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# L$^q$ inequalities for the $s^{th}$ derivative of a polynomial

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**Abstract**

Let $f(z)$ be an analytic function on the unit disk $\{z \in \mathbb{C},\ |z| \leq 1\}$, for each $q > 0$, the $\|f\|_q$ is defined as follows

$$\|f\|_q := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^q d\theta \right\}^{1/q}, \quad 0 < q < \infty,$$
$$\|f\|_\infty := \max_{|z|=1} |f(z)|.$$

Govil and Rahman [*Functions of exponential type not vanishing in a half-plane and related polynomials*, Trans. Amer. Math. Soc. 137 (1969) 501–517] proved that if $p(z)$ is a polynomial of degree $n$, which does not vanish in $|z| < k$, where $k \geq 1$, then for each $q > 0$,

$$\|p'\|_q \leq \frac{n}{\|k+z\|_q} \|p\|_q.$$

In this paper, we shall present an interesting generalization and refinement of this result which include some previous results.

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## 1. Introduction

Let $\mathcal{P}_n$ be the set of polynomials of degree at most $n$ with complex coefficients. For $p \in \mathcal{P}_n$, define

$$\|p\|_q := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) \right|^q d\theta \right\}^{1/q}, \quad 0 < q < \infty,$$
$$\|p\|_\infty := \max_{|z|=1} |p(z)|.$$

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If $p \in \mathcal{P}_n$, then a famous result due to Bernstein [4], states that

$$\|p'\|_\infty \leq n \|p\|_\infty . \tag{1.1}$$

The inequality (1.1) can be obtained by letting $q \to \infty$ in

$$\|p'\|_q \leq n \|p\|_q , \quad 0 < q < \infty. \tag{1.2}$$

The inequality (1.2) for $q \geq 1$ and $0 < q < 1$ is due to Zygmund [16] and Arestov [1] respectively.

For the class of polynomials having no zeros in $|z| < 1$, Erdös conjectured and later proved by Lax [10] that

$$\|p'\|_\infty \leq \frac{n}{2} \|p\|_\infty . \tag{1.3}$$

The inequality (1.3) can be obtained by letting $q \to \infty$ in

$$\|p'\|_q \leq \frac{n}{\|1+z\|_q} \|p\|_q \ , \text{ for } q > 0. \tag{1.4}$$

The inequality (1.4) demonstrated by De-Brujin [5] for the case $q \geq 1$. Rahman and Schmeisser [14] have shown that the inequality (1.4) remains true for $0 < q < 1$ as well.

As an extension of (1.3), Malik [11] proved that if $p(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1+k} \|p\|_\infty , \tag{1.5}$$

whereas under the same assumption, Govil and Rahman [9] proved that

$$\|p'\|_q \leq \frac{n}{\|k+z\|_q} \|p\|_q \ , \text{ for } q > 0. \tag{1.6}$$

The inequality (1.5) is also generalized by Govil and Rahman [9] for the $s^{th}$ derivative of $p(z)$. They specifically proved that if $p(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then for $1 \leq s < n$,

$$\left\|p^{(s)}\right\|_\infty \leq \frac{n(n-1)\cdots(n-s+1)}{1+k^s} \|p\|_\infty . \tag{1.7}$$

As a refinement of (1.7), Govil [8] proved that if $p(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then for $1 \leq s < n$, one gets

$$\left\|p^{(s)}\right\|_\infty \leq \frac{n(n-1)\cdots(n-s+1)}{1+k^s} \left\{ \|p\|_\infty - \min_{|z|=k} |p(z)| \right\}. \tag{1.8}$$

The following result, proposes a refinement and generalization to inequalities (1.6) and (1.8).

