Int. J. Nonlinear Anal. Appl. 8 (2017) No. 2, 223-241
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.10778.1523

Analytical aspects of the interval unilateral quadratic matrix equations and their united solution sets

Tayyebe Haqiri[a,c,*], Azim Rivaz[b], Mahmoud Mohseni Moghadam[b]

[a] *School of Mathematics and Computer Science, Damghan University, Damghan, Iran*
[b] *Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran*
[c] *Member of Young Researchers Society of Shahid Bahonar University of Kerman, Kerman, P.O. Box 76169-14111, Iran*

(Communicated by S. Babaie-Kafaki)

Abstract

This paper introduces the *interval unilateral quadratic matrix equation*, $\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C} = 0$ and attempts to find various analytical results on its AE-solution sets in which $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are known real interval matrices, while X is an unknown matrix. These results are derived from a generalization of some results of Shary. We also give sufficient conditions for non-emptiness of some quasi-solution sets, provided that $\mathbf{A}$ is regular. As the most common case, the united solution set has been studied and two direct methods for computing an outer estimation and an inner estimation of the united solution set of an interval unilateral quadratic matrix equation are proposed. The suggested techniques are based on nonlinear programming as well as sensitivity analysis.

Keywords: AE-solution sets; interval unilateral quadratic matrix equation; united solution set; nonlinear programming; sensitivity analysis.
2010 MSC: Primary 65G40; Secondary 65F30.

1. Introduction

Given A, B and $C \in \mathbb{R}^{n\times n}$, consider the *unilateral quadratic matrix equation* (UQME)

$$F(X) := AX^2 + BX + C = 0, \tag{1.1}$$

*Corresponding author
Email addresses: Haqiri@math.uk.ac.ir, thaqiri@gmail.com (Tayyebe Haqiri), arivaz@uk.ac.ir (Azim Rivaz), mohseni@uk.ac.ir (Mahmoud Mohseni Moghadam)

Received: March 2017 *Revised:* July 2017

where $X \in \mathbb{R}^{n\times n}$ is unknown. Even though this equation is not the only possible definition for a quadratic matrix equation, it deserves a particular consideration since more common matrix equations can be reduced to this equation [3]. Moreover, the special features of equation (1.1), as the simplest nonlinear matrix equation, enable one to prove more specific results on the solutions [3]. Other forms of a quadratic matrix equation can be considered, for example, $X^2A+XB+C=0$ and the algebraic Riccati equation $XAX+BX+XC+D=0$. For the first one, a similar approach is possible. The theory and numerical methods of the second are also well developed in [4, 19, 26].

This UQME often occurs in many areas of scientific computing and engineering applications such as the quasi-birth-death process [3], the quadratic eigenvalue problem [31]

$$Q(\lambda, A, B, C)x := (\lambda^2 A + \lambda B + C)x = 0,$$

structural systems [32] and vibration problems [20].

Recall that any solution X of $AX^2+BX+C=0$ is called a *right solvent* or briefly a solvent to distinguish from a *left solvent*, which is a solution of the related quadratic matrix equation $X^2A+XB+C=0$. The existence and characterization of solvents can be related to the quadratic eigenvalue problem [13, 14, 26]. In fact, if we assume that A in (1.1) is invertible, then $Q(\lambda, A, B, C) = \lambda^2 A+\lambda B+C$ has $2n$ finite eigenvalues that can be ordered by their absolute values as

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{2n}|. \tag{1.2}$$

In order to study uniqueness and solvability aspects, we need to introduce Definition 1.1 and Theorem 1.2.

Definition 1.1. (See e.g. [18, Definition 2.2.1]) Let S_1 and S_2 be two solvents of UQME (1.1) and the eigenvalues of $Q(\lambda, A, B, C)$ are ordered as in (1.2). If S_1 has the spectrum $\sigma(S_1) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and S_2 has the spectrum $\sigma(S_2) = \{\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{2n}\}$, both satisfy the condition $|\lambda_n| > |\lambda_{n+1}|$, then S_1 and S_2 are called *dominant* solvent and *minimal* solvent, respectively.

Theorem 1.2. (See e.g. [18, Theorem 2.2.2]) Assume that the eigenvalues of $Q(\lambda, A, B, C)$, ordered according to (1.2) satisfy $|\lambda_n| > |\lambda_{n+1}|$ and that corresponding to $\{\lambda_i\}_{i=1}^n$ and $\{\lambda_i\}_{i=n+1}^{2n}$ there are two sets of linearly independent eigenvectors $\{v_1, v_2, \ldots, v_n\}$, $\{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$. Then there exists a dominant solvent and a minimal solvent of (1.1). If, further, the eigenvectors of $Q(\lambda, A, B, C)$ are distinct then the dominant and minimal solvents are unique.

Remark 1.3. It follows from the theory of λ-matrices that a dominant solvent and a minimal solvent of UQME (1.1), if they exist, are unique provided that A is nonsingular [18].

Remark 1.4. For real coefficients A, B and C, the dominant solvent of the UQME $AX^2+BX+C=0$ can not ever be complex, since the complex eigenvalues come in conjugate pairs with the same modulus, so if one takes the dominant eigenvalues you necessarily pick both entries of a conjugate pair.

For solving the quadratic matrix equation (1.1), Davis [5, 6] considered Newton's method in detail. Higham and Kim [13] incorporated exact line searches into Newton's method to improve the global convergence of Newton's method. Newton's method has improved with $\grave{S}$amanskii technique to acquire faster convergence in the work of Long, Hu and Zhang [21]. Two good references about numerical methods and algorithms for solving quadratic matrix equations are [18] and [26] and the references given. The problem of reducing an algebraic Riccati equation $XCX-AX-XD+B=0$ to a UQME of kind of equation (1.1) is also analyzed [4]. The problem of computing verified solutions to the quadratic matrix equations has been addressed before in the literature, see for instance [10, 12, 22]. But there are only a few works like as [11, 29] concerning the interval forms of these quadratic matrix equations. Besides, except when we are computing, nearly all measurements contain uncertainty as well as experiments and models of real life or physical phenomena. Interval analysis is one of the methods of representing uncertainty and/or ambiguity in mathematics. In interval analysis, uncertain parameters are described by a lower and upper bound then, sharp or nearly sharp bounds on the solution(s) are computed. You can find pointers to items concerning interval computations on the *interval computations website* `http://www.cs.utep.edu/interval-comp`.

Since the elements of A, B and C in equation (1.1) almost always contain doubt, they would represent in interval form to guarantee bounds on the set of possible result values. Interval analysis deals incidentally with evaluating the errors in answer resulting from the errors in the initial data. Thus, the following **interval unilateral quadratic matrix equation** (**IUQME**) should be solved

$$\mathbf{F}(X) := \mathbf{A}X^2 + \mathbf{B}X + \mathbf{C} = 0, \tag{1.3}$$

where $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are known real interval matrices and $A \in \mathbf{A}, B \in \mathbf{B}$ and $C \in \mathbf{C}$. Some special cases such as the interval quadratic equations and the interval univariate polynomials have also been investigated, see for example [9, 11]. In what follows, we assume that $\mathbf{A}$ in IUQME (1.3) is *regular* which means that each matrix A belongs to $\mathbf{A}$ is nonsingular. Up to our knowledge, this paper is the first attempt to address IUQME in *general* form.

Notice one essential point: due to the fact that the associativity of multiplication is not true with respect to the existing interval arithmetic, $\mathbf{A}X^2 := \mathbf{A}(XX)$ is not necessarily equal to $(\mathbf{A}X)X$.

The rest of this paper is organized as follows. In the next section, we review some facts and notations from interval arithmetic. Section 3 contains our main results concerning various AE-solution sets including the generalization of the AE-solution sets to IUQMEs and the characterization of a few main solution sets to IUQMEs. Explicit explanation of the united solution set and two methods for discovering outer and inner interval estimations of the united solution set are studied in Section 4. In Section 5, we demonstrate our results by means of some numerical examples. Section 6 is devoted to our conclusions and suggestions for further works.

2. Notations and preliminary concepts

We use $\mathbb{K}$ to denote either of the fields of real, $\mathbb{R}$ or complex numbers, $\mathbb{C}$. With the notations $\mathbb{K}^n, \mathbb{K}^{n\times n}, \mathbb{IK}^n$ and $\mathbb{IK}^{n\times n}$, we denote, respectively, the space of n-dimensional vectors, the space of $n \times n$ matrices, the set of all n-dimensional interval vectors (boxes) and the set of all $n \times n$ interval matrices, all over $\mathbb{K}$. In the present paper, all interval quantities will be typeset in boldface whereas lower case will imply scalar quantities or vectors and upper case will denote matrices. Under-scores and over-scores will show lower bounds and upper bounds of interval quantities, correspondingly.

For $\mathbf{x} \in \mathbb{IR}$, we write $\mathbf{x} = [\underline{x}, \overline{x}]$ where $\underline{x} := \min \mathbf{x}$ and $\overline{x} := \max \mathbf{x}$. The absolute value of $\mathbf{x}$ is shown as $|\mathbf{x}| := \max\{|x| \mid x \in \mathbf{x}\} = \max\{|\underline{x}|, |\overline{x}|\}$, the radius of $\mathbf{x}$ is defined by $\operatorname{rad} \mathbf{x} := 1/2(\overline{x}-\underline{x})$ and the midpoint of $\mathbf{x}$ is given by $\operatorname{mid} \mathbf{x} := 1/2(\overline{x}+\underline{x})$. The hull of two intervals $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{IR}$, $\square(\mathbf{x}, \mathbf{y})$, is the tightest interval $\mathbf{z}$ which encloses both $\mathbf{x}$ and $\mathbf{y}$. The intersection of two intervals $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{IR}$ is empty if they are disjoint, otherwise $\mathbf{x} \cap \mathbf{y} := \{z \in \mathbb{R} | z \in \mathbf{x}, z \in \mathbf{y}\} = [\max\{\underline{x}, \underline{y}\}, \min\{\overline{x}, \overline{y}\}]$. For $\mathbb{IR}$, real interval arithmetic is usually done via the set theory, namely, for any arithmetic operation $op \in \{+, -, *, /\}$, one has $\mathbf{x}\ op\ \mathbf{y} := \{x\ op\ y | x \in \mathbf{x}, y \in \mathbf{y}\}$.

