Dynamics of higher order rational difference equation $x_{n+1} = (\alpha + \beta x_n)/(A + Bx_n + Cx_{n-k})$

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Abstract

The main goal of this paper is to investigate the periodic character, invariant intervals, oscillation and global stability and other new results of all positive solutions of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

where the parameters α, β, A, B and C are positive, and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are positive real numbers and $k \in \{1, 2, 3, \ldots\}$. We give a detailed description of the semi-cycles of solutions and determine conditions under which the equilibrium points are globally asymptotically stable. In particular, our paper is a generalization of the rational difference equation that was investigated by Kulenovic et al. [The Dynamics of $x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}$, Facts and Conjectures, Comput. Math. Appl. 45 (2003) 1087–1099].

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1. Introduction

The dynamical system is the study of phenomena that evolve in space and/ or time by looking at the dynamic behavior or the geometrical and topological properties of the solution, whether a particular system comes from Economics, Biology, Physics, Chemistry, or even Social Science such as population models, disease and infection models, etc. Dynamical system in point of view of mathematics is a

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system whose behavior at given time depends, in some sense, on its behavior at one or more previous times.

One of the important models in biology and computer science are the rational models. The dynamical system is the subject that provides the mathematical tools for its analysis. In this paper, we have completely investigated the semi-cycle and the global stability of a class of higher order difference equations that generalize equations of second orders that have been investigated in the references mainly in [23], [24], [5].

In this paper, we have completely investigated the semi-cycle and the global stability of a class of higher order difference equations that generalize equations of second orders that have been investigated in the references mainly in [23], [24], [5]. We will study the dynamics of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + B x_n + C x_{n-k}}, \ n = 0, 1, 2, \dots. \tag{1.1}$$

A special case of Eq. (1.1) was studied in [24], when $k = 1$. We have completely investigated the semi-cycle and the global stability of a class of higher order difference equations that generalize equations of second orders that have been investigated in the references mainly in [23], [24], [5].

The aim of this paper is to study equilibrium points, local stability and global stability, periodic solution, semicycles and boundedness of the solutions of this equation. We are particulary interested in the asymptotic behavior of the solutions. Some of recent research on rational difference equations and nonlinear difference equations can be found in [23], [24], [1], [2], [17], [16], [12], [15], [15], [39], [43], [20], [26], [40], [30], [35], [36], [38], [41].

Our model generalizes the growth model and the logistic maps and some of its generalizations that have been studies by many scholars in the past several years.

2. Basic theory and definitions

Here, we list below some definitions and basic results that will be useful in our investigation.

Definition 2.1. Let I be some interval of real numbers and let

$$f : I^{k+1} \longrightarrow I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, \dots x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \ n = 0, 1, \dots \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 2.2. A point $\overline{x}$ is called an equilibrium point of Eq. (2.1) if

$$\overline{x} = f(\overline{x}, \overline{x}, \dots, \overline{x}),$$

that is

$$x_n = \overline{x}, \ for \ n \geq 0$$

is a solution of Eq. (2.1), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2.3. The solution $\{y_n\}_{n=-k}^{\infty}$ of the difference equation $y_{n+1} = f(y_n, y_{n-1}, \dots, y_{n-k})$ is periodic if there exists a positive integer p such that $y_{n+p} = y_n$. The smallest such integer p is called the prime period of the solution of the difference equation.

Definition 2.4. Let $\overline{x}$ be an equilibrium point of Eq. (2.1)

1. The equilibrium point $\overline{x}$ of Eq. (2.1) is called stable if for every ϵ, there exists δ such that if

$$x_{-k}, \ldots x_{-1}, x_0 \in I$$

   and

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \delta$$

   then

$$|x_n - \overline{x}| < \epsilon$$

   for all $n \geq -k$.

2. The equilibrium point $\overline{x}$ of Eq. (2.1) is called locally asymptotically stable if is it stable and if there exists $\gamma > 0$ such that if

$$x_{-k}, \ldots, x_{-1}, x_0 \in I$$

   and

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \gamma,$$

   then

$$\lim_{n \to \infty} x_n = \overline{x}.$$

3. The equilibrium point $\overline{x}$ of Eq. (2.1) is called global attractor if for every

$$x_{-k}, \ldots, x_{-1}, x_0 \in I,$$

   we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

4. The equilibrium point $\overline{x}$ of Eq. (2.1) is called global asymptotically stable it is stable and is global attractor.

5. The equilibrium point $\overline{x}$ of Eq. (2.1) is called unstable if it is not stable

6. The equilibrium point $\overline{x}$ of Eq. (2.1) is called repeller if there exists $r > 0$ such that if $x_{-k}, \ldots, x_{-1}, x_0 \in I$ and

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \gamma,$$

   then there exists $N > -k$ such that
$$|x_N - \overline{x}| > r.$$

Clearly, a repeller is an unstable equilibrium point.