**Theorem 1.1.** If $p \in \mathcal{P}_n$ and $p(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then for every complex number $\beta$ with $|\beta| \leq 1$, $q > 0$ and $1 \leq s < n$, we have

$$\left\| p^{(s)}(z) + \beta \frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}} m \right\|_q \leq \frac{n(n-1)\cdots(n-s+1)}{\|\Lambda_{k,s}+z\|_q} \|p\|_q , \tag{1.9}$$

where $\Lambda_{k,s} = \frac{\binom{n}{s}(|a_0|-m)k^{s+1}+|a_s|k^{2s}}{\binom{n}{s}(|a_0|-m)+|a_s|k^{s+1}}$, and $m = \min_{|z|=k} |p(z)|$.

**Remark 1.2.** By applying the inequality (2.6) from Lemma 2.7, we get $\Lambda_{k,s} \geq k^s$, resulting (1.9) to be a refinement and generalization of (1.6).

Let $q \to \infty$ and choose argument of $\beta$ suitably such that $|\beta| = 1$, then the inequality (1.9) reduces to the following result which recently obtained by Mir [12].

**Corollary 1.3.** If $p \in \mathcal{P}_n$ and $p(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then for $1 \leq s < n$,

$$\left\|p^{(s)}\right\|_\infty \leq \frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}} \left\{\|p\|_\infty - m\right\}, \tag{1.10}$$

where $\Lambda_{k,s} = \frac{\binom{n}{s}(|a_0|-m)k^{s+1}+|a_s|k^{2s}}{\binom{n}{s}(|a_0|-m)+|a_s|k^{s+1}}$, and $m = \min\limits_{|z|=k} |p(z)|$.

**Remark 1.4.** By applying the inequality (2.6) from Lemma 2.7, we get $\Lambda_{k,s} \geq k^s$, resulting (1.10) to be a refinement of (1.8).

**Remark 1.5.** For $s = 1$, the inequality (1.10) reduces to a result which has been recently proved by Gardner, Govil and Weems [7].

**Example 1.6.** Consider the polynomial $p(z) = (z+k)^n$, where $k \geq 1$, then $m = \min\limits_{|z|=k} |p(z)| = 0$ and $\Lambda_{k,s} = k^s$. Now by Corollary 1.3, the inequality (1.10) reduce to the following inequality which is sharp

$$(1+k)^{n-s} \leq \frac{(1+k)^n}{1+k^s}.$$

If we take $k = 1$ then, $\Lambda_{k,s} = 1$ in Theorem 1.1, giving rise to the following generalization of (1.3).

**Corollary 1.7.** If $p \in \mathcal{P}_n$ and $p(z)$ does not vanish in $|z| < 1$, then for $1 \leq s < n$,

$$\left\|p^{(s)}\right\|_\infty \leq \frac{n(n-1)\cdots(n-s+1)}{2} \left\{\|p\|_\infty - \min_{|z|=1} |p(z)|\right\}. \tag{1.11}$$

The inequality is sharp and equality holds for the polynomials $p(z) = z^n + 1$.

**Remark 1.8.** The inequality (1.11) has been studied by Zireh [15, Corollary 1.6].

## 2. Lemmas

For the proof of main theorem, we need the following lemmas. The first lemma is due to Aziz et al. [3].

**Lemma 2.1.** If $p \in \mathcal{P}_n$ and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for each $\alpha,\ 0 \leq \alpha < 2\pi$, and $q > 0$,

$$\int_0^{2\pi}\int_0^{2\pi} \left|q'(e^{i\theta}) + e^{i\alpha}p'(e^{i\theta})\right|^q d\theta d\alpha \leq 2\pi n^q \int_0^{2\pi} \left|p(e^{i\theta})\right|^q d\theta.$$

**Lemma 2.2.** *If $p \in \mathcal{P}_n$ and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for each $\alpha$, $0 \leq \alpha < 2\pi$, and $0 \leq s < n$, $q > 0$, we have*

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} \left|q^{(s)}(e^{i\theta}) + e^{i\alpha}p^{(s)}(e^{i\theta})\right|^q d\theta d\alpha \leq \\ 2\pi(n-s+1)^q(n-s+2)^q\cdots(n-1)^q n^q \int_0^{2\pi} \left|p(e^{i\theta})\right|^q d\theta.\end{aligned} \tag{2.1}$$