In this paper, we also use circular complex intervals which have better algebraic properties than rectangular complex interval arithmetic [2]. Hence, a square interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ will write as $\mathbf{A} := [\operatorname{mid}\mathbf{A} - \operatorname{rad}\mathbf{A}, \operatorname{mid}\mathbf{A} + \operatorname{rad}\mathbf{A}]$ in which $\operatorname{mid}\mathbf{A}$ and $\operatorname{rad}\mathbf{A}$ are in $\mathbb{R}^{n\times n}$ with $\operatorname{rad}\mathbf{A} \geq 0$. Then, the extended definitions of addition, subtraction and multiplication of interval matrices $\mathbf{A}$ and $\mathbf{B}$ with compatible sizes are as follows

$$\mathbf{A}\ op\ \mathbf{B} := \square\{A\ op\ B | A \in \mathbf{A}, B \in \mathbf{B}\}.$$

In particular,

$$\mathbf{A}X^2 = \mathbf{A}(XX) := \square\{AX^2 | A \in \mathbf{A}\},$$

while

$$(\mathbf{A}X)X := \square\{GX | G \in \mathbf{A}X\} = \square\{GX | G \in \square\{AX | A \in \mathbf{A}\}\}.$$

Meanwhile, for interval vectors and matrices, $\operatorname{mid}, \operatorname{rad}, |.|, \square$ and intersection will be applied component-wise.

Let $\mathbf{A} = (\mathbf{A}_{ij})$ and $\mathbf{B} = (\mathbf{B}_{ij})$ be two real interval matrices with the same size. Then, we write $\mathbf{A} \subseteq \mathbf{B}$ whenever $\mathbf{A}_{ij} \subseteq \mathbf{B}_{ij}$ for all i, j where $\subseteq$ is a partial ordering defined as $\mathbf{A}_{ij} \subseteq \mathbf{B}_{ij}$ if and only if $\underline{B}_{ij} \leq \underline{A}_{ij}$ and $\overline{A}_{ij} \leq \overline{B}_{ij}$. Matrix inequalities as $A \leq B$ $(\geq)$ or $A < B$ $(>)$ are understood component-wise. Let $X = (X_{ij})$ belonging to $\mathbb{R}^{n\times n}$ be such that $X_{ij} \in \mathbf{A}_{ij}$ for all i, j, then we write $X \in \mathbf{A}$. For any bounded set of real $m \times n$ matrices such as Σ, the *interval hull* of Σ, $\square\Sigma := [\inf(\Sigma), \sup(\Sigma)]$, is the tightest interval matrix enclosing Σ.

Elementary properties of $\operatorname{mid}, \operatorname{rad}$ and $\square$ are provided by two next lemmas.

Lemma 2.1. (See e.g. [23]) Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ real interval matrices and X be a matrix with real elements and compatible size. Then,

1. $\mathbf{A} \subseteq \mathbf{B} \Leftrightarrow |\operatorname{mid}\mathbf{B} - \operatorname{mid}\mathbf{A}| \leq \operatorname{rad}\mathbf{B} - \operatorname{rad}\mathbf{A}$,
2. $\operatorname{mid}(\mathbf{A} \pm \mathbf{B}) = \operatorname{mid}\mathbf{A} \pm \operatorname{mid}\mathbf{B}$,
3. $\operatorname{rad}(\mathbf{A} \pm \mathbf{B}) = \operatorname{rad}\mathbf{A} + \operatorname{rad}\mathbf{B}$,
4. $\operatorname{mid}(\mathbf{A}X) = (\operatorname{mid}\mathbf{A})X$,
5. $\operatorname{rad}(\mathbf{A}X) = (\operatorname{rad}\mathbf{A})|X|$.

Lemma 2.2. [23, Proposition 3.1.8] Let $\mathbf{A}$ be a real interval matrix and Φ and Ω be two bounded sets of real point matrices, all of the same size. Then,

1. $\Phi \subseteq \Omega \Rightarrow \square\Phi \subseteq \square\Omega$,
2. $\Phi \subseteq \mathbf{A} \Rightarrow \square\Phi \subseteq \mathbf{A}$.

3. Description of the Generalized AE-solution Sets for IUQME

Shary introduced the concept of generalized solution sets and AE(AllExist)-solution sets to an interval linear system of equations [30]. In [30] quantifiers are used to describe and recognize various kinds of interval uncertainty in the course of modeling. By a similar convention, we consider the different possible styles of describing the uncertainty type distributions with respect to the interval parameters of IUQME (1.3). According to [30], we say that the interval $\mathbf{I}$ is of *A-uncertainty* whenever a certain property holds for all members from the given interval **I** while *E-uncertainty* implies that only some members of the interval **I** have that property, not necessarily all. For some recent works see, for example, [16] and [17] and the references given.

Here, we extend the concept of AE-solution sets to the IUQME (1.3).

Definition 3.1. We define the *AE-solution set of type* α, β, γ or $\alpha\beta\gamma$*-solution set* to the IUQME (1.3) as

$$\begin{aligned}\Sigma_{\alpha\beta\gamma}(\mathbf{A},\mathbf{B},\mathbf{C}) := \{X \in \mathbb{R}^{n\times n} | ((\forall A_{\pi'_1} \in \mathbf{A}_{\pi'_1}) \ldots (\forall A_{\pi'_p} \in \mathbf{A}_{\pi'_p})\\ (\forall B_{\theta'_1} \in \mathbf{B}_{\theta'_1}) \ldots (\forall B_{\theta'_r} \in \mathbf{B}_{\theta'_r})(\forall C_{\psi'_1} \in \mathbf{C}_{\psi'_1}) \ldots (\forall C_{\psi'_t} \in \mathbf{C}_{\psi'_t})\\ (\exists A_{\pi''_1} \in \mathbf{A}_{\pi''_1}) \ldots (\exists A_{\pi''_q} \in \mathbf{A}_{\pi''_q})(\exists B_{\theta''_1} \in \mathbf{B}_{\theta''_1}) \ldots (\exists B_{\theta''_s} \in \mathbf{B}_{\theta''_s})\\ (\exists C_{\psi''_1} \in \mathbf{C}_{\psi''_1}) \ldots (\exists C_{\psi''_u} \in \mathbf{C}_{\psi''_u})(AX^2 + BX + C = 0))\},\end{aligned}$$

where $\alpha = (\alpha_{ij}), \beta = (\beta_{ij})$ and $\gamma = (\gamma_{ij})$ are $n \times n$ *quantifier matrices* defined as

$$\alpha_{ij} := \begin{cases} \forall & \text{if } (i,j) \in \Pi', \\ \exists & \text{if } (i,j) \in \Pi'', \end{cases} \quad \beta_{ij} := \begin{cases} \forall & \text{if } (i,j) \in \Theta', \\ \exists & \text{if } (i,j) \in \Theta'', \end{cases}$$

and

$$\gamma_{i,j} := \begin{cases} \forall & \text{if } (i,j) \in \Psi', \\ \exists & \text{if } (i,j) \in \Psi'', \end{cases}$$

and with this attention that whenever $\Pi' = \emptyset$ we shall write $\alpha = \exists$ at the same time as $\Pi'' = \emptyset$ will be interpreted as $\alpha = \forall$. We also assume that the set of the index pairs (i, j) corresponding to $\mathbf{A}_{ij}$s of $\mathbf{A}$ is separated into two disjoint parts $\Pi' = \{\pi'_1, \pi'_2, \ldots, \pi'_p\}$ and $\Pi'' = \{\pi''_1, \pi''_2, \ldots, \pi''_q\}$ with $p + q = n^2$. These sets possess this limitation that (i, j) belongs to Π' if and only if the parameter $\mathbf{A}_{ij}$ of $\mathbf{A}$ is of A-uncertainty while being in Π'' means $\mathbf{A}_{ij}$ is of E-uncertainty and vice versa. In a comparable manner, we introduce two disjoint sets $\Theta' = \{\theta'_1, \theta'_2, \ldots, \theta'_r\}$ and $\Theta'' = \{\theta''_1, \theta''_2, \ldots, \theta''_s\}$ in which $r + s = n^2$ matching to β and $\Psi' = \{\psi'_1, \psi'_2, \ldots, \psi'_t\}$ and $\Psi'' = \{\psi''_1, \psi''_2, \ldots, \psi''_u\}$ in which $t + u = n^2$ tallied to γ.

Thus, there are two possibilities for the quantifier corresponding to the universal quantifier "$\forall$"and the existential quantifier "$\exists$". But the order of them is such that all the occurrences of the universal quantifier precede all the occurrences of the existential quantifier which clarifies the name *AE-form*.

To describe some particular cases, we determine disjoint decompositions of the interval matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. For this purpose, we define two interval matrices in a general form: "universal", $\mathbf{I}^{\forall} = (\mathbf{I}^{\forall}_{ij})$ and "existential", $\mathbf{I}^{\exists} = (\mathbf{I}^{\exists}_{ij})$ of the size n as

$$\mathbf{I}^{\forall}_{ij} := \begin{cases} \mathbf{I}_{ij}, & \text{if } \delta_{ij} = \forall, \\ 0, & \text{o.w.}, \end{cases} \quad \text{and} \quad \mathbf{I}^{\exists}_{ij} := \begin{cases} \mathbf{I}_{ij}, & \text{if } \delta_{ij} = \exists, \\ 0, & \text{o.w.}, \end{cases} \tag{3.1}$$

in which $\delta = (\delta_{ij})$ is the quantifier matrix associated to $\mathbf{I} = (\mathbf{I}_{ij})$. Thus, $\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists}, \mathbf{B} = \mathbf{B}^{\forall} + \mathbf{B}^{\exists}$ and $\mathbf{C} = \mathbf{C}^{\forall} + \mathbf{C}^{\exists}$. In addition, for all $i, j, 1 \le i, j \le n$, we have $\mathbf{A}^{\forall}_{ij}\mathbf{A}^{\exists}_{ij} = 0, \mathbf{B}^{\forall}_{ij}\mathbf{B}^{\exists}_{ij} = 0$ and $\mathbf{C}^{\forall}_{ij}\mathbf{C}^{\exists}_{ij} = 0$ which prove that $\mathbf{A}^{\forall}$ and $\mathbf{A}^{\exists}, \mathbf{B}^{\forall}$ and $\mathbf{B}^{\exists}$ as well as $\mathbf{C}^{\forall}$and $\mathbf{C}^{\exists}$ form disjoint decompositions for $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, separately. Now, we consider some special cases of Definition 3.1 as follows:

- The most interesting case among various possible choices occurs when we pick out the existential quantifier for all the components of quantifier matrices α, β and γ, i.e., where $\Pi' = \Theta' = \Psi' = \emptyset$: The *united solution set* to the IUQME (1.3),

$$\Sigma_{\exists\exists\exists}(\mathbf{A}, \mathbf{B}, \mathbf{C}) := \{X \in \mathbb{R}^{n\times n} | \\ ((\exists A \in \mathbf{A})(\exists B \in \mathbf{B})(\exists C \in \mathbf{C})(AX^2 + BX + C = 0))\}, \tag{3.2}$$

formed by all possible solutions of all point UQMEs, $AX^2 + BX + C = 0$ with $A \in \mathbf{A}, B \in \mathbf{B}$ and $C \in \mathbf{C}$. We may refer to each element of this solution set as the *weak solution* or briefly a *solution*.