Definition 2.5. Let $a = \frac{\partial f}{\partial x}(\overline{x}, \overline{x})$ and $b = \frac{\partial f}{\partial y}(\overline{x}, \overline{x})$ where $f(x, y)$ is the function in Eq. (2.1) and $\overline{x}$ is the equilibrium of Eq. (2.1). Then the equation

$$z_{n+1} = az_n + bz_{n-k}, \ n = 0, 1, 2, \ldots \tag{2.2}$$

is called linearized equation associated with Eq. (2.1) about the equilibrium point $\overline{x}$, and its characteristic equation is

$$\lambda^{k+1} + a\lambda^k + b = 0. \tag{2.3}$$

Theorem 2.6. [29]:*(Linearized Stability)*

1. If all the roots of Eq. (2.3) lie in open disk $|\lambda| < 1$, then the equilibrium point $\overline{x}$ of Eq. (2.1) is asymptotically stable.
2. If at least one root of Eq. (2.3) has absolute value greater than 1, then the equilibrium $\overline{x}$ of Eq. (2.1) is unstable.

Theorem 2.7. [6] Assume $a, b \in R$ and $k \in \{1, 2, \ldots\}$. Then

$$|a| + |b| < 1 \tag{2.4}$$

is sufficient condition for asymptotic stability of the difference equation

$$x_{n+1} - ax_n + bx_{n-k} = 0, \ n = 0, 1, 2, \ldots . \tag{2.5}$$

Suppose in addition that one of the following two cases holds:

1. k is odd and $b < 0$.
2. k is even and $ab < 0$.

Then Eq. (2.4) is a necessary condition for asymptotic stability of Eq. (2.5).

Theorem 2.8. [23] Let $I = [a, b]$ be an interval of real numbers and assume

$$f : [a, b] \times [a, b] \rightarrow [a, b]$$

is continuous function satisfying the following properties:

1. $f(x, y)$ is nondecreasing in x for each $y \in [a, b]$ and $f(x, y)$ is nonincreasing in y for each $x \in [a, b]$.
2. If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$m = f(m, M)$$
$$M = f(M, m)$$

then $m = M$.

Then Eq. (2.1) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq. (2.1) converges to $\overline{x}$.

Theorem 2.9. [23] Let $I = [a, b]$ be an interval of real numbers and assume

$$f : [a, b] \times [a, b] \rightarrow [a, b]$$

is continuous function satisfying the following properties:

1. $f(x, y)$ is non increasing in each of its arguments;
2. If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$m = f(M, M)$$
$$M = f(m, m)$$

then $m = M$.

Then Eq. (2.1) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq. (2.1) converges to $\overline{x}$.

Theorem 2.10. [23] consider the difference Eq. (2.1). Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is continuous function satisfying the following properties:

1. $f(x, y)$ is non increasing in x for each $y \in [a, b]$, and $f(x, y)$ is non increasing in y for each $x \in [a, b]$.
2. If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$m = f(M, M)$$
$$M = f(m, m)$$

then $m = M$.
3. The equation $f(x, y) = x$ has a unique positive solution.

Then Eq. (2.1) has a unique positive solution and every positive solution of Eq. (2.1) converges to $\overline{x}$.

Theorem 2.11. [23] Assume that $f \in C\left[(0, \infty) \times (0, \infty), (0, \infty)\right]$ and that $f(x, y)$ is decreasing in both arguments. Let $\overline{x}$ be a positive equilibrium of equation Eq. (2.1), then every oscillatory solution of Eq. (2.1) has semicycle of length at most $k + 1$.

3. The main results.

3.1. Local stability and invariant intervals of $x_{n+1} = (\alpha + \beta x_n)/(A + B x_n + C x_{n-k})$

In this subsection we investigate local stability and and the invariant intervals of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + B x_n + C x_{n-k}}, n = 0, 1, 2, \ldots, \qquad (3.1)$$

where the parameters α, β, A, B and C are positive, and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are arbitrary non-negative real numbers and $k \in \{1, 2, 3, \ldots\}$.

Our concentration in this subsection is on the local stability, invariant intervals, periodic character, and the character of semicycles. In the next subsection we study global asymptotic stability of Eq. (3.1).

It is worth mentioning that the results in [24] are special case of our main results. Where the global stability of Eq. (3.1) for $k = 1$ has been investigated in it. They showed that in respect to variation of the parameters, the positive equilibrium point is globally asymptotically stable or every solution lies eventually in an invariant interval.

Dehghan in [6] investigated the global stability, invariant intervals, the character of semi-cycles, and boundedness of the equation

$$x_{n+1} = \frac{x_n + p}{B x_n + q x_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

where the parameters p and q and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are positive real numbers, $k \in \{1, 2, 3, \ldots\}$.

