u_t(t)||^2_{\mathcal{D}^{1,2}} \to \infty$, *as* $t \to T_-$.

Proof . For the proof we refer to [10, Theorem 3.2]. □

Next, to prove the existence of an absorbing set in the space $\mathcal{X}_0$, we set $v = u_t + \varepsilon u$ for sufficiently small ε. Then following [13, page 207], for calculation needs we rewrite (3.1) as follows

$$v_t - \varepsilon v + \left(-\phi(x)||\nabla m||^2 \Delta + \varepsilon^2\right) u + f(m) = 0. \tag{3.3}$$

Lemma 3.2. *Assume that* $f(u)$ *is a* C^1*-function,* $a \geq 0$, $N \geq 3$. *If the initial data* $(u_0,\ u_1) \in D(A) \times \mathcal{D}^{1,2}$ *and satisfy the condition*

$$||\nabla u_0|| > 0, \tag{3.4}$$

then we have that

$$||\nabla u(t)|| > 0, \ \ for\ all\ \ t \geq 0. \tag{3.5}$$

Proof . Let $u(t)$ be a unique solution of the problem (1.1) in the sense of Theorem 3.1 on $[0, T)$. Multiplying the first equation in (1.1) by $-2\Delta u_t$ and integrating it over $\mathcal{R}^N$, we have

$$\begin{aligned} \frac{d}{dt}||\nabla u_t(t)||^2 \ &+ \ ||\nabla u(t)||^2 \frac{d}{dt} ||u(t)||^2_{D(A)} \\ &+ \ 2(f(u(t)),\ \Delta u_t(t)) = 0 \end{aligned} \tag{3.6}$$

Since $||\nabla u_0|| > 0$, we see that $||\nabla u(t)|| > 0$ near $t = 0$. Let

$$T =: \sup\{t \in [0, +\infty) : ||\nabla u(s)|| > 0 \ \text{ for } \ 0 \leq s < t\},$$

then $T > 0$ and $||\nabla u(t)|| > 0$ for $0 \leq t < T$. By contradiction we may prove that $T = +\infty$. □

For the existence of the absorbing set we have to prove the following Theorem.

Theorem 3.3. *Assume that* $0 \leq a < 2/(N-2)$, $N \geq 3$, $||\nabla u_0|| > 0$, $(u_0,\ u_1) \in D(A) \times \mathcal{D}^{1,2}$ *and*

$$\rho_1 > 4\alpha^{-1/2} R^2 c_3^2, \tag{3.7}$$

where $\rho_1 = \min(-\varepsilon, \varepsilon, 2\varepsilon)$, $c_3 =: \left(\max\left\{1,\ M_0^{-1}\right\}\right)^{1/2}$. *Then the ball* $\mathcal{B}_0$ *is an absorbing set in the energy space* $\mathcal{X}_0$. *We also obtain that the unique local solution defined by Theorem 3.1 exists globally in time.*

Proof . Given the constants $T > 0$, $R > 0$, we introduce the two parameter space of solutions

$$X_{T,R} =: \quad \left\{m \in C\left(0, T; D(A)\right) : m_t \in C\left(0, T; \mathcal{D}^{1,2}\right), m(0) = u_0, \right. \\ \left. m_t(0) = u_1,\ e(m) \leq R^2,\ t \in [0,T]\right\},$$

where $e(m) =: ||m_t||^2_{\mathcal{D}^{1,2}} + ||m||^2_{D(A)}$. Also u_0 satisfies the nondegenerate condition $||\nabla u_0|| > 0$. The set $X_{T,R}$ is a complete metric space under the distance $d(u,\ v) =: \sup_{0\leq t\leq T} e(u(t) - v(t))$. We may introduce the notation

$$M_0 =: \frac{1}{2}||\nabla u_0||^2,\ T_0 =: \sup\left\{t \in [0,\ \infty) :\ ||\nabla m(s)||^2 > M_0,\ 0 \leq s \leq t\right\}.$$