- The *tolerable solution set* to the IUQME (1.3),

$$\Sigma_{\forall\forall\exists}(\mathbf{A}, \mathbf{B}, \mathbf{C}) := \{X \in \mathbb{R}^{n\times n} | \\ ((\forall A \in \mathbf{A})(\forall B \in \mathbf{B})(\exists C \in \mathbf{C})(AX^2 + BX + C = 0))\}.$$

Hence, the tolerable solution set is the set in which if $X \in \mathbb{R}^{n\times n}$ belongs to this set, then for each $A \in \mathbf{A}$ and each $B \in \mathbf{B}$, there exists at least one $C \in \mathbf{C}$ with $AX^2 + BX + C = 0$ or $AX^2 + BX \in -\mathbf{C}$.

- The *controllable solution set* to the IUQME (1.3),

$$\Sigma_{\exists\exists\forall}(\mathbf{A}, \mathbf{B}, \mathbf{C}) := \{X \in \mathbb{R}^{n\times n} | \\ ((\forall C \in \mathbf{C})(\exists A \in \mathbf{A})(\exists B \in \mathbf{B})(AX^2 + BX + C = 0))\},$$

formed by all matrices as $X \in \mathbb{R}^{n\times n}$ such that for any $C \in \mathbf{C}$, one could determine some $A \in \mathbf{A}$ and some $B \in \mathbf{B}$ satisfying $AX^2 + BX + C = 0$.

Remark 3.2. Note that always $\Sigma_{\alpha\beta\gamma}(\mathbf{A}, \mathbf{B}, \mathbf{C}) \subseteq \Sigma_{\exists\exists\exists}(\mathbf{A}, \mathbf{B}, \mathbf{C})$, that is, the united solution set of IUQME (1.3) is the widest solution set of all possible AE-solution sets of IUQME (1.3). An evident outcome is that if we have already found out that $\Sigma_{\exists\exists\exists}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is empty, we can conclude that the tolerable solution set and the controllable solution set to the IUQME (1.3) are also empty.

We now show with an example that there can be some cases in which the solutions to an IUQME can not be enclosed by a finite interval matrix.

Remark 3.3. It is going to be difficult to find a case in which we can bound all solvents of UQME (1.1), unless we impose additional restrictions on this matrix equation. This means that the solution sets of IUQME (1.3) can *not* be necessarily included in a finite interval matrix even in a trivial example. For instance, consider $\mathbf{A}_1 X^2 + \mathbf{B}_1 X + \mathbf{C}_1 = 0$ for which $\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1 \in \mathbb{IR}^{2\times 2}$ and $I \in \mathbf{A}_1, 0 \in \mathbf{B}_1, -I \in \mathbf{C}_1$ where I denotes the identity matrix of the compatible size. Then, for any arbitrary $a, b \in \mathbb{R}$ satisfying $b \neq 0$, the matrix

$$\frac{1}{1+a^2}\begin{bmatrix} a & b \\ \frac{1}{b} & -a \end{bmatrix}, \tag{3.3}$$

is a solvent of $X^2 = I$. So, the two off-diagonal entries are unbounded because one can take b large or small at will. It is obvious that the UQME $X^2 = I$ is one of the UQMEs belonging to $\Sigma_{\exists\exists\exists}(\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1)$. Hence, there are solutions with arbitrarily large or/and small entries. On the other hand, according to Definition 1.1, for any a, b with $b \neq 0$, (3.3) is neither dominant nor minimal solvent, since $Q(\lambda, I, 0, -I)$ has two eigenvalues 1 and -1, having the same multiplicity 2, and so it is impossible to find two sets of these eigenvalues such as $\{\lambda_1, \lambda_2\}$ and $\{\lambda_3, \lambda_4\}$ satisfying $\min\{|\lambda_1|, |\lambda_2|\} > \max\{|\lambda_3|, |\lambda_4|\}$.

From this point on, whenever we are talking about the solvents of (1.1), particularly in the definition of the solution sets, we mean only the *unique* dominant solvent which is the one of interest in most applications. For instance, we consider the union of all *dominant* solvents of all possible UQMEs $AX^2 + BX + C = 0$ with $A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}$ instead of (3.2).

Theorem 3.4.

$$\begin{aligned}\Sigma_{\alpha\beta\gamma}(\mathbf{A},\mathbf{B},\mathbf{C}) = \cap_{A'\in\mathbf{A}^{\forall}} \cap_{B'\in\mathbf{B}^{\forall}} \cap_{C'\in\mathbf{C}^{\forall}} \cup_{A''\in\mathbf{A}^{\exists}} \cup_{B''\in\mathbf{B}^{\exists}} \cup_{C''\in\mathbf{C}^{\exists}} \\ \{X \in \mathbb{R}^{n\times n} | ((A' + A'')X^2 + (B' + B'')X + (C' + C'') = 0)\}.\end{aligned}$$

In particular,

$$\Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C}) = \cup_{A\in\mathbf{A}} \cup_{B\in\mathbf{B}} \cup_{C\in\mathbf{C}} \{X \in \mathbb{R}^{n\times n} | AX^2 + BX + C = 0\},$$

$$\Sigma_{\forall\forall\exists}(\mathbf{A},\mathbf{B},\mathbf{C}) = \cap_{A\in\mathbf{A}} \cap_{B\in\mathbf{B}} \cup_{C\in\mathbf{C}} \{X \in \mathbb{R}^{n\times n} | AX^2 + BX + C = 0\},$$

$$\Sigma_{\exists\exists\forall}(\mathbf{A},\mathbf{B},\mathbf{C}) = \cap_{C\in\mathbf{C}} \cup_{A\in\mathbf{A}} \cup_{B\in\mathbf{B}} \{X \in \mathbb{R}^{n\times n} | AX^2 + BX + C = 0\}.$$

Proof . According to Definition 3.1, we can rewrite the $\alpha\beta\gamma$-solution set to the IUQME (1.3) as

$$\begin{aligned}&\Sigma_{\alpha\beta\gamma}(\mathbf{A},\mathbf{B},\mathbf{C}) = \{X \in \mathbb{R}^{n\times n} | ((\forall A' \in \mathbf{A}^{\forall})(\forall B' \in \mathbf{B}^{\forall})(\forall C' \in \mathbf{C}^{\forall}) \\ &(\exists A'' \in \mathbf{A}^{\exists})(\exists B'' \in \mathbf{B}^{\exists})(\exists C'' \in \mathbf{C}^{\exists}) \\ &((A' + A'')X^2 + (B' + B'')X + (C' + C'') = 0))\} \\ &= \bigcap_{A'\in\mathbf{A}^{\forall}} \bigcap_{B'\in\mathbf{B}^{\forall}} \bigcap_{C'\in\mathbf{C}^{\forall}} \{X \in \mathbb{R}^{n\times n} | ((\exists A'' \in \mathbf{A}^{\exists})(\exists B'' \in \mathbf{B}^{\exists})(\exists C'' \in \mathbf{C}^{\exists})| \\ &((A' + A'')X^2 + (B' + B'')X + (C' + C'') = 0))\} \\ &= \bigcap_{A'\in\mathbf{A}^{\forall}} \bigcap_{B'\in\mathbf{B}^{\forall}} \bigcap_{C'\in\mathbf{C}^{\forall}} \bigcup_{A''\in\mathbf{A}^{\exists}} \bigcup_{B''\in\mathbf{B}^{\exists}} \bigcup_{C''\in\mathbf{C}^{\exists}} \\ &\{X \in \mathbb{R}^{n\times n} | ((A' + A'')X^2 + (B' + B'')X + (C' + C'') = 0)\},\end{aligned}$$

in which the second and third equalities are due to the definitions of intersection and union of sets, respectively. The characterizations of the united, tolerable and controllable solution sets are evident results of the general case above. □

Theorem 3.5. $X \in \mathbb{R}^{n\times n}$ *belongs to* $\Sigma_{\alpha\beta\gamma}(\mathbf{A},\mathbf{B},\mathbf{C})$ *if and only if*

$$\begin{aligned}&\{A'X^2 + B'X + C' | A' \in \mathbf{A}^{\forall}, B' \in \mathbf{B}^{\forall}, C' \in \mathbf{C}^{\forall}\} \subseteq \\ &\{-(A''X^2 + B''X + C'') | A'' \in \mathbf{A}^{\exists}, B'' \in \mathbf{B}^{\exists}, C'' \in \mathbf{C}^{\exists}\}.\end{aligned} \tag{3.4}$$

Proof . Again, by exploiting the matrices $\mathbf{A}^{\forall}, \mathbf{A}^{\exists}, \mathbf{B}^{\forall}, \mathbf{B}^{\exists}, \mathbf{C}^{\forall}$ and $\mathbf{C}^{\exists}$, one can recompose the definition of $\Sigma_{\alpha\beta\gamma}(\mathbf{A},\mathbf{B},\mathbf{C})$ in the following equivalent form

$$\begin{aligned}&\Sigma_{\alpha\beta\gamma}(\mathbf{A},\mathbf{B},\mathbf{C}) = \{X \in \mathbb{R}^{n\times n} | \\ &((\forall A' \in \mathbf{A}^{\forall})(\forall B' \in \mathbf{B}^{\forall})(\forall C' \in \mathbf{C}^{\forall}) \\ &(\exists A'' \in \mathbf{A}^{\exists})(\exists B'' \in \mathbf{B}^{\exists})(\exists C'' \in \mathbf{C}^{\exists}) \\ &((A' + A'')X^2 + (B' + B'')X + (C' + C'') = 0))\}.\end{aligned} \tag{3.5}$$