Theorem 3.1. *The change of variable*

$$x_n = \frac{A}{B} y_n$$

reduces Eq. (3.1) to the difference equation

$$y_{n+1} = \frac{p + q y_n}{1 + y_n + r y_{n-k}}, n = 0, 1, 2, 3, \ldots, \tag{3.2}$$

where

$$p = \frac{\alpha B}{A^2}, q = \frac{\beta}{A}, r = \frac{C}{B}$$

with

$$p, q, r \in (0, \infty)$$

and

$$y_k, y_{-k+1}, \ldots, y_{-1}, y_0 \in (0, \infty).$$

Proof . Let $x_n = \frac{A}{B} y_n$, then $x_{n+1} = \frac{A}{B} y_{n+1}$ and $x_{n-k} = \frac{A}{B} y_{n-k}$. Substitute in the Eq. (3.1) and simplifying, we get

$$y_{n+1} = \frac{p + q y_n}{1 + y_n + r y_{n-k}}.$$

$\square$

3.2. Equilibrium points and local stability

We are going to find the unique positive equilibrium point of the nonlinear difference equation

$$y_{n+1} = \frac{p + q y_n}{1 + y_n + r y_{n-k}}, \quad n = 0, 1, \ldots, \tag{3.3}$$

where the parameters p, q, r and the initial conditions $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0$ are positive real numbers, and $k \in \{1, 2, \ldots\}$. To find the equilibrium point, we solve the following equation

$$\overline{y} = \frac{p + q \overline{y}}{1 + \overline{y} + r \overline{y}},$$

hence

$$\overline{y} \left(1 + \overline{y} + r \overline{y}\right) = p + q \overline{y};$$

by rearranging the terms, we get:

$$(1 + r) \overline{y}^2 + (1 - q) \overline{y} - p = 0.$$

Solving this quadratic equation, we get the equilibrium points

$$\overline{y} = \frac{(q - 1) \pm \sqrt{(q - 1)^2 + 4p(r + 1)}}{2(r + 1)}.$$

The only positive solution is:

$$\overline{y} = \frac{(q - 1) + \sqrt{(q - 1)^2 + 4p(r + 1)}}{2(r + 1)}.$$

To find the linearized equation of our problem, consider $f(x, y) = \frac{p+qx}{1+x+ry}$, then

$$\frac{\partial f}{\partial x} = \frac{q(1+x+ry) - (p+qx)}{(1+x+ry)^2} = \frac{q+x+qry-p-qx}{(1+x+ry)^2} = \frac{q-p+qry}{(1+x+ry)^2},$$

which implies that

$$\frac{\partial f}{\partial x}(\overline{y}, \overline{y}) = \frac{q-p+qr\overline{y}}{(1+\overline{y}+r\overline{y})^2} = \frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2}.$$

Similarly,

$$\frac{\partial f}{\partial y} = \frac{0(1+x+ry) - r(p+qx)}{(1+x+ry)^2} = \frac{-r(p+qx)}{(1+x+ry)^2}.$$

Thus,

$$\frac{\partial f}{\partial y}(\overline{y}, \overline{y}) = \frac{-r(p+q\overline{y})}{(1+(1+r)\overline{y})^2},$$

so the linearized equation which is associated to Eq. (3.3) about the equilibrium point $\overline{y}$ is:

$$z_{n+1} = \frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2}z_n - \frac{r(p+q\overline{y})}{(1+(1+r)\overline{y})^2}z_{n-k},$$

i.e,

$$z_{n+1} - \frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2}z_n + \frac{r(p+q\overline{y})}{(1+(1+r)\overline{y})^2}z_{n-k} = 0 \tag{3.4}$$

and its characteristic equation is:

$$\lambda^{n+1} - \frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2}\lambda^n + \frac{r(p+q\overline{y})}{(1+(1+r)\overline{y})^2}\lambda^{n-k} = 0, \tag{3.5}$$

which implies

$$\lambda^{k+1} - \frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2}\lambda^k + \frac{r(p+q\overline{y})}{(1+(1+r)\overline{y})^2} = 0. \tag{3.6}$$

The following two lemmas are important for the study of local stability.

Lemma 3.2. [14], [22] Assume that $a, b \in \Re$ and $k \in \{1, 2, 3, \ldots\}$. Then a necessary and sufficient condition for asymptotic stability of the equation

$$x_{n+1} + ax_n + bx_{n-k} = 0, n = 0, 1, 2, \ldots$$

is that

$$|a| < 1 + b < 2.$$

Lemma 3.3. [23], [14] Assume that all the roots of the characteristic equation of the above equation lie inside the unit circle, then the positive equilibrium point is locally asymptotically stable.