By condition $||\nabla u_0|| > 0$, we may see that $M_0 > 0$, $T_0 > 0$ and $||\nabla m(t)|| > M_0 > 0$, for all $t \in [0, T_0]$. Multiplying equation (3.1) by

$$gAv = g(-\varphi\Delta)v = -\Delta v = -\Delta(u_t + \varepsilon u),$$

and integrating over $\mathcal{R}^N$, we obtain (using Hölder inequality with $p^{-1} = \frac{1}{N}$, $q^{-1} = \frac{N-2}{2N}$, $r^{-1} = \frac{1}{2}$)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&\left\{||m||^2_{\mathcal{D}^{1,2}}||u||^2_{D(A)} + ||v||^2_{\mathcal{D}^{1,2}} - \frac{\varepsilon^2}{2}||u||^2_{\mathcal{D}^{1,2}}\right\} \\ &- \varepsilon||v||^2_{\mathcal{D}^{1,2}} + \varepsilon||m||^2_{\mathcal{D}^{1,2}}||u||^2_{D(A)} - \varepsilon^3||u||^2_{\mathcal{D}^{1,2}} \\ &\leq \left|\left(\frac{d}{dt}||m||^2_{\mathcal{D}^{1,2}}\right)||u||^2_{D(A)}\right| + k_2||m||^a_{L^{Na}}||\nabla m||_{L^{\frac{2N}{N-2}}}||\nabla v||. \end{aligned} \tag{3.8}$$

We observe that

$$\theta(t) =: ||m||^2_{\mathcal{D}^{1,2}}||u||^2_{D(A)} + ||v||^2_{\mathcal{D}^{1,2}} - \frac{\varepsilon^2}{2}||u||^2_{\mathcal{D}^{1,2}} \geq M_0||u||^2_{D(A)} + ||u_t||^2_{\mathcal{D}^{1,2}} \geq c_3^{-2}e(u), \tag{3.9}$$

with $c_3 =: \left(\max\left\{1,\ M_0^{-1}\right\}\right)^{1/2}$. We also have

$$\begin{aligned} \left|\left(\frac{d}{dt}||m||^2_{\mathcal{D}^{1,2}}\right)||u||^2_{D(A)}\right| &= \left|\left(2\int_{\mathcal{R}^N}\Delta m m_t\varphi g dx\right)||u||^2_{D(A)}\right| \\ &\leq 2\left(||m||^2_{D(A)}\right)^{1/2}\left(||m_t||^2_{L^2_g}\right)^{1/2}||u||^2_{D(A)} \\ &\leq 2\alpha^{-1/2}R^2 e(u) \leq 2\alpha^{-1/2}R^2c_3^2\ \theta(t). \end{aligned} \tag{3.10}$$

By the relations (3.9) and (3.10), applying Young's inequality in the last term of (3.8) and using the estimates

$$||m||^a_{L^{Na}} \leq R^a \quad \text{and} \quad ||\nabla m||_{L^{\frac{2N}{N-2}}} \leq ||m||_{D(A)} \leq R, \tag{3.11}$$

the inequality (3.8) becomes

$$\frac{d}{dt}\theta(t) + C_*\theta(t) \leq C(R), \tag{3.12}$$

where $C_* = \frac{1}{2}\left(\rho_1 - 4\alpha^{-1/2}R^2c_3^2\right) > 0$, $\rho_1 = \min(-\varepsilon,\ \varepsilon,\ 2\varepsilon)$ and $C(R) = k_2R^{2(a+1)}$. Applying Gronwall's Lemma in (3.12) we get

$$\theta(t) \leq \theta(0)\, e^{-C_*t} + \frac{1-e^{-C_*t}}{C_*}C(R). \tag{3.13}$$

By using the nondegenerate condition $||\nabla u_0|| > 0$ and relation (3.5), we may assume that $||\nabla m(s)|| > M_0 > 0, \quad 0 \leq s \leq t, \quad t \in [0, +\infty)$. Letting $t \to \infty$, in relation (3.13) we conclude that

$$\lim_{t\to\infty} \sup \theta(t) \leq \frac{C(R)}{C_*} =: R_*^2. \tag{3.14}$$

So, the ball $B_0 =: B_{\mathcal{X}_0}(0, \bar{R}_*)$ for any $\bar{R}_* > R_*$, where R_* defined by (3.14), is an absorbing set for the associated semigroup $S(t)$ in the energy space $\mathcal{X}_0 \subset \mathcal{X}_1$, compactly. Also, from inequality (3.14) and following the arguments of Theorem 3.1 (see [10]), we conclude that the solution of (3.1) exists globally in time. □

4. Global strong attractor in the space $\mathcal{X}_1$

In this section we intend to study the problem (1.1) in a dynamical system point of view. An important remark is that we were unable to show that the operator $S(t) : \mathcal{X}_0 \to \mathcal{X}_0$, which is associated to the problem (1.1), is continuous. For this reason we will study our problem as a dynamical system in the space $\mathcal{X}_1 =: \mathcal{D}^{1,2}(\mathcal{R}^N) \times L_g^2(\mathcal{R}^N)$. We need the following results.