Thus, for all $A' \in \mathbf{A}^{\forall}, B' \in \mathbf{B}^{\forall}, C' \in \mathbf{C}^{\forall}$, one can find some $A'' \in \mathbf{A}^{\exists}, B'' \in \mathbf{B}^{\exists}, C'' \in \mathbf{C}^{\exists}$ such that

$$A'X^2 + B'X + C' = -(A''X^2 + B''X + C''). \tag{3.6}$$

Now, let $X \in \Sigma_{\alpha\beta\gamma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and $D \in \{A'X^2 + B'X + C' | A' \in \mathbf{A}^{\forall}, B' \in \mathbf{B}^{\forall}, C' \in \mathbf{C}^{\forall}\}$, so there exist some matrices like as $A'_1 \in \mathbf{A}^{\forall}, B'_1 \in \mathbf{B}^{\forall}$ and $C'_1 \in \mathbf{C}^{\forall}$ for which $D = A'_1X^2 + B'_1X + C'_1$. (3.6) implies that for adequate matrices $A''_1 \in \mathbf{A}^{\exists}, B''_1 \in \mathbf{B}^{\exists}$ and $C''_1 \in \mathbf{C}^{\exists}$, we see that $D = -(A''_1X^2 + B''_1X + C''_1)$. Thus, (3.4) is achieved. Now, assume that (3.4) holds. According to (3.5), it is sufficient to show that for all $A' \in \mathbf{A}^{\forall}, B' \in \mathbf{B}^{\forall}$ and $C' \in \mathbf{C}^{\forall}$, there exist $A'' \in \mathbf{A}^{\exists}, B'' \in \mathbf{B}^{\exists}$ and $C'' \in \mathbf{C}^{\exists}$ such that

$$(A' + A'')X^2 + (B' + B'')X + (C' + C'') = 0.$$

Applying (3.4), there are $A'' \in \mathbf{A}^{\exists}, B'' \in \mathbf{B}^{\exists}$ and $C'' \in \mathbf{C}^{\exists}$ for which (3.6) is true or $(A' + A'')X^2 + (B' + B'')X + (C' + C'') = 0$. $\square$

So, the fundamental theorem for characterizations of the AE-solution sets of an interval linear system ($x \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ if and only if $\mathbf{A}^{\forall}x - \mathbf{b}^{\forall} \subseteq \mathbf{b}^{\exists} - \mathbf{A}^{\exists}x$) [30, Theorem 3.4] can not be completely generalized here.

Remark 3.6. As a special case of the definition of the interval arithmetic operations, only the weak equality

$$\mathbf{D}Y = \square\{DY \mid D \in \mathbf{D}\} \tag{3.7}$$

is valid for any $\mathbf{D} \in \mathbb{IR}^{n\times k}$ and any $Y \in \mathbb{R}^{k\times m}$ unless when Y is a real *vector*, i.e., $m = 1$ [23, Proposition 3.1.4]. Since the expression $\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}$ is a *single-use expression* with respect to all interval variables, if we put $\mathbf{D} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]$ and $Y = [X^2, X, I]^T$ where T denotes the transpose, then $\mathbf{D} \in \mathbb{IR}^{n\times 3n}, Y \in \mathbb{IR}^{3n\times n}$ and also we have

$$\begin{aligned} \mathbf{A}X^2 + \mathbf{B}X + \mathbf{C} = \mathbf{D}Y &:= \square\{DY | D \in \mathbf{D}\} \\ = \square\{AX^2 + BX + C | A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}\}. \end{aligned} \tag{3.8}$$

Now, we characterize the solution sets $\Sigma_{\exists\exists\gamma}$ (*quasi-controllable*) and $\Sigma_{\forall\forall\gamma}$ (*quasi-tolerable*) which are indeed two generalizations of the controllable and tolerable solution sets, respectively.

Theorem 3.7. *Suppose $X \in \mathbb{R}^{n\times n}$ belongs to the AE-solution set $\Sigma_{\exists\exists\gamma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Then*

$$-\mathbf{C}^{\forall} \subseteq \mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}^{\exists}. \tag{3.9}$$

Proof . Let $X \in \Sigma_{\exists\exists\gamma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Then, (3.4) leads to

$$\{C' | C' \in \mathbf{C}^{\forall}\} \subseteq \{-(A''X^2 + B''X + C'') | A'' \in \mathbf{A}, B'' \in \mathbf{B}, C'' \in \mathbf{C}^{\exists}\}.$$

Lemma 2.2 part 1 and (3.8) give us

$$\begin{aligned} &\mathbf{C}^{\forall} = \square\{C' | C' \in \mathbf{C}^{\forall}\} \subseteq \\ &\square\{-(A''X^2 + B''X + C'') | A'' \in \mathbf{A}, B'' \in \mathbf{B}, C'' \in \mathbf{C}^{\exists}\}. \end{aligned}$$

A similar argument for the proof of (3.8) implies

$$\mathbf{C}^{\forall} \subseteq -(\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}^{\exists}).$$

On the other hand, for two arbitrary intervals, we have [2] $\mathbf{x} \subseteq \mathbf{y} \Rightarrow -\mathbf{x} \subseteq -\mathbf{y}$. Therefore, we deduce that $-\mathbf{C}^{\forall} \subseteq \mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}^{\exists}$. $\square$

Here is an elementary consequent of Theorem 3.7.

Corollary 3.8. If $X \in \Sigma_{\exists\exists\forall}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ then $-\mathbf{C} \subseteq \mathbf{A}X^2 + \mathbf{B}X$.

Theorem 3.9. *If*

$$\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}^{\forall} \subseteq -\mathbf{C}^{\exists}, \tag{3.10}$$

then $X \in \Sigma_{\forall\forall\gamma}$.

Proof . Suppose (3.10) holds. It is obvious that

$$\{AX^2 + BX + C' | A \in \mathbf{A}, B \in \mathbf{B}, C' \in \mathbf{C}^{\forall}\} \subseteq$$
$$\square\{AX^2 + BX + C' | A \in \mathbf{A}, B \in \mathbf{B}, C' \in \mathbf{C}^{\forall}\}.$$

Similar to (3.8), we write

$$\square\{AX^2 + BX + C' | A \in \mathbf{A}, B \in \mathbf{B}, C' \in \mathbf{C}^{\forall}\} =$$
$$\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}^{\forall} \subseteq -\mathbf{C}^{\exists} = \{-C'' | C'' \in \mathbf{C}^{\exists}\}.$$

We have thus proved

$$\{AX^2 + BX + C' | A \in \mathbf{A}, B \in \mathbf{B}, C' \in \mathbf{C}^{\forall}\} \subseteq \{-C'' | C'' \in \mathbf{C}^{\exists}\}.$$

Now, Theorem 3.5 for $\alpha = \beta = \forall$ allows us to conclude $X \in \Sigma_{\forall\forall\gamma}$. $\square$

Corollary 3.10. If $\mathbf{A}X^2 + \mathbf{B}X \subseteq -\mathbf{C}$ then $X \in \Sigma_{\forall\forall\exists}(\mathbf{A}, \mathbf{B}, \mathbf{C})$.

Next consequences are new characterizations of the AE-solution sets $\Sigma_{\exists\exists\gamma}$ and $\Sigma_{\forall\forall\gamma}$ in terms of midpoint and radius matrices.

Theorem 3.11. *If* $X \in \Sigma_{\exists\exists\gamma}$ *then*

$$|(\operatorname{mid}\mathbf{A})X^2 + (\operatorname{mid}\mathbf{B})X + \operatorname{mid}\mathbf{C}| \leq$$
$$(\operatorname{rad}\mathbf{A})X^2 + (\operatorname{rad}\mathbf{B})|X| + \operatorname{rad}\mathbf{C}^{\exists} - \operatorname{rad}\mathbf{C}^{\forall}. \tag{3.11}$$

Proof . Since $X \in \Sigma_{\exists\exists\gamma}$, Theorem 3.7 together with Lemma 2.1 part 1 follows

$$|(\operatorname{mid}(\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}^{\exists}) - \operatorname{mid}(-\mathbf{C}^{\forall})| \leq$$
$$\operatorname{rad}(\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}^{\exists}) - \operatorname{rad}(-\mathbf{C}^{\forall}).$$

By repeatedly using lemma 2.1, we have

$$|(\operatorname{mid}\mathbf{A})X^2 + (\operatorname{mid}\mathbf{B})X + \operatorname{mid}\mathbf{C}^{\exists} + \operatorname{mid}\mathbf{C}^{\forall}| \leq$$
$$(\operatorname{rad}\mathbf{A})X^2 + (\operatorname{rad}\mathbf{B})|X| + \operatorname{rad}\mathbf{C}^{\exists} - \operatorname{rad}\mathbf{C}^{\forall}.$$

In view of

$$\operatorname{mid}\mathbf{C}^{\exists} + \operatorname{mid}\mathbf{C}^{\forall} = \operatorname{mid}(\mathbf{C}^{\exists} + \mathbf{C}^{\forall}) = \operatorname{mid}\mathbf{C},$$

the proof is completed. $\square$

Corollary 3.12. Let $X \in \Sigma_{\exists\exists\forall}(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Then,

$$|(\operatorname{mid}\mathbf{A})X^2 + (\operatorname{mid}\mathbf{B})X + \operatorname{mid}\mathbf{C}| \leq (\operatorname{rad}\mathbf{A})X^2 + (\operatorname{rad}\mathbf{B})|X| - \operatorname{rad}\mathbf{C}.$$

Theorem 3.13. *If*

$$|(\operatorname{mid}\mathbf{A})X^2 + (\operatorname{mid}\mathbf{B})X + \operatorname{mid}\mathbf{C}| \le \operatorname{rad}\mathbf{C}^{\exists} - \operatorname{rad}\mathbf{C}^{\forall} - (\operatorname{rad}\mathbf{A})X^2 - (\operatorname{rad}\mathbf{B})|X|, \tag{3.12}$$

then $X \in \Sigma_{\forall\forall\gamma}$.