Theorem 3.4. *The positive fixed point $\overline{y}$ of Eq. (3.3) is asymptotically stable.*

Proof . Considering the linearized equation Eq. (3.4) with

$$a = -\frac{q-p+qr\overline{y}}{(1+\overline{y}(1+r))^2}, \quad b = \frac{r(p+q\overline{y})}{(1+(1+r)\overline{y})^2},$$

we find that the condition in Lemma 3.2 using mathematica code, is satisfied for p, q and r are any positive numbers. $\square$

3.3. Boundedness of solutions and invariant intervals

Theorem 3.5. *Every solution of Eq. (3.3) is bounded from above and from below by a positive constant.*

Proof . Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (3.3), so clearly if the solution is bounded from above by a constant M, then

$$y_{n+1} \geq \frac{p}{1 + (1+r)M}$$

and so its bounded from below.

Now assume for the sake of contradiction that the solution is not bounded from above, then there exists a subsequence $\{y_{n_m+1}\}_{m=0}^{\infty}$ such that $\lim_{m\to\infty} y_{n_m} = \infty$, $\lim_{m\to\infty} y_{n_m+1} = \infty$ and $y_{n_m+1} = \max\{y_n : n \leq n_m\}$ for $m \geq 0$. So for Eq. (3.3), we see that

$$y_{n+1} < p + qy_n, \text{ for } n \geq 0$$

and so

$$\lim_{m\to\infty} y_{n_m} = \lim_{m\to\infty} y_{n_m-1} = \infty,$$

hence, for sufficiently large m,

$$0 < y_{n_m+1} - y_{n_m} = \frac{p + [(q-1) - y_{n_m} - ry_{n_m-1}]y_{n_m}}{1 + y_{n_m} + ry_{n_m-1}} < 0,$$

which is a contradiction. $\square$

Definition 3.6. Invariant Interval of the difference equation Eq. (2.1) is an interval with the property that if $k+1$ consecutive terms of the solution fall in I then all subsequent terms of the solution also belong to I. In other words, I is an invariant interval for Eq. (2.1) if $y_{N-k+1}, \ldots, y_{N-1}, y_N \in I$ for some $N \geq 0$, then $y_n \in I$ for every $n > N$.

Theorem 3.7. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (3.3). Then the following statements are true:*

1. *Suppose $p \leq q$ and assume that for some $N \geq 0$,*

$$y_{N-k}, \ y_{N-k+1}, \ \ldots, y_N \in [0, \frac{q-1+\sqrt{(q-1)^2+4p}}{2}],$$

   then $y_n \in [0, \frac{q-1+\sqrt{(q-1)^2+4p}}{2}]$ for all $n > N$.
2. *Suppose $q < p < q(rq+1)$ and assume that for some $N \geq 0$,*

$$y_{N-k}, \ y_{N-k+1}, \ \ldots, y_N \in [\frac{p-q}{qr}, q]$$

   then $y_n \in [\frac{p-q}{qr}, q]$ for all $n > N$.
3. *Suppose $p > q(rq+1)$ and assume that for some $N \geq 0$,*

$$y_{N-k}, \ y_{N-k+1}, \ \ldots, y_N \in [q, \frac{p-q}{qr}]$$

   then $y_n \in [q, \frac{p-q}{qr}]$ for all $n > N$.

Proof .

1. Set

$$g(x) = \frac{p+qx}{1+x} \text{ and } b = \frac{(q-1) + \sqrt{(q-1)^2 + 4p}}{2}$$

   and observe that g is an increasing function and $g(b) = b$, using Eq. (3.3), we see that when $y_{N-k},\ y_{N-k+1},\ \ldots, y_N \in [0, b]$, then

$$y_{N+1} = \frac{p + qy_N}{1 + y_N + ry_{N-k}} \leq \frac{p + qy_N}{1 + y_N} = g(y_N) \leq b.$$

   The proof follows by induction.

2. Take the function

$$f(x, y) = \frac{p + qx}{1 + x + ry},$$

   it's clear that this function is increasing in x for $y > \frac{p-q}{qr}$. Using Eq. (3.3), we see that if $y_{N-k},\ y_{N-k+1},\ \ldots, y_N \in [\frac{p-q}{qr}, q]$, then

$$y_{N+1} = \frac{p + qy_N}{1 + y_N + ry_{N-k}} = f(y_N, y_{N-k}) \leq f(q, \frac{p-q}{qr}) = q$$

   and by using the condition $p < q(rq + 1)$, we obtain

$$y_{N+1} = \frac{p + qy_N}{1 + y_N + ry_{N-k}} = f(y_N, y_{N-k}) \geq f(\frac{p-q}{qr}, q) = \frac{q(pr + p - q)}{(rq)^2 + rq + p - q} > \frac{p-q}{qr}.$$

   The proof follows by induction.