**Proof .** Let $h(z) = q(z)+e^{i\alpha}p(z)$, then the $s$th derivative is $h^{(s)}(z) = q^{(s)}(z)+e^{i\alpha}p^{(s)}(z)$ for $1 \leq s < n$. Using the inequality (1.2) repeatedly, it follows that for each $q > 0$,

$$\begin{aligned}&\int_0^{2\pi} \left|q^{(s)}(e^{i\theta}) + e^{i\alpha}p^{(s)}(e^{i\theta})\right|^q d\theta \\ &\quad\leq (n-s+1)^q \int_0^{2\pi} \left|q^{(s-1)}(e^{i\theta}) + e^{i\alpha}p^{(s-1)}(e^{i\theta})\right|^q d\theta \\ &\quad\vdots \\ &\quad\leq (n-s+1)^q(n-s+2)^q\cdots(n-1)^q \int_0^{2\pi} \left|q'(e^{i\theta}) + e^{i\alpha}p'(e^{i\theta})\right|^q d\theta.\end{aligned}$$

Now, integrating the above inequality with respect to $\alpha$ and applying Lemma 2.1, it yields

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi} \left|q^{(s)}(e^{i\theta}) + e^{i\alpha}p^{(s)}(e^{i\theta})\right|^q d\theta d\alpha \\ &\leq (n-s+1)^q(n-s+2)^q\cdots(n-1)^q \int_0^{2\pi}\int_0^{2\pi} \left|q'(e^{i\theta}) + e^{i\alpha}p'(e^{i\theta})\right|^q d\theta d\alpha \\ &\leq 2\pi(n-s+1)^q(n-s+2)^q\cdots(n-1)^q n^q \int_0^{2\pi} \left|p(e^{i\theta})\right|^q d\theta.\end{aligned}$$

□

The following lemma is due to Aziz et al. [3].

**Lemma 2.3.** If $p \in \mathcal{P}_n$, $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, and $p(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then for $1 \leq s < n$ and $|z| = 1$,

$$\delta_{k,s}\left|p^{(s)}(z)\right| \leq \left|q^{(s)}(z)\right|, \tag{2.2}$$

and

$$\frac{1}{\binom{n}{s}}\left|\frac{a_s}{a_0}\right| k^s \leq 1, \tag{2.3}$$

where

$$\delta_{k,s} = \frac{\binom{n}{s}|a_0|k^{s+1} + |a_s|k^{2s}}{\binom{n}{s}|a_0| + |a_s|k^{s+1}}.$$

**Lemma 2.4.** *The function*

$$S(x) = \frac{\binom{n}{s}xk^{s+1} + |a_s|k^{2s}}{\binom{n}{s}x + |a_s|k^{s+1}}$$

*for $k \geq 1$ is a non-decreasing function of $x$.*

**Proof .** The proof follows by considering the first derivative test for $S(x)$. $\square$

**Lemma 2.5.** If $p \in \mathcal{P}_n$ and $p(z)$ does not vanish in $|z| < k$, where $k > 0$, then $m < |p(z)|$ for $|z| < k$, and in particular $m < |a_0|$, where $m = \min\limits_{|z|=k} |p(z)|$.

The above lemma is due to Gardner, Govil and Musukula [6].

**Lemma 2.6.** If $p \in \mathcal{P}_n$ and $p(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then for $|z| = 1$,

$$\left|q^{(s)}(z)\right| \geq n(n-1)\cdots(n-s+1) \min_{|z|=k} |p(z)|, \tag{2.4}$$

where $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$.

The above lemma is due to Govil [8].