Proof . Since $\mathbf{C} = \mathbf{C}^{\exists} + \mathbf{C}^{\forall}$, (3.12) is equivalent to

$$\begin{aligned} &|\operatorname{mid}(\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}^{\forall}) - \operatorname{mid}(-\mathbf{C}^{\exists})| \le \\ &\operatorname{rad}(-\mathbf{C}^{\exists}) - \operatorname{rad}(\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}^{\forall}). \end{aligned} \tag{3.13}$$

Next, Lemma 2.1 part 1 yields $\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}^{\forall} \subseteq -\mathbf{C}^{\exists}$. Now, Theorem 3.9 implies that $X \in \Sigma_{\forall\forall\gamma}$. $\square$

Corollary 3.14. Suppose

$$|(\operatorname{mid}\mathbf{A})X^2 + (\operatorname{mid}\mathbf{B})X + \operatorname{mid}\mathbf{C}| \le \operatorname{rad}\mathbf{C} - (\operatorname{rad}\mathbf{A})X^2 - (\operatorname{rad}\mathbf{B})|X|.$$

Then, $X \in \Sigma_{\forall\forall\exists}(\mathbf{A}, \mathbf{B}, \mathbf{C})$.

Following the same idea for describing solvability of a system of interval linear equations $\mathbf{A}x = \mathbf{b}$ [15], we define the solvability concept for $\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C} = 0$.

Definition 3.15. The interval unilateral quadratic matrix equation (1.3) is called *solvable* if there exists $A \in \mathbf{A}, B \in \mathbf{B}$ and $C \in \mathbf{C}$ such that UQME $AX^2 + BX + C = 0$ has a (dominant) solvent.

Obviously, solvability of IUQME is equivalent to the existence of a weak solution or non-emptiness of the united solution set.

Theorem 3.16. Consider IUQME (1.3). Assume that for any $C \in \mathbf{C}$, there are some $A \in \mathbf{A}$ and some $B \in \mathbf{B}$ such that $Q(\lambda, A, B, C)$ has distinct eigenvalues satisfying the conditions of Theorem 1.2. Then, $\Sigma_{\exists\exists\gamma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is nonempty.

Proof . Suppose $C_1 \in \mathbf{C}$. So, there exist $A_1 \in \mathbf{A}$ and $B_1 \in \mathbf{B}$ such that $Q(\lambda, A_1, B_1, C_1)$ has distinct eigenvalues satisfying the conditions of Theorem 1.2 which are sufficient conditions for the existence and uniqueness of dominant (and minimal) solvents. This completes the proof. $\square$

It is obvious that if for any members A, B and C of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, respectively, $Q(\lambda, A, B, C)$ has distinct eigenvalues which satisfy the conditions of Theorem 1.2, then $\Sigma_{\forall\forall\gamma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ will also be nonempty.

4. Detailed characterization of the united solution set of IUQME

The united solution set has many applications in the field of scientific computing such as computational optimization, numerical simulation and verification in system engineering [1]. Remark 3.2 displays another reason to be interested in the united solution set. Accordingly, we focus on the united solution set of IUQME in this part. The characterization of the solution sets is, however, hard to derive since it is indeed equivalent to the characterization of standard interval systems with dependencies which is hard to deal with; see e.g. [28].

Theorem 4.1.
$$\Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C}) \subseteq \{X \in \mathbb{R}^{n\times n} | (\mathbf{A}X^2 + \mathbf{B}X) \cap (-\mathbf{C}) \neq \emptyset\},$$

and

$$\Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C}) \subseteq \{X \in \mathbb{R}^{n\times n} | 0 \in \mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}\}.$$

Moreover,

$$\begin{aligned}&\Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C}) \subseteq \{X \in \mathbb{R}^{n\times n} | \\ &|(\operatorname{mid}\mathbf{A})X^2 + (\operatorname{mid}\mathbf{B})X + \operatorname{mid}\mathbf{C}| \\ &\le (\operatorname{rad}\mathbf{A})X^2 + (\operatorname{rad}\mathbf{B})|X| + \operatorname{rad}\mathbf{C}\}.\end{aligned} \tag{4.1}$$

Proof . Let $X \in \Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C})$. So, $AX^2 + BX = -C$ for some $A \in \mathbf{A}, B \in \mathbf{B}$ and $C \in \mathbf{C}$; hence $-C \in (\mathbf{A}X^2 + \mathbf{B}X) \cap (-\mathbf{C})$. Since, $(\mathbf{A}X^2 + \mathbf{B}X) \cap (-\mathbf{C}) \neq \emptyset$ if and only if $0 \in \mathbf{A}X^2 + \mathbf{B}X + \mathbf{C}$, the first and second parts of the theorem follow. For the third part, put $\mathbf{D} := \mathbf{A}X^2 + \mathbf{B}X$. Then,

$$\begin{aligned}&(\mathbf{A}X^2 + \mathbf{B}X) \cap (-\mathbf{C}) \neq \emptyset \Leftrightarrow \mathbf{D} \cap (-\mathbf{C}) \neq \emptyset \\ &\Leftrightarrow |\operatorname{mid}\mathbf{D} - \operatorname{mid}(-\mathbf{C})| \le \operatorname{rad}\mathbf{D} + \operatorname{rad}(-\mathbf{C}).\end{aligned}$$

Since $\mathbf{D} = \mathbf{A}X^2 + \mathbf{B}X$ and by means of Lemma 2.1 we conclude that

$$|(\operatorname{mid}\mathbf{A})X^2 + (\operatorname{mid}\mathbf{B})X + \operatorname{mid}\mathbf{C}| \le (\operatorname{rad}\mathbf{A})X^2 + (\operatorname{rad}\mathbf{B})|X| + \operatorname{rad}\mathbf{C}.$$

One can also put $\mathbf{C}^{\forall} = 0$ in Theorem 3.11 to prove the last part. □

None of the AE-solution sets, $\Sigma_{\alpha\beta\gamma}(\mathbf{A},\mathbf{B},\mathbf{C})$, is generally an interval matrix. Thus, it is natural to look for some interval matrices which are either contained in the solution set (*inner interval estimate problem*) or contain the solution set (*outer interval estimate problem*) where the inclusions are as sharp as possible. Besides, only some estimate of the rigorous solution set suffices for factual goals in real life conditions. From now on, we confine ourselves to find some interval estimations only for nonempty and bounded AE-solution sets of type $\exists\exists\exists$.

4.1. Outer estimation of $\Sigma_{\exists\exists\exists}$ via a nonlinear programming approach

By definition, $\square\Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C})$ is the tightest interval matrix enclosing $\Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C})$. Therefore, it could be supposed as the sharpest outer estimation for $\Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C})$. Since we have assumed that the solution set $\Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C})$ is *nonempty* and *bounded*, we can define its *exact* interval hull as

$$\square\Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C}) := [\underline{X}, \overline{X}],$$

where for $i, j = 1 : n$,

$$\begin{cases} \underline{X} = (\underline{X}_{ij}), \quad \underline{X}_{ij} = \inf\{X_{ij} | X = (X_{ij}) \in \Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C})\}, \\ \overline{X} = (\overline{X}_{ij}), \quad \overline{X}_{ij} = \sup\{X_{ij} | X = (X_{ij}) \in \Sigma_{\exists\exists\exists}(\mathbf{A},\mathbf{B},\mathbf{C})\}. \end{cases}$$

The inequality appeared in the last part of Theorem 4.1 provides us with an approach to discover an outer estimation $\mathbf{X}_{\text{nlp}} := [\underline{X}_{\text{nlp}}, \overline{X}_{\text{nlp}}]$ for the interval hull which thus is an outer estimation for $\Sigma_{\exists\exists\exists}$ as well. Indeed, (4.1) turns out to

$$\begin{cases} (\operatorname{mid}\mathbf{A})X^2 - (\operatorname{rad}\mathbf{A})X^2 + (\operatorname{mid}\mathbf{B})X - (\operatorname{rad}\mathbf{B})|X| \le -\underline{C}, \\ (\operatorname{mid}\mathbf{A})X^2 + (\operatorname{rad}\mathbf{A})X^2 + (\operatorname{mid}\mathbf{B})X + (\operatorname{rad}\mathbf{B})|X| \ge -\overline{C}, \end{cases} \tag{4.2}$$

in which $X = (X_{ij})$ is an arbitrary member of $\Sigma_{\exists\exists\exists}(\mathbf{A}, \mathbf{B}, \mathbf{C})$. If $S = (S_{ij})$ denotes the *sign matrix* of X, then $|X| = S \odot X$ where $\odot$ denotes the so-called *Hadamard* or *component-wise product*. Now, in order to determine the lower and upper bound for each element X_{ij} of X, the following nonlinear programming problems for all possible cases of S and fixed $i, j, 1 \le i, j \le n$ should be solved:

$$\begin{gathered} \min / \max \quad X_{ij} \\ \text{s.t.} \\ \begin{cases} \underline{A}X^2 + (\operatorname{mid}\mathbf{B})X - (\operatorname{rad}\mathbf{B})(S \odot X) \le -\underline{C}, \\ \overline{A}X^2 + (\operatorname{mid}\mathbf{B})X + (\operatorname{rad}\mathbf{B})(S \odot X) \ge -\overline{C}. \end{cases} \end{gathered} \tag{4.3}$$

In 1964, Oettli and Prager [24] also proved a nice characterization of the weak solutions of standard interval systems of linear equations. The main merit of the Oettli-Prager theorem consists in the fact that it describes the set of all weak solutions by means of a single nonlinear inequality.

Therefore, we are required to solve $2n^2 \times 2^{n^2}$ nonlinear programming problems which may grow exponentially as the dimension n increases, causing this algorithm cumbersome and time-consuming for matrices of high order. Another drawback, however, is that we will not be able to consider only dominant solvents, so the enclosure is expected to be wider than it actually is. It is worth

Algorithm 1 A nonlinear programming algorithm to compute an enclosure $\mathbf{X}_{\text{nlp}}$ for the interval hull of the united solution set to the IUQME (1.3).

```
Given A, B, C
for k = 1, ..., 2^{n^2} do
    Compute S_k as k-th possible matrix for the sign of an n × n matrix
    for i, j = 1, ..., n do
        Solve the nonlinear programming problems appeared in (4.3) to maximize and minimize
        X_{ij}^{(k)} using S_k instead of S
    end for
end for
(X̄_nlp)_{ij} = max_k {X_{ij}^{(k)}}
(X̲_nlp)_{ij} = min_k {X_{ij}^{(k)}}
Output X_nlp = [X̲_nlp, X̄_nlp]
```

pointing out that in view of Theorem 3.11, a similar scheme can also be exploited to discover an outer estimation for the solution sets $\Sigma_{\exists\exists\gamma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$.