3. Take the function

$$f(x, y) = \frac{p + qx}{1 + x + ry},$$

   it's clear that this function is decreasing in x for $y < \frac{p-q}{qr}$. Using Eq. (3.3), we see that for $y_{N-k},\ y_{N-k+1},\ \ldots, y_N \in [q, \frac{p-q}{qr}]$, then

$$y_{N+1} = \frac{p + qy_N}{1 + y_N + ry_{N-k}} = f(y_N, y_{N-k}) \geq f(\frac{p-q}{qr}, \frac{p-q}{qr}) = q$$

   and by using the condition $p > q(rq + 1)$, we obtain

$$y_{N+1} = \frac{p + qy_N}{1 + y_N + ry_{N-k}} = f(y_N, y_{N-k}) \leq f(q, q) = \frac{p + q^2}{1 + (r+1)q} < \frac{p-q}{qr}.$$

   The proof follows by induction.

□

3.4. Existence of two cycles

Definition 3.8. Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (3.3). We say that the solution has a prime period *two* if the solution eventually takes the form:

$$\ldots, \phi, \psi, \phi, \psi, \phi, \psi, \ldots,$$

where $\phi,\ \psi$ are distinct and positive.

Theorem 3.9. *If k is even, then Eq. (3.3) has no nonnegative distinct prime period two solution.*

Proof . Let k be even, and assume for the sake of contradiction that there is distinct nonnegative real numbers ϕ, ψ such that

$$\ldots, \phi, \psi, \phi, \psi, \phi, \psi, \ldots$$

is prime period two solution of Eq. (3.3), then ϕ, ψ satisfy

$$\psi = \frac{p + q\phi}{1 + \phi + r\phi}$$

and

$$\phi = \frac{p + q\psi}{1 + \psi + r\psi}$$

then by substituting ϕ into the equation of ψ, we get easily by a simple calculation that

$$(1 + r + q)\psi^2 + (1 - q^2)\psi - (p + qp) = 0.$$

Solving this quadratic equation for ψ, we get

$$\psi = \frac{(q^2 - 1) \pm \sqrt{(q^2 - 1)^2 + 4(p + qp)(1 + r + q)}}{2(1 + r + q)},$$

but

$$\sqrt{(q^2 - 1)^2 + 4(p + qp)(1 + r + q)} > (q^2 - 1)$$

and ψ is nonnegative, then

$$\psi = \frac{(q^2 - 1) + \sqrt{(q^2 - 1)^2 + 4(p + qp)(1 + r + q)}}{2(1 + r + q)}.$$

Now again the same steps for ϕ, substituting ψ into ϕ, we get that

$$\phi = \frac{(q^2 - 1) + \sqrt{(q^2 - 1)^2 + 4(p + qp)(1 + r + q)}}{2(1 + r + q)},$$

which implies $\psi = \phi$, which contradicts the hypothesis that ψ and ϕ are distinct nonnegative real numbers. $\square$

Theorem 3.10. *If k is odd, then Eq. (3.3), has no nonnegative distinct prime period two solution.*

Proof . Let k be odd, and assume that for the sake of contradiction that there is distinct nonnegative real numbers ϕ and ψ such that

$$\ldots, \phi, \psi, \phi, \psi, \phi, \psi, \ldots$$

is prime period two solution of Eq. (3.3), then ϕ, ψ satisfy

$$\phi = \frac{p + q\psi}{1 + \psi + r\phi}, \quad \psi = \frac{p + q\phi}{1 + \phi + r\psi}.$$

By multiplying, we get

$$\phi + \phi\psi + r\phi^2 = p + q\psi, \quad \psi + \phi\psi + r\psi^2 = p + q\phi.$$

By rearranging the above equation by some algebra we get

$$q(\psi - \phi) + (\psi - \phi) + r(\psi^2 - \phi^2) = 0$$

and

$$q(\psi - \phi) + (\psi - \phi) + r(\psi - \phi)(\psi + \phi) = 0.$$

We can divide the above equation by $(\psi - \phi)$, since $\phi \neq \psi$, then $q + 1 + r(\psi + \phi) = 0$ which implies that $\psi + \phi = \frac{-q-1}{r}$ which is a contradiction for that ψ and ϕ are both nonnegative. $\square$

Corollary 3.11. Eq. (3.3) posses no prime period two solution.

3.5. Analysis of semicycles and oscillation

Analysis of semicycles of the solution of Eq. (3.3) is a powerful tool for a detailed understanding of the entire character of solutions.