**Lemma 2.7.** *If $p \in \mathcal{P}_n$ and $p(z)$ does not vanish in $|z| < k$, where $k \geq 1$, then for $1 \leq s < n$ and $|z| = 1$,*

$$\Lambda_{k,s} \left|p^{(s)}(z)\right| \leq \left|q^{(s)}(z)\right| - \{n(n-1)\cdots(n-s+1)m\}, \tag{2.5}$$

*where*

$$\Lambda_{k,s} = \frac{\binom{n}{s}(|a_0| - m)\, k^{s+1} + |a_s| k^{2s}}{\binom{n}{s}(|a_0| - m) + |a_s| k^{s+1}}$$

*and*

$$\frac{1}{\binom{n}{s}} \frac{|a_s|}{|a_0| - m} k^s \leq 1, \tag{2.6}$$

*where $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$ and $m = \min\limits_{|z|=k} |p(z)|$.*

**Proof .** Let $\lambda$ be a complex number with $|\lambda| < 1$, then $|\lambda m| < |p(z)|$ for $|z| = k$. From Rouche's Theorem, the polynomial $p(z) - \lambda m = (a_0 - \lambda m) + \sum_{i=1}^{n} a_i z^i$ has no zeros in $|z| < k$. Hence from Lemma 2.3, we get

$$A_{k,s} \left|p^{(s)}(z)\right| \leq \left|q^{(s)}(z) - \overline{\lambda} m n(n-1)\cdots(n-s+1) z^{n-s}\right| \quad \text{on} \quad |z| = 1, \tag{2.7}$$

where

$$A_{k,s} = \frac{\binom{n}{s}(|a_0 - \lambda m|)\, k^{s+1} + |a_s| k^{2s}}{\binom{n}{s}(|a_0 - \lambda m|) + |a_s| k^{s+1}}.$$

Since for every $\lambda, |\lambda| \leq 1$ we have

$$|a_0 - \lambda m| \geq |a_0| - |\lambda| m \geq |a_0| - m. \tag{2.8}$$

From (2.8) and making use of Lemmas 2.4 and 2.5 it yields

$$A_{k,s} \geq \Lambda_{k,s}. \tag{2.9}$$

Combining (2.7) and (2.9), for every $\lambda$ where $|\lambda| \leq 1$, we obtain

$$\Lambda_{k,s} \left|p^{(s)}(z)\right| \leq \left|q^{(s)}(z) - \overline{\lambda} mn(n-1)\cdots(n-s+1)z^{n-s}\right| \quad \text{on} \quad |z| = 1, \tag{2.10}$$

where

$$\Lambda_{k,s} = \frac{\binom{n}{s}(|a_0| - m)k^{s+1} + |a_s|k^{2s}}{\binom{n}{s}(|a_0| - m) + |a_s|k^{s+1}}. \tag{2.11}$$

Also by Lemma 2.6, we have that $|q^{(s)}(z)| \geq mn(n-1)\cdots(n-s+1)$. Hence we can choose argument $\lambda$ suitably so that

$$\begin{aligned} &\left|q^{(s)}(z) - \overline{\lambda} mn(n-1)\cdots(n-s+1)z^{n-s}\right| = \\ &\left|q^{(s)}(z)\right| - |\lambda|\, mn(n-1)\cdots(n-s+1)\left|z^{n-s}\right|. \end{aligned} \tag{2.12}$$

Combining (2.12) with (2.10) and let $|\lambda| \to 1$, we get the inequality (2.5). Now by applying the inequality (2.3) for the polynomial $p(z) - \lambda m = (a_0 - \lambda m) + \sum_{i=1}^{n} a_i z^i$ we have

$$\frac{1}{\binom{n}{s}} \frac{|a_s|}{|a_0 - \lambda m|} k^s \leq 1. \tag{2.13}$$

Since $\lambda$ is arbitrary, we can choose argument $\lambda$ suitably so that $|a_0 - \lambda m| = |a_0| - |\lambda| m$. letting $|\lambda| \to 1$, gives the result. $\square$

The following lemma is due to Aziz and Rather [2].