Proposition 4.1. *Assume that* $(u_0,\ u_1) \in \mathcal{X}_0$ *and* $0 \leq a \leq 4/(N-2)$, *where* $N \geq 3$. *Then the linear wave equation* (3.1) *has solutions such that*

$$u \in C\left(0, T; \mathcal{D}^{1,2}\right) \text{ and } u_t \in C\left(0, T; L_g^2\right).$$

Proof . The proof follows the lines of [5, Proposition 3.1]. □

Theorem 4.2. *Assume that* $f(u) = |u|^a u$ *is a nonlinear* C^1 *function such that* $|f(u)| \leq k_1|u|^{a+1}$ *and* $0 \leq a \leq 4/(N-2)$, *where* $N \geq 3$. *If* $(u_0,\ u_1) \in D(A) \times \mathcal{D}^{1,2}$ *and satisfy the nondegenerate condition*

$$||\nabla u_0|| > 0, \tag{4.1}$$

then there exists $T > 0$ *such that the problem* (1.1) *admits local weak solutions* u *satisfying*

$$u \in C(0, T; \mathcal{D}^{1,2}) \text{ and } u_t \in C(0, T; L_g^2). \tag{4.2}$$

Proof . The proof follows the lines of [10, Theorem 3.2] (see also [11]). In this case, because of the compact embedding $\mathcal{X}_0 \subset \mathcal{X}_1$ we obtain for the associated norms that

$$e_1(u(t)) \leq e(u(t)),$$

where $e_1(u(t)) =: ||u||^2_{\mathcal{D}^{1,2}} + ||u_t||^2_{L_g^2}$ and $e(u(t)) =: ||u||^2_{D(A)} + ||u_t||^2_{\mathcal{D}^{1,2}}$. Following the same steps as in Theorem 3.1 we take the inequality

$$e_1(u(t)) \leq e(u(t)) \leq R^2,$$

where R is a positive parameter. So, u is a solution such that

$$u \in L^\infty(0, T;\ \mathcal{D}^{1,2}), \quad u_t \in L^\infty(0, T;\ L_g^2).$$

The continuity properties (4.2), are also proved with the methods indicated in [13, Sections II.3 and II.4]. □

Next, we prove a useful lemma.

Lemma 4.3. *The mapping* $S(t) : \mathcal{X}_0 \subset \mathcal{X}_1 \to \mathcal{X}_1$ *is continuous, for all* $t \geq 0$.

Proof . Let u, v two solutions of our problem such that

$$u_{tt} - \phi(x)||\nabla u||^2 \Delta u = -f(u),$$
$$v_{tt} - \phi(x)||\nabla v||^2 \Delta v = -f(v).$$

Let $w = u - v$. So, we have that

$$\begin{aligned} w_{tt} - \phi||\nabla u||^2 \Delta w &= \phi \left\{||\nabla u||^2 - ||\nabla v||^2\right\} \Delta v - (f(u) - f(v)) \\ w(0) = 0, \qquad & w_t(0) = 0. \end{aligned}$$

Multiplying the previous equation by $2gw_t$ and integrating over $\mathcal{R}^N$, we get

$$\begin{aligned} \int_{\mathcal{R}^N} g w_t w_{tt} dx - 2\int_{\mathcal{R}^N} ||\nabla u||^2 \Delta w w_t dx \\ &= \left\{||\nabla u||^2 - ||\nabla v||^2\right\} \int_{\mathcal{R}^N} \Delta v w_t dx \\ &- 2\int_{\mathcal{R}^N} g\left(f(u) - f(v)\right) w_t dx. \end{aligned} \tag{4.3}$$

Hence we get

$$\begin{aligned} \frac{d}{dt} e^*(w) = (\frac{d}{dt}||\nabla u||^2)||\nabla w||^2 + 2\left\{||\nabla u||^2 - ||\nabla v||^2\right\} \\ \times (\Delta v\ , w_t) - 2(f(u) - f(v)\ , w_t)_{L^2_g} \\ \equiv I_1(t) + I_2(t) + I_3(t), \end{aligned} \tag{4.4}$$

So

$$\frac{d}{dt} e^*(w) = I_1(t) + I_2(t) + I_3(t), \tag{4.5}$$

where $e^*(w) = ||w_t||^2_{L^2_g} + C_u||w||^2_{\mathcal{D}^{1,2}}$ and $C_u = ||u||^2_{\mathcal{D}^{1,2}}$. To estimate the above integrals, we observe that we need more smoothness for the solutions u, v. From Theorem 3.1, we have unique local solution in the space $\mathcal{X}_0$, if $(u_0, u_1) \in \mathcal{X}_0$.