Definition 3.12. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (2.1) and $\overline{x}$ be a positive equilibrium point. We now give the definitions of positive and negative semicycle of a solution of Eq. (2.1) relative to the equilibrium point $\overline{x}$

- A positive semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (2.1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ such that

$$either\ l = -1,\ or\ l < -1\ and\ x_{l-1} < \overline{x}$$

  and

$$either\ m = \infty,\ or\ m < \infty\ and\ x_{m+1} < \overline{x}.$$

- A negative semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (2.1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$either\ l = -1,\ or\ l < -1\ and\ x_{l-1} \geq \overline{x}$$

  and

$$either\ m = \infty,\ or\ m < \infty\ and\ x_{m+1} \geq \overline{x}.$$

Definition 3.13. (Oscillation)

1. A sequence $\{x_n\}$ is said to oscillate a bout zero or simply to oscillate if the terms x_n are neither eventually all positive nor eventually all negative. Otherwise the sequence is called nonoscillatory. A sequence is called strictly oscillatory if for n_0, there exist $n_1, n_2 \geq n_0$ such that $x_{n_1} x_{n_2} < 0$.

2. A sequence x_n is said to oscillate about $\overline{x}$ if the sequence $x_n - \overline{x}$ oscillates. The sequence x_n is called strictly oscillatory about $\overline{x}$ if the sequence $x_n - \overline{x}$ is strictly oscillatory.

Next, we give analysis of the semicycles of solution of Eq. (3.3) relative to equilibrium point $\overline{y}$ and based on invariant interval of Eq. (3.3).

Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (3.3). Then observe that the following identities are true:

$$y_{n+1} - q = (qr) \left[\frac{\frac{p-q}{qr} - y_{n-k}}{1 + y_n + r y_{n-k}} \right], \tag{3.7}$$

$$y_{n+1} - \left(\frac{p-q}{qr}\right) = \frac{qr(q - \frac{p-q}{qr})y_n + qr(y_{n-k} - \frac{p-q}{qr}) + pr(q - y_{n-k})}{qr(1 + y_n + ry_{n-k})}, \tag{3.8}$$

$$y_n - y_{n+4} = \frac{M(y_n - q)}{(1 + y_{n+3})(1 + y_{n+1} + ry_n) + r(p + qy_{n+1})} + qr(y_n - \frac{p-q}{qr})y_{n+1} + y_n + ry_n^2 - p \tag{3.9}$$

and

$$y_{n+1} - \overline{y} = \frac{(\overline{y} - q)(\overline{y} - y_n) + \overline{y}r(\overline{y} - y_{n-k})}{1 + y_n + ry_{n-k}}, \tag{3.10}$$

where

$$M = y_{n+1}y_{n+3} + y_{n+3} + y_{n+1} + ry_n y_{n+3}.$$

So the proof of the following lemmas are straight forward consequence of the above identities.

Lemma 3.14. Suppose that $p > q(qr + 1)$ and let $\{y_n\}_{n=-k}^\infty$ be solutions of Eq. (3.3), then the following statements are true:

1. If for some $N \geq 0$, $y_{N-k} < \frac{p-q}{qr}$. Then $y_{N+1} > q$.
2. If for some $N \geq 0$, $y_{N-k} = \frac{p-q}{qr}$. Then $y_{N+1} = q$.
3. If for some $N \geq 0$, $y_{N-k} > \frac{p-q}{qr}$. Then $y_{N+1} < q$.
4. If for some $N \geq 0$, $q < y_{N-k} < \frac{p-q}{qr}$. Then $q < y_{N+1} < \frac{p-q}{qr}$.
5. If for some $N \geq 0$, $q < y_{N-k}, \ldots, y_{N-1}, y_N < \frac{p-q}{qr}$. Then $q < y_n < \frac{p-q}{qr}$. That is $[q, \frac{p-q}{qr}]$ is an invariant interval for Eq. (3.3).
6. If for some $N \geq 0$, $\overline{y} < y_{N-k}$, and $\overline{y} < y_N$. Then $y_{N+1} < \overline{y}$.
7. If for some $N \geq 0$, $\overline{y} \geq y_{N-k}$ and $\overline{y} \geq y_N$. Then $y_{N+1} \geq \overline{y}$.
8. $q < \overline{y} < \frac{p-q}{qr}$.

Theorem 3.15. [23] Assume that $f \in C\left[(0, \infty) \times (0, \infty), (0, \infty)\right]$ and that $f(x, y)$ is decreasing in both arguments. Let $\overline{x}$ be a positive equilibrium of Eq. (3.3), then every oscillatory solution of Eq. (3.3) has semicycle of length at most $k + 1$.

Corollary 3.16. Every nontrivial and oscillatory solution of Eq. (3.3) which lies in the interval $[q, \frac{p-q}{qr}]$ oscillates about the equilibrium point $\overline{y}$, with semicycle of length at most $k + 1$.