4.2. Inner estimation of $\Sigma_{\exists\exists\exists}$ via a sensitivity analysis technique

In this section the sensitivity of X in $AX^2 + BX + C$ to changes in A, B and C will be our main concern. Clearly, we may assume that X is a (differentiable) function of A, B and C and it is always assumed that the elements of X are *not* dependent, namely X has no specific structure. This technique comes from [7, Chapters 1.6–2.3] and primarily involves solving some linear systems.

By means of the sensitivity analysis technique, we consider a finite number of adequate unilateral quadratic matrix equations of form (1.1) instead of interval form (1.3). In fact, we want to find a way to approach as much as possible the end corners of $\mathbf{X}_{\text{sns}} := [\underline{X}_{\text{sns}}, \overline{X}_{\text{sns}}]$. As mentioned in [7], we expect that even for large values of $\operatorname{rad}\mathbf{A}, \operatorname{rad}\mathbf{B}$ and $\operatorname{rad}\mathbf{C}$, the error in $\mathbf{X}_{\text{sns}}$ depends only on the machine precision. To discover these adequate UQMEs, we consider the signs of some calculated partial derivatives. This point is explained in more detail in the following paragraphs.

From [25], we have the following results about the first order derivative of differentiable matrices.

Lemma 4.2. [25] Let R and S be two differentiable matrices of compatible sizes. Then,

1. $\partial(R+S) = \partial R + \partial S$,
2. $\partial(RS) = (\partial R)S + R(\partial S)$,
3. $\frac{\partial R}{\partial R_{ij}} = J^{ij}$,

in which J^{ij} is the single entry matrix, 1 at (i,j) and zero elsewhere.

In what follows, we assume that the selected matrices $A \in \mathbf{A}, B \in \mathbf{B}$ and $C \in \mathbf{C}$ are always mutually independent. Then, with the use of Lemma 4.2 for finding partial derivatives from the common equation $AX^2 + BX + C = 0$ with respect to A_{ij}, B_{ij} and C_{ij}, $1 \le i, j \le n$, we obtain these ordinary matrix equations

$$\begin{cases} \dfrac{\partial A}{\partial A_{ij}} X^2 + A\dfrac{\partial X}{\partial A_{ij}} X + AX\dfrac{\partial X}{\partial A_{ij}} + B\dfrac{\partial X}{\partial A_{ij}} = 0, \\ A\dfrac{\partial X}{\partial B_{ij}} X + AX\dfrac{\partial X}{\partial B_{ij}} + \dfrac{\partial B}{\partial B_{ij}} X + B\dfrac{\partial X}{\partial B_{ij}} = 0, \\ A\dfrac{\partial X}{\partial C_{ij}} X + AX\dfrac{\partial X}{\partial C_{ij}} + B\dfrac{\partial X}{\partial C_{ij}} + \dfrac{\partial C}{\partial C_{ij}} = 0. \end{cases} \tag{4.4}$$

We recall that for a point matrix $G \in \mathbb{K}^{p\times q}$, the vector $\mathrm{vec}(G) \in \mathbb{K}^{pq}$ denotes the column-wise vectorization whereby the successive columns of G are stacked one below the other, beginning with the first column and ending with the last. Also, the Kronecker product of two matrices $G \in \mathbb{K}^{p\times q}$ and $H \in \mathbb{K}^{r\times s}$, denoted by $G\otimes H$, is defined as $G\otimes H := [G_{ij}H] \in \mathbb{K}^{pr\times qs}$. Now, (4.4) can be written in the vector form as

$$\begin{cases} \left(X^T \otimes A + I \otimes (AX+B)\right) \mathrm{vec}\left(\dfrac{\partial X}{\partial A_{ij}}\right) = -\,\mathrm{vec}\left(J^{ij}X^2\right), \\ \left(X^T \otimes A + I \otimes (AX+B)\right) \mathrm{vec}\left(\dfrac{\partial X}{\partial B_{ij}}\right) = -\,\mathrm{vec}\left(J^{ij}X\right), \\ \left(X^T \otimes A + I \otimes (AX+B)\right) \mathrm{vec}\left(\dfrac{\partial X}{\partial C_{ij}}\right) = -\,\mathrm{vec}\left(J^{ij}\right), \end{cases} \tag{4.5}$$

in which we have used the fact that for multiplication of three matrices G, H, K of compatible sizes, the equality $\mathrm{vec}(GHK) = (K^T \otimes G)\,\mathrm{vec}\,(H)$ holds, see for instance [12].

Each of the equations in (4.5) illustrates n^2 linear matrix equations to be separately solved for $\mathrm{vec}\left(\frac{\partial X}{\partial A_{ij}}\right)$, $\mathrm{vec}\left(\frac{\partial X}{\partial B_{ij}}\right)$ and $\mathrm{vec}\left(\frac{\partial X}{\partial C_{ij}}\right)$ while A and B are replaced by $\mathrm{mid}\,\mathbf{A}$ and $\mathrm{mid}\,\mathbf{B}$, respectively, and X is the unique dominant solvent for $(\mathrm{mid}\,\mathbf{A})X^2 + (\mathrm{mid}\,\mathbf{B})X + \mathrm{mid}\,\mathbf{C} = 0$. If the UQME $(\mathrm{mid}\,\mathbf{A})X^2 + (\mathrm{mid}\,\mathbf{B})X + \mathrm{mid}\,\mathbf{C} = 0$ does not have a dominant solvent, we may use a dominant solvent of any UQME $AX^2 + BX + C = 0$ with $A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}$ and update the other parts according to the new choice.

Now, by considering the signs of the matrices $\frac{\partial X}{\partial A_{ij}}$, $\frac{\partial X}{\partial B_{ij}}$ and $\frac{\partial X}{\partial C_{ij}}$, required suitable matrices corresponding to A, B and C in $AX^2+BX+C = 0$ are determined. Indeed, these UQMEs are the ones which should be solved to discover the extremes of $\mathbf{X}_{\mathrm{sns}}$. For example, if for fixed $l, k, 1 \le l, k \le n$,

$\frac{\partial X_{lk}}{\partial A_{ij}} > 0, \frac{\partial X_{lk}}{\partial B_{ij}} > 0$ and $\frac{\partial X_{lk}}{\partial C_{ij}} > 0$, then the choices $\overline{A}_{ij}, \overline{B}_{ij}$ and $\overline{C}_{ij}$ for A, B and C, respectively, will make us approach lk-th element in $\overline{X}_{\text{sns}}$ while the options $\underline{A}_{ij}, \underline{B}_{ij}$ and $\underline{C}_{ij}$ will lead us to lk-th element in $\underline{X}_{\text{sns}}$. This means that, to approximate $(\overline{X}_{\text{sns}})_{lk}, 1 \le l, k \le n$, we set $A^{lk,\text{up}} := (A_{ij}^{lk,\text{up}})$ as

$$A_{ij}^{lk,\text{up}} := \begin{cases} \overline{A}_{ij}, & \frac{\partial X_{lk}}{\partial A_{ij}} > 0, \\ \underline{A}_{ij}, & \frac{\partial X_{lk}}{\partial A_{ij}} < 0, \\ (\operatorname{mid} \mathbf{A})_{ij}, & \frac{\partial X_{lk}}{\partial A_{ij}} = 0. \end{cases} \tag{4.6}$$

To define $A^{lk,\text{down}} := (A_{ij}^{lk,\text{down}})$ and then approximate $(\underline{X}_{\text{sns}})_{lk}, 1 \le l, k \le n$, we select in the opposite way, that is

$$A_{ij}^{lk,\text{down}} := \begin{cases} \underline{A}_{ij}, & \frac{\partial X_{lk}}{\partial A_{ij}} > 0, \\ \overline{A}_{ij}, & \frac{\partial X_{lk}}{\partial A_{ij}} < 0, \\ (\operatorname{mid} \mathbf{A})_{ij}, & \frac{\partial X_{lk}}{\partial A_{ij}} = 0. \end{cases} \tag{4.7}$$

In the particular case where $\frac{\partial X_{lk}}{\partial G_{ij}} = 0$, the variable X_{lk} is said to be *insensitive* to any perturbation in G_{ij} where $G \in \{A, B, C\}$. Besides, the superscripts "up" and "down"emphasis on approaching $\overline{X}_{\text{sns}}$ and $\underline{X}_{\text{sns}}$, respectively. In the same way, we specify $B_{ij}^{lk,\text{up}}, B_{ij}^{lk,\text{down}}$ and $C_{ij}^{lk,\text{up}}, C_{ij}^{lk,\text{down}}$. Afterwards, for the calculated matrices $A^{lk,\text{up}}, B^{lk,\text{up}}$, $C^{lk,\text{up}}$, and $A^{lk,\text{down}}, B^{lk,\text{down}}$, $C^{lk,\text{down}}$, $2n^2$ UQMEs

$$A^{lk,\text{up}}X^2 + B^{lk,\text{up}}X + C^{lk,\text{up}} = 0, \tag{4.8}$$

and

$$A^{lk,\text{down}}X^2 + B^{lk,\text{down}}X + C^{lk,\text{down}} = 0, \tag{4.9}$$

should be solved to discover the bounds of $X_{lk} \in [(\underline{X}_{\text{sns}})_{lk}, (\overline{X}_{\text{sns}})_{lk}]$. More precisely, for estimating $(\overline{X}_{\text{sns}})_{lk}$ one needs to solve (4.8) and then set the maximum among all real lk-th entries of all solvents of the UQME (4.8) as $(\overline{X}_{\text{sns}})_{lk}$. To estimate the lower bound $(\underline{X}_{\text{sns}})_{lk}$, we apply *minimum* rather than *maximum*. It is worth mentioning that the idea of utilizing global/local monotonicity in solving standard interval linear systems comes originally from [27]. The exact strategy is shown in Algorithm 2.

Thus, when estimating the united solution set of IUQME via the sensitivity analysis approach, $3n^2$ linear matrix equations of the form $MY = N$ and $2n^2 + 1$ UQMEs of form (1.1) need to be solved. Some different techniques are described in [18] to get an approximate dominant solvent of (1.1). [8, 13] and [14] also give some algorithms for computing the solvents of (1.1). Example 5.1 also illustrates this technique in more details.

5. Examples

We illustrate our methods on three problems. Moreover, the algorithms are tested in MATLAB 2013a with INTLAB v6 and run on a laptop with 1GB main memory.