Under of these lights of remarks, we assume that $(u_0, u_1) \in \mathcal{X}_1$. Then, again from Theorem 3.1, we take that $(u, u_t) \in \mathcal{X}_1$. Therefore we have that

$$\begin{aligned} I_1(t) &= (2\int_{\mathcal{R}^N} \Delta u u_t \phi(x) g(x) dx)||\nabla w||^2 \\ &\leq 2(||u||^2_{D(A)})^{1/2}(||u_t||^2_{L^2_g})^{1/2}||\nabla w||^2 \\ &\leq 2R_* k(||u_t||^2_{\mathcal{D}^{1,2}})^{1/2}||\nabla w||^2 \\ &\leq 2R_*^2 k||\nabla w||^2 \leq C_2 e^*(w), \end{aligned} \tag{4.6}$$

where $C_2 = 2R_*^2 k$. We also obtain the following estimation

$$\begin{aligned} I_3(t) \leq |I_3(t)| &\leq k_1\alpha^{-1}(||\nabla u||^2 - ||\nabla v||^2)||\nabla(u - v)||\ ||w_t||_{L^2_g} \\ &\leq k_1\alpha^{-1} 2R_*^2||w||_{\mathcal{D}^{1,2}}\ ||w_t||_{L^2_g} \\ &\leq C_A(\frac{C_u}{2C_u}||w||^2_{\mathcal{D}^{1,2}} + \frac{1}{2}||w_t||^2_{L^2_g}) \ \text{(Young's inequality)} \\ &\leq C_A C_B e^*(w), \end{aligned} \tag{4.7}$$

where $C_A = 2k_1\alpha^{-1}R_*^2$, $C_B = \max(\frac{1}{2}, \frac{1}{2C_u})$.

Finally,

$$\begin{aligned} I_2(t) &\leq (||\nabla u|| + ||\nabla v||)(||\nabla(u-v)||)\left(\int_{\mathcal{R}^N} \Delta v w_t dx\right) \\ &\leq 2R_*||w||_{\mathcal{D}^{1,2}}(||v||^2_{D(A)})^{1/2}(||w_t||^2_{L^2_g})^{1/2} \\ &\leq 2R_*^2||w||_{\mathcal{D}^{1,2}}(||w_t||^2_{L^2_g})^{1/2} \\ &\leq 2R_*^2(\frac{C_u}{2C_u}||w||^2_{\mathcal{D}^{1,2}} + \frac{1}{2}||w_t||^2_{L^2_g}) \\ &\leq C_\Gamma C_B e^*(w), \end{aligned} \tag{4.8}$$

where $C_\Gamma = 2R_*^2$. So using relations (4.6)-(4.8), estimation (4.5) becomes

$$\begin{aligned} \frac{d}{dt}e^*(w) &\leq (C_2 + C_A C_B + C_\Gamma C_B)e^*(w) \\ &\leq C_{**}e^*(w), \end{aligned} \tag{4.9}$$

where $C_{**} = C_2 + C_A C_B + C_\Gamma C_B$. The proof of lemma is now completed. $\square$

Relation (4.9) shows that we have unique solution in $\mathcal{X}_1$, if we assume smoother initial data. More precisely, we take $(u_0, u_1) \in \mathcal{X}_1$. Therefore, if we set $\widehat{u}_a = (u_0, u_1)$, $\widehat{u}_b = (u_0', u_1')$, from the last inequality (4.9) we take

$$||S(t)\widehat{u}_a - S(t)\widehat{u}_b||_{\mathcal{X}_1} \leq C(||\widehat{u}_a||_{\mathcal{X}_0}, ||\widehat{u}_b||_{\mathcal{X}_0})||\widehat{u}_a - \widehat{u}_b||_{\mathcal{X}_1}. \tag{4.10}$$

Since we have uniqueness only for smoother data, see relation (4.9), many trajectories can start from the initial value $\widehat{u}_a \in \mathcal{X}_1$. Let us now denote these trajectories by $[x^\beta(\tau, \widehat{u}_a)]_{\tau\in[0,\delta_1]}$, for short $x^\beta(\cdot, \widehat{u}_a)$, $\beta \in \Gamma_{\widehat{u}_a}$, where $\Gamma_{\widehat{u}_a}$ is the set of indices marking trajectories starting from $\widehat{u}_a$.