Lemma 3.17. Suppose that $q < p < q(qr + 1)$ and let $\{y_n\}_{n=-k}^\infty$ be solution of equation (3.3), then the following statements are true:

1. If for some $N \geq 0$, $y_{N-k} < \frac{p-q}{qr}$. Then $y_{N+1} > q$.
2. If for some $N \geq 0$, $y_{N-k} = \frac{p-q}{qr}$. Then $y_{N+1} = q$.
3. If for some $N \geq 0$, $y_{N-k} > \frac{p-q}{qr}$. Then $y_{N+1} < q$.
4. If for some $N \geq 0$, $\frac{p-q}{qr} < y_{N-k} < q$. Then $\frac{p-q}{qr} < y_{N+1} < q$.
5. If for some $N \geq 0$, $\frac{p-q}{qr} < y_{N-k}, \ldots, y_{N-1}, y_N < q$. Then $\frac{p-q}{qr} < y_n < q$. That is $[\frac{p-q}{qr}, q]$ is an invariant interval for Eq. (3.3).
6. $\frac{p-q}{qr} < \overline{y} < q$.

Theorem 3.18. [23] Assume that $f \in C\left[(0, \infty) \times (0, \infty), (0, \infty)\right]$ is such that $f(x, y)$ is increasing in x for each fixed y, and $f(x, y)$ is decreasing in y for each fixed x. Let $\overline{y}$ be a positive equilibrium of Eq. (2.1) then except possibly for the first semicycle, every oscillatory solution of Eq. (2.1) has semicycle of length at least $k + 1$.

Corollary 3.19. Every nontrivial and oscillatory solution $\{y_n\}_{n=-k}^{\infty}$ of Eq. (3.3) which lies in the interval $[\frac{p-q}{qr}, q]$ oscillates about the equilibrium point $\overline{y}$, with semicycle of length at least $k + 1$.

Finally, we will discuss thoroughly the analysis of semicycles of the solution $\{y_n\}_{n=-k}^{\infty}$ under the assumption that $p = q(qr + 1)$. In this case, Eq. (3.3) reduces to

$$y_{n+1} = \frac{q + rq^2 + qy_n}{1 + y_n + ry_{n-k}}$$

and the equilibrium point is $\overline{y} = q$, then the identities(3.7) through (3.10) reduces $y_{n+1} - q$ to

$$y_{n+1} - q = \frac{qr}{1 + y_n + ry_{n-k}}(q - y_{n-k}). \tag{3.11}$$

Furthermore, if $qr < 1$, then

$$\lim_{n \to \infty} y_n = \overline{y}. \tag{3.12}$$

Remark 3.20.   1. Identity (3.11) follows by straight forward computation.
   2. Limit in (3.12) is a consequence of the fact that in this case $qr \in (0, 1)$ and Eq. (3.3) has no prime period two solution.

Lemma 3.21. Suppose that $p = q(qr + 1)$ and let $\{y_n\}_{n=-k}^{\infty}$ be solutions of Eq. (3.3), then the following statements are true:

   1. If for some $N \geq 0$, $y_{N-k} < q$. Then $y_{N+1} > q$.
   2. If for some $N \geq 0$, $y_{N-k} = q$. Then $y_{N+1} = q$.
   3. If for some $N \geq 0$, $y_{N-k} > q$. Then $y_{N+1} < q$.

Theorem 3.22. Suppose that $p = q(qr + 1)$ and let $\{y_n\}_{n=-k}^{\infty}$ be a nontrivial solution of Eq. (3.3), then $\{y_n\}_{n=-k}^{\infty}$ oscillates about the equilibrium point q.

Now, assume that the solutions does not eventually lie in the invariant interval. Assume that $p > q(qr + 1)$, let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq. (3.3) which does not eventually lie in the interval $I = [q, \frac{p-q}{qr}]$, then it can be observed that the solution oscillates about the equilibrium point relative to $[q, \frac{p-q}{qr}]$ essentially in one of the following two ways:

   • $k + 1$ consecutive terms in $(\frac{p-q}{qr}, \infty)$, are followed by $k + 1$ consecutive terms in $(\frac{p-q}{qr}, \infty)$, and so on. The solution never meets the interval $(q, \frac{p-q}{qr})$.

   • There exists exactly k terms in $(\frac{p-q}{qr}, \infty)$ which is followed by k terms in $(q, \frac{p-q}{qr})$ which is followed by k terms in $(0, 1)$ which is followed by k terms in $(q, \frac{p-q}{qr})$ which is followed by k terms in $(\frac{p-q}{qr}, \infty)$ and so on. The solution meets consecutively the intervals:

$$\ldots, (\frac{p-q}{qr}, \infty), (q, \frac{p-q}{qr}), (0, 1), (q, \frac{p-q}{qr}), (\frac{p-q}{qr}, \infty), \ldots$$

   in order with k terms per interval. The situation is essentially the same relative to the interval $(\frac{p-q}{qr}, q)$, when $q < p < q(qr + 1)$ and the same thing is done when $p = q(qr + 1)$.