Example 5.1. Consider the example mentioned in [18], $X^2 + X + \begin{bmatrix} -6 & -5 \\ 0 & -6 \end{bmatrix} = 0$. Now, suppose that the elements of A, B and C have been measured with a certain uncertainty, so that we obtain the IUQME $\mathbf{A}X^2 + \mathbf{B}X + \mathbf{C} = 0$ with

$$\mathbf{A} = \mathbf{B} = \begin{bmatrix} [0.9000, \ \ 1.1000] & [-0.0100, \ \ 0.0100] \\ [-0.0100, \ \ 0.0100] & [0.9000, \ \ 1.1000] \end{bmatrix},$$

Algorithm 2 A sensitivity analysis algorithm to compute an inner estimation $\mathbf{X}_{\text{sns}}$ for the interval hull of the united solution set to the IUQME (1.3).

Given $\mathbf{A}, \mathbf{B}, \mathbf{C}$
Compute a dominant solvent $\tilde{X}$ of $(\operatorname{mid}\mathbf{A})X^2 + (\operatorname{mid}\mathbf{B})X + \operatorname{mid}\mathbf{C} = 0$
$M = \tilde{X}^T \otimes \operatorname{mid}\mathbf{A} + I \otimes ((\operatorname{mid}\mathbf{A})\tilde{X} + \operatorname{mid}\mathbf{B})$
for $l, k = 1, \ldots, n$ **do**
  for $i, j = 1, \ldots, n$ **do**
    Solve the linear systems $MY = N$ for $N = -\operatorname{vec}(J^{ij}\tilde{X}^2), -\operatorname{vec}(J^{ij}\tilde{X})$ and $-\operatorname{vec}(J^{ij})$ to determine $\operatorname{vec}\left(\frac{\partial X}{\partial A_{ij}}\right), \operatorname{vec}\left(\frac{\partial X}{\partial B_{ij}}\right)$ and $\operatorname{vec}\left(\frac{\partial X}{\partial C_{ij}}\right)$, respectively
  end for
  Set $A^{lk,\text{up}}, A^{lk,\text{down}}, B^{lk,\text{up}}, B^{lk,\text{down}}, C^{lk,\text{up}}, C^{lk,\text{down}}$ according to (4.6) and (4.7)
  Find all solvents $X^{lk,\text{up}}$(s) of UQME (4.8)
  $(\overline{X}_{\text{sns}})_{lk} = \max\{X_{lk}^{lk,\text{up}} \in \mathbb{R} | X^{lk,\text{up}} \text{ is a solvent for (4.8)}\}$
  Find all solvents $X^{lk,\text{down}}$(s) of UQME (4.9)
  $(\underline{X}_{\text{sns}})_{lk} = \min\{X_{lk}^{lk,\text{down}} \in \mathbb{R} | X^{lk,\text{down}} \text{ is a solvent for (4.9)}\}$
end for
Output $\mathbf{X}_{\text{sns}} = [\underline{X}_{\text{sns}}, \overline{X}_{\text{sns}}]$

and

$$\mathbf{C} = \begin{bmatrix} [-6.2000, \ -5.8000] & [-5.2000, \ -4.8000] \\ [0, \ 0] & [-6.2000, \ -5.8000] \end{bmatrix}.$$

With the sensitive analysis approach, we obtain

$$\mathbf{X}_{\text{sns}} = \begin{bmatrix} [-3.3089, \ 1.9240] & [-54.5893, \ 1.0009] \\ [-0.0122, \ 0.3325] & [-3.2964, \ 1.9283] \end{bmatrix}.$$

To better illustrate the sensitivity analysis technique, let us show how the element $(\mathbf{X}_{\text{sns}})_{22} = \left[(\underline{X}_{\text{sns}})_{22}, (\overline{X}_{\text{sns}})_{22}\right]$ is calculated. First, note that $\begin{bmatrix} -3 & -1 \\ 0 & -3 \end{bmatrix}$ is the dominant solvent for the mid-point system $X^2 + X + \begin{bmatrix} -6 & -5 \\ 0 & -6 \end{bmatrix} = 0$ [18]. Consequently, for $i, j = 2$, (4.5) is as follows

$$\begin{cases} \left(\begin{bmatrix} -3 & 0 \\ -1 & -3 \end{bmatrix} \otimes I + I \otimes \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}\right) \operatorname{vec}\left(\dfrac{\partial X}{\partial A_{22}}\right) = -\operatorname{vec}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} X^2\right), \\ \left(\begin{bmatrix} -3 & 0 \\ -1 & -3 \end{bmatrix} \otimes I + I \otimes \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}\right) \operatorname{vec}\left(\dfrac{\partial X}{\partial B_{22}}\right) = -\operatorname{vec}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} X\right), \\ \left(\begin{bmatrix} -3 & 0 \\ -1 & -3 \end{bmatrix} \otimes I + I \otimes \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}\right) \operatorname{vec}\left(\dfrac{\partial X}{\partial C_{22}}\right) = -\operatorname{vec}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right). \end{cases} \tag{5.1}$$

The following results are obtained by solving (5.1):

$$\frac{\partial X}{\partial A_{22}} = \begin{bmatrix} 0 & 0 \\ -0.3600 & 1.8000 \end{bmatrix}, \frac{\partial X}{\partial B_{22}} = \begin{bmatrix} 0 & 0 \\ 0.1200 & -0.6000 \end{bmatrix}, \frac{\partial X}{\partial C_{22}} = \begin{bmatrix} 0 & 0 \\ -0.0400 & 0.2000 \end{bmatrix}. \tag{5.2}$$

Now, we try to find $(\underline{X}_{\text{sns}})_{22}$ by considering the signs of the above calculated derivatives: $\frac{\partial X}{\partial A_{22}}$, $\frac{\partial X}{\partial B_{22}}$, $\frac{\partial X}{\partial C_{22}}$. From (4.7) and (5.2), we conclude that

$$A^{22,\text{down}} = \begin{bmatrix} 1 & 0 \\ 0.0100 & 0.9000 \end{bmatrix}, B^{22,\text{down}} = \begin{bmatrix} 1 & 0 \\ -0.0100 & 1.1000 \end{bmatrix}, C^{22,\text{down}} = \begin{bmatrix} -6 & -5 \\ 0 & -6.2000 \end{bmatrix}.$$

Thus, we should solve the UQME

$$A^{22,\text{down}}X^2 + B^{22,\text{down}}X + C^{22,\text{down}} = \begin{bmatrix} 1 & 0 \\ 0.0100 & 0.9000 \end{bmatrix} X^2 + \begin{bmatrix} 1 & 0 \\ -0.0100 & 1.1000 \end{bmatrix} X + \begin{bmatrix} -6 & -5 \\ 0 & -6.2000 \end{bmatrix} = 0,$$

which has six solvents, namely,

$$\left\{ \begin{bmatrix} 2.0008 & 0.9844 \\ -0.0042 & 2.0783 \end{bmatrix}, \begin{bmatrix} -3.0050 & -0.9432 \\ 0.0263 & -3.2964 \end{bmatrix}, \begin{bmatrix} 3.9747 + 2.2472i & -44.7473 + 3.3625i \\ 0.1603 + 0.4615i & -5.0858 - 2.2555i \end{bmatrix}, \begin{bmatrix} 1.7565 - 2.1104i & -22.5650 + 22.4999i \\ 0.0863 - 0.3360i & -2.8676 + 1.9996i \end{bmatrix}, \begin{bmatrix} 1.7565 + 2.1104i & -22.5650 - 22.4999i \\ 0.0863 + 0.3360i & -2.8676 - 1.9996i \end{bmatrix}, \begin{bmatrix} 3.9747 - 2.2472i & -44.7473 - 3.3625i \\ 0.1603 - 0.4615i & -5.0858 + 2.2555i \end{bmatrix} \right\}.$$

Thus,

$$(\underline{X}_{\text{sns}})_{22} = \min\{2.0783, \ -3.2964\} = -3.2964.$$

Likewise, it follows from (4.6) and (5.2) that

$$A^{22,\text{up}} = \begin{bmatrix} 1 & 0 \\ -0.0100 & 1.1000 \end{bmatrix}, B^{22,\text{up}} = \begin{bmatrix} 1 & 0 \\ 0.0100 & 0.9000 \end{bmatrix}, C^{22,\text{up}} = \begin{bmatrix} -6 & -5 \\ 0 & -5.8000 \end{bmatrix}.$$

So, for obtaining $(\overline{X}_{\text{sns}})_{22}$ we should find all solvents of the UQME

$$A^{22,\text{up}}X^2 + B^{22,\text{up}}X + C^{22,\text{up}} = \begin{bmatrix} 1 & 0 \\ -0.0100 & 1.1000 \end{bmatrix} X^2 + \begin{bmatrix} 1 & 0 \\ 0.0100 & 0.9000 \end{bmatrix} X + \begin{bmatrix} -6 & -5 \\ 0 & -5.8000 \end{bmatrix} = 0. \tag{5.3}$$

This is the complete set of solvents for (5.3):

$$\left\{ \begin{bmatrix} 1.9992 & 1.0147 \\ 0.0038 & 1.9283 \end{bmatrix}, \begin{bmatrix} -2.9953 & -1.0536 \\ -0.0221 & -2.7503 \end{bmatrix}, \begin{bmatrix} 4.3511 & 23.2677 \\ -0.7428 & -5.1362 \end{bmatrix}, \begin{bmatrix} 1.5367 & 13.9504 \\ 0.1507 & -2.1783 \end{bmatrix}, \begin{bmatrix} -1.1004 & -28.3136 \\ -0.2080 & -0.0762 \end{bmatrix}, \begin{bmatrix} 7.4408 & -151.1871 \\ 0.3757 & -8.4739 \end{bmatrix} \right\}$$

Therefore,

$$(\overline{X}_{\text{sns}})_{22} = \max\{1.9283, -2.7503, -5.1362, -2.1783, -0.0762, -8.4739\} = 1.9283.$$

Finally, we conclude that $(\mathbf{X}_{\text{sns}})_{22} = \left[(\underline{X}_{\text{sns}})_{22}, (\overline{X}_{\text{sns}})_{22}\right] = [-3.2964, \ 1.9283]$ which is the same result that we had previously claimed. The total time required to compute the whole matrix $\mathbf{X}_{\text{sns}}$ is 1.5374 seconds. Instead, the result obtained from the nonlinear programming method is

$$\mathbf{X}_{\text{nlp}} = \begin{bmatrix} [-3.4181, \ 2.1742] & [-54.8029, \ 1.2464] \\ [-0.5391, \ 0.4153] & [-3.8349, \ 2.1799] \end{bmatrix}.$$

However, this time, the result is obtained after 9.3424 seconds.