Definition 4.4. (Set of Short Trajectories). Let $\delta_1 > 0$. We define

$$X_{\delta_1} \equiv \bigcup_{\widehat{u}_a \in \mathcal{X}_1} \bigcup_{\beta\in\Gamma_{\widehat{u}_a}} x^\beta(\cdot, \widehat{u}_a),$$
$$X^s_{\delta_1} \equiv \overline{X}_{\delta_1},$$

where the closure is with respect to the norm $L^2(0, \delta_1; \mathcal{X}_1)$.

Then the space $X^s_{\delta_1}$ equipped by the topology of $L^2(0, \delta_1; \mathcal{X}_1)$ is a metric space. Let us define the operators $L_t : X^s_{\delta_1} \to X^s_{\delta_1}$ by the relation

$$L_t(x^\beta(., \widehat{u}_a)) = x(., x^\beta(t, \widehat{u}_a)) : L^2(0, \delta_1; \mathcal{X}_1) \to L^2(0, \delta_1; \mathcal{X}_1),$$

if $(x^\beta(., \widehat{u}_a)) \in X_{\delta_1}$ and by the natural extension (as a limit of a Cauchy sequence) if $x^\beta(\cdot) \in X^s_{\delta_1} \backslash X_{\delta_1}$. Due to Lipschitz continuity of L_t we will work with elements of X_{δ_1}.

Lemma 4.5.

(i) The operators $(L_t)_{t\geq 0}$ form a semigroup on $X^s_{\delta_1}$.

(ii) The mapping $L_t : X^s_{\delta_1} \to X^s_{\delta_1}$ is continuous, for all $t \geq 0$.

Proof . (i) Follows from the fact that $x(\delta_1, \widehat{u}_a) \in \mathcal{X}_0$ and the operators $S(t) : \mathcal{X}_0 \to \mathcal{X}_0$ defined by $S(t)(\widehat{u}_a) = \widehat{u}_b$, have the semigroup property. (But it is not clear whether these operators are continuous for fixed $t \geq 0$ with respect to the $\mathcal{X}_0$ topology.)
(ii) We can rewrite relation (4.10) as

$$||L_t u_0 - L_t u_1||_{\mathcal{X}_1} \leq C(||u_0||_{\mathcal{X}_0},\ ||u_1||_{\mathcal{X}_0})||u_0 - u_1||_{\mathcal{X}_1}. \tag{4.11}$$

The above inequality implies that L_t is Lipschitz continuous and the lemma is proved. □

Remark 4.6. According to Theorem 3.3 we have that the ball $\mathcal{B}_0 =: B_{\mathcal{X}_0}(0, \overline{R}_*)$ is an absorbing set in the space $\mathcal{X}_0 \subset \mathcal{X}_1$, compactly. Putting $\rho_0 = \delta_1 \overline{R}_*$ we get that for $\rho' > \rho_0$, the ball $\mathcal{B}^0_{\rho'}$ is an absorbing set in $X^s_{\delta_1}$.

So, we obtain the following theorem.

Theorem 4.8. *The dynamical system given by the semigroup $(L_t)_{t\geq 0}$, possesses the global attractor, denoted by*

$$\mathcal{A}_{X^s_{\delta_1}} = \bigcap_{t\geq 0} \bigcup_{s\geq t} \overline{L_s \mathcal{B}^0_{\rho'}} \subset X^s_{\delta_1}.$$

Proof . For the proof we use the above results and the ideas developed in [13, Theorem 1.1]. We also refer to [8, Theorem 4.14]). □

Remark 4.9. We must remark that the semigroup generated by the problem (1.1) actually possesses an *attractor like*. To be more precise this problem possesses an *attractor like* invariant set because it doesn't attract all the trajectories. We call the set $\mathcal{A}_{X^s_{\delta_1}}$ an attractor just for simplicity reasons.

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