3.6. Global asymptotic stability

The next results are about the global stability for the positive equilibrium of Eq. (3.3).

Theorem 3.23. [1] Let $I = [a, b]$ be an interval of real numbers and assume

$$f : [a, b] \times [a, b] \to [a, b]$$

is continuous function satisfying the following properties

1. $f(x, y)$ is non increasing in each of its arguments;
2. If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$m = f(M, M)$$
$$M = f(m, m)$$

then $m = M$.

Then

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1, \dots \tag{3.13}$$

has a unique equilibrium $\overline{y} \in [a, b]$ and every solution of Eq. (3.13) converges to $\overline{y}$.

Theorem 3.24. [1] Let $I = [a, b]$ be an interval of real numbers and assume

$$f : [a, b] \times [a, b] \to [a, b]$$

is continuously function satisfying the following properties

1. $f(x, y)$ is non decreasing in x for each $y \in [a, b]$ and $f(x, y)$ is non increasing in y for each $x \in [a, b]$.
2. If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$m = f(m, M)$$
$$M = f(M, m),$$

then $m = M$.

Then Eq. (3.13) has a unique equilibrium $\overline{y} \in [a, b]$ and every solution of Eq. (3.13) converges to $\overline{y}$.

Now we will apply these theorems on our equation.

Theorem 3.25. *Assume that $p > q(qr+1)$, then the positive equilibrium of Eq. (3.3) on the interval* $[q, \frac{p-q}{qr}]$ *is globally asymptotically stable.*

Proof . This proof can easily done depending on theorem (3.23). Assume that $p > q(qr + 1)$ and consider the function $f(x, y) = \frac{p+qx}{1+x+ry}$. First, note that $f(x, y)$ on the interval $[q, \frac{p-q}{qr}]$ is non increasing function in both of its arguments x, y. Second, Let $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(m, m) = M \text{ and } f(M, M) = m,$$

then

$$M = \frac{p + qm}{1 + m + rm},$$

and
$$m = \frac{p + qM}{1 + M + rM}.$$

But we showed before that our equation has no periodic two solution, then the only solution is $m = M$.

Then both conditions of theorem (3.23) hold, therefore if $\overline{y}$ is an equilibrium point of Eq. (3.3), then every solution of Eq. (3.3) converges to $\overline{y}$ in the interval $[q, \frac{p-q}{qr}]$. As $\overline{y}$ is asymptotically stable, then it is globally asymptotically stable on $[q, \frac{p-q}{qr}]$. $\square$

Theorem 3.26. *Assume that $q < p < q(qr + 1)$, then the positive equilibrium of Eq. (3.3) on the interval $[\frac{p-q}{qr}, q]$ is globally asymptotically stable.*

Proof . This proof can be easily done depending on theorem (3.24). Assume that $q < p < q(qr + 1)$ and consider the function $f(x, y) = \frac{p+qx}{1+x+ry}$. First, note that $f(x, y)$ on the interval $[\frac{p-q}{qr}, q]$ is nondecreasing function in x, and nonincreasing y. Second, let $(m, M) \in [a, b] \times [a, b]$ is a solution of the system
$$f(m, M) = m \; and f(M, m) = M,$$

then
$$m = \frac{p + qm}{1 + m + rM},$$

and
$$M = \frac{p + qM}{1 + M + rm}.$$

But we showed before that our equation has no periodic two solution, then the only solution is $m = M$.

Then both conditions of theorem (3.24) hold, so if $\overline{y}$ is an equilibrium point of Eq. (3.3), then every solution of Eq. (3.3) converges to $\overline{y}$ in the Interval $[\frac{p-q}{qr}, q]$. As $\overline{y}$ is asymptotically stable, then it is globally asymptotically stable on $[\frac{p-q}{qr}, q]$. $\square$

Theorem 3.27. *Assume that $p \leq q$, then the positive equilibrium of Eq. (3.3) on the interval $\left[0, \frac{(q-1)+\sqrt{(q-1)^2+4p}}{2}\right]$ is globally asymptotically stable.*

Proof . The same proof of theorem (3.26). $\square$

4. Conclusion

In this paper, we have completely investigated the semi-cycle and the global stability of a class of higher order rational difference equations
$$x_{n+1} = \frac{\alpha + \beta x_n}{A + B x_n + C x_{n-k}}, \; n = 0, 1, 2, \ldots .$$

that generalize equations of second and third orders that have been investigated in the references mainly in [23], [24], [5]. Our model generalizes the growth model and the logistic maps and some of its generalizations that have several applications in biology, economics and computer science.

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