Example 5.2. Consider the UQME

$$X^2 + BX + C = 0, \tag{5.4}$$

with $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$ which has exactly one dominant solvent [13]. We perturb $A = I$, B and C, so that IUQME (1.3) should be solved with $\operatorname{mid}\mathbf{A} = A, \operatorname{mid}\mathbf{B} = B, \operatorname{mid}\mathbf{C} = C$ and

$$\operatorname{rad}\mathbf{A} = \operatorname{rad}\mathbf{B} = \operatorname{rad}\mathbf{C} = \begin{bmatrix} 0.1000 & 0.1000 \\ 0.1000 & 0.1000 \end{bmatrix}.$$

The sensitivity analysis method computed the inner approximation below in 1.6888 seconds, while the nonlinear programming approach took 10.4209 seconds to find an enclosure for the united solution set:

$$\mathbf{X}_{\text{sns}} = \begin{bmatrix} [-54.3825, \ \ 1.0053] & [-0.63, \ \ 0] \\ [-1.4857, \ \ -0.3173] & [-0.8889, \ \ 0] \end{bmatrix}, \mathbf{X}_{\text{nlp}} = \begin{bmatrix} [-54.5001, \ \ 1.3389] & [-3.2484, \ \ 0.1517] \\ [-1.5973, \ \ 0.9884] & [-2.2263, \ \ 0.5986] \end{bmatrix}.$$

Our next aim is to provide an example to verify the accuracy of the presented methods and especially to observe the quality of obtained results in a simple case, i.e., for one-dimensional real interval matrices or real closed intervals.

Example 5.3. Consider the interval quadratic equation

$$[1,2]x^2 + [-2,1]x + [-3,-1] = [0,0]. \tag{5.5}$$

We can rewrite (5.5) as

$$[x^2 - 2x - 3, 2x^2 + x - 1] = [0,0],$$

when $x \geq 0$ and as

$$[x^2 + x - 3, 2x^2 - 2x - 1] = [0,0],$$

when $x \leq 0$. If there exists a value of x such that

$$x^2 - 2x - 3 \leq 0 \leq 2x^2 + x - 1, \tag{5.6}$$

or

$$x^2 + x - 3 \leq 0 \leq 2x^2 - 2x - 1, \tag{5.7}$$

then there exist $a \in [1,2], b \in [-2,1]$ and $c \in [-3,-1]$ such that $ax^2 + bx + c = 0$ for this value of x. Therefore, $x \in \Sigma_{\exists\exists\exists}([1,2],[-2,1],[-3,-1])$. (5.6) and (5.7) imply, respectively, that $x \in [0.5000, 3]$ and $x \in [-2.3028, -0.3660]$. Thus the united solution set of this interval quadratic equation is $[0.5000, 3] \cup [-2.3028, -0.3660]$.

In the case of nonlinear programming, (4.3) is as follows

$$\begin{aligned} &\min / \max \quad x \\ &\qquad \text{s.t.} \\ &\begin{cases} x^2 - \dfrac{1}{2}x - \dfrac{3}{2}|x| \leq 3, \\[2ex] 2x^2 - \dfrac{1}{2}x + \dfrac{3}{2}|x| \geq 1. \end{cases} \end{aligned}$$

Taking into account the sign of x, the above equations are exactly the same as (5.6) and (5.7). Then, the result of nonlinear programming method coincides with the exact united solution set. The total execution time is also 0.7015 seconds.

Now, if we suppose that $x \geq 0$ and compute $\frac{4}{3}$ as the positive solution (dominant solvent) of the midpoint system $\mathrm{mid}([1,2])x^2 + \mathrm{mid}([-2,1])x + \mathrm{mid}([-3,-1]) = 0$, then the sensitivity analysis approach gives $[0.5000, 3]$. The same method produces the interval $[-2.3028, -0.3660]$ when we choose the negative solution (minimal solvent) of the midpoint system, -1, in Line 2 of Algorithm 2. Note that this time we have assumed that x in (5.5) is less than or equal to 0. An inner estimation for $\Sigma_{\exists\exists\exists}([1,2],[-2,1],[-3,-1])$ is thus $[0.5000, 3] \cup [-2.3028, -0.3660]$ which coincides with the result obtained from the exact computation above. The total execution time is 0.1728 seconds which is again much less than the time needed for nonlinear programming technique.

The results of Example 5.3 lead us to the conjecture that in the case of one-dimensional interval matrices, the results of both nonlinear programming and sensitivity analysis method coincide exactly. However, we have neither a mathematical proof nor a counterexample to disprove it.

6. Conclusions and Future Works

In this work, we have generalized Shary's results about AE-solution sets for IUQMEs. Then, we have proved some characterization theorems. We have also attained a sufficient condition under which some particular cases of these solution sets are nonempty. The most significant result of our work is focused on generating an outer estimation and an inner estimation for the united solution set to IUQME. To this end, we have proposed two methods: the nonlinear programming technique and the sensitivity analysis approach.

There are still several problems that can be tackled: one open problem is to develop outer and inner estimations for different AE-solution sets to interval matrix polynomials of any degree, not only quadratic ones. It would be desirable to investigate the conditions under which the AE-solution sets are bounded. Moreover, no attempt has been made in the present paper to develop verification methods which is another interesting problem.

Acknowledgements

The first author would like to thank Dr. Federico Poloni and Dr. Somayyeh Zangoei Zadeh for their helpful collaboration.

References

[1] E. Adams and U. Kulisch, *Scientific computing with automatic result verification*, Academic Press, San Diego, 1993.

[2] G. Alefeld and J. Herzberger, *Introduction to interval computations*, Academic Press, New York, 1983.

[3] D.A. Bini, G. Latouche and B. Meini, *Numerical methods for structured Markov chains*, Oxford University Press, New York, 2005.

[4] D.A. Bini, B. Meini and F. Poloni, *Transforming algebraic Riccati equations into unilateral quadratic matrix equations*, Numer. Math. 116 (2010) 553–578.

[5] G.J. Davis, *Numerical solution of a quadratic matrix equation*, SIAM J. Sci. Stat. Comp. 2 (1981) 164–175.

[6] G.J. Davis, *Algorithm 598: An algorithm to compute solvents of the matrix equation* $AX^2 + BX + C = 0$, ACM. T. Math. Software. 9 (1983) 246–254.

[7] A.S. Deif, *Sensitivity analysis in linear systems*, Springer-Verlag, Berlin, 1986.

[8] J.E. Dennis, J.F. Traub and R.P. Weber, *Algorithms for solvents of matrix polynomials*, SIAM J. Numer. Anal. 15 (1978) 523–533.
[9] F. Xuchuan, D. Jiansong and C. Falai, *Zeros of univariate interval polynomials*, J. Comput. App. Math. 216 (2008) 563–573.
[10] T. Haqiri and F. Poloni, *Methods for verified solutions to continuous-time algebraic Riccati equations*, J. Comput. Appl. Math. 313 (2017) 515–535.
[11] E. Hansen and G.W. Walster, *Global optimization using interval analysis: revised and expanded*, CRC Press, 2003.
[12] B. Hashemi and M. Dehghan, *Efficient computation of enclosures for the exact solvents of a quadratic matrix equation*, Electron. J. Linear Algebra 20 (2010) 519–536.
[13] N.J. Higham and H.M. Kim, *Solving a quadratic matrix equation by Newton's method with exact line searches*, SIAM J. Matrix Anal. Appl. 23 (2001) 303–316.
[14] N.J. Higham and H.M. Kim, *Numerical analysis of a quadratic matrix equation*, IMA J. Numer. Anal. 20 (2000) 499–519.
[15] M. Hladík, *Weak and strong solvability of interval linear systems of equations and inequalities*, Linear Algebra Appl. 438 (2013) 4156–4165.
[16] M. Hladík, *AE solutions and AE solvability to general interval linear systems*, Linear Algebra Appl. 465 (2015) 221–238.
[17] M. Hladík and E. D. Popova, *Maximal inner boxes in parametric AE-solution sets with linear shape*, Appl. Math. Comput. 270 (2015) 606–619.
[18] H.M Kim, *Numerical methods for solving a quadratic matrix equation*, PhD thesis, University of Manchester, 2000.
[19] P. Lancaster and L. Rodman, *Algebraic Riccati equations*, Oxford University Press, New York, 1995.
[20] P. Lancaster, *Lambda-matrices and vibrating systems*, Pergamon Press, Oxford, 1966.
[21] J.H. Long, X.Y. Hu and L. Zhang, *Improved Newton's method with exact line searches to solve quadratic matrix equation*, J. Comput. Appl. Math. 222 (2008) 645–654.
[22] S. Miyajima, *Fast verified computation for solutions of continuous-time algebraic Riccati equations*, Jpn. J. Ind. Appl. Math. 32 (2015) 529–544.
[23] A. Neumaier, *Interval methods for systems of equations*, Cambridge University Press, Cambridge, 1990.
[24] W. Oettli and W. Prager, *Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides*, Numer. Math. 6 (1964) 405–409.
[25] K.B. Petersen and M.S. Pedersen, *The matrix cookbook*, Technical University of Denmark, Denmark, 2012.
[26] F. Poloni, *Algorithms for quadratic matrix and vector equations*, PhD thesis, University of Pisa, Italy, 2011.
[27] E.D. Popova. *Computer-assisted proofs in solving linear parametric problems*, 12th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics, Duisburg, Germany, 2006.
[28] E.D. Popova, *Solvability of parametric interval linear systems of equations and inequalities*, SIAM J. Matrix Anal. Appl. 36 (2015) 615–633.
[29] N.P. Seif, S.A. Hussein and A.S. Deif, *The interval Sylvester equation*, Computing. 52 (1994) 233–244.
[30] S.P. Shary, *A New technique in systems analysis under interval uncertainty and ambiguity*, Reliab. Comput. 8 (2002) 321–418.
[31] F. Tisseur and K. Meerbergen, *The quadratic eigenvalue problem*, SIAM Rev. 43 (2001) 235–286.
[32] Z.C. Zheng, G.X. Ren and W.J. Wang, *A reduction method for large scale unsymmetric eigenvalue problems in structural dynamics*, J. Sound Vibration. 199 (1997) 253–268.