**Lemma 2.8.** Let $A,\ B,\ C$ are non-negative real numbers such that $B + C \leq A$, then for every real $\alpha$,

$$\left|(B + C) + e^{i\alpha}(A - C)\right| \leq \left|B + e^{i\alpha} A\right|. \tag{2.14}$$

## 3. The proof of the main theorem

**Proof .** By the assumptions, $p(z)$ does not vanish in $|z| < k$ where $k \geq 1$, therefore by Lemma 2.7, for $|z| = 1$ and $1 \leq s < n$ we have

$$\Lambda_{k,s} \left|p^{(s)}(z)\right| \leq \left|q^{(s)}(z)\right| - \left\{n(n-1)\cdots(n-s+1)m\right\}.$$

This inequality can be rewritten as

$$\begin{aligned} &\Lambda_{k,s} \left\{\left|p^{(s)}(z)\right| + \frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}} m\right\} \\ &\qquad \leq \left|q^{(s)}(z)\right| - \left\{\frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}} m\right\}. \end{aligned} \tag{3.1}$$

Taking $A = \left|q^{(s)}(z)\right|,\ B = \left|p^{(s)}(z)\right|$ and $C = \frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}} m$ in Lemma 2.8, and noting that $\Lambda_{k,s} \geq 1$, by (3.1), $B + C \leq A - C \leq A$. Thus, for every real $\alpha$, we obtain

$$\begin{aligned} &\Bigg|\left|p^{(s)}(e^{i\theta})\right| + \frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}} m + \\ &\qquad e^{i\alpha}\left|q^{(s)}(e^{i\theta})\right| - \left\{\frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}} m\right\}\Bigg| \\ &\qquad \leq \left|\left|p^{(s)}(e^{i\theta})\right| + e^{i\alpha}\left|q^{(s)}(e^{i\theta})\right|\right|. \end{aligned} \tag{3.2}$$

This implies for each $q > 0$,

$$\int_0^{2\pi} \left|f(\theta) + e^{i\alpha} g(\theta)\right|^q d\theta \le \int_0^{2\pi} \left|\left|p^{(s)}(e^{i\theta})\right| + e^{i\alpha}\left|q^{(s)}(e^{i\theta})\right|\right|^q d\theta, \tag{3.3}$$

where

$$f(\theta) = \left|p^{(s)}(e^{i\theta})\right| + \frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}} m \tag{3.4}$$

and

$$g(\theta) = \left|q^{(s)}(e^{i\theta})\right| - \left\{\frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}}\right\} m.$$

Integrating from both sides of (3.3) with respect to $\alpha$ from 0 to $2\pi$, gives

$$\begin{aligned}
\int_0^{2\pi}\int_0^{2\pi} \left|f(\theta) + e^{i\alpha} g(\theta)\right|^q d\theta d\alpha &\le \int_0^{2\pi}\int_0^{2\pi} \left|\left|p^{(s)}(e^{i\theta})\right| + e^{i\alpha}\left|q^{(s)}(e^{i\theta})\right|\right|^q d\theta d\alpha \\
&= \int_0^{2\pi}\int_0^{2\pi} \left|e^{i\alpha}\left|p^{(s)}(e^{i\theta})\right| + \left|q^{(s)}(e^{i\theta})\right|\right|^q d\theta d\alpha \\
&= \int_0^{2\pi} \left\{\int_0^{2\pi} \left|e^{i\alpha}\left|p^{(s)}(e^{i\theta})\right| + \left|q^{(s)}(e^{i\theta})\right|\right|^q d\alpha\right\} d\theta \\
&= \int_0^{2\pi} \left\{\int_0^{2\pi} \left|e^{i\alpha} p^{(s)}(e^{i\theta}) + q^{(s)}(e^{i\theta})\right|^q d\alpha\right\} d\theta \\
&= \int_0^{2\pi} \left\{\int_0^{2\pi} \left|e^{i\alpha} p^{(s)}(e^{i\theta}) + q^{(s)}(e^{i\theta})\right|^q d\theta\right\} d\alpha.
\end{aligned}$$

This result in conjunction with the inequality (2.1) concludes that

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \left|f(\theta) + e^{i\alpha} g(\theta)\right|^q d\theta d\alpha \le \\
&2\pi(n-s+1)^q(n-s+2)^q\cdots(n-1)^q n^q \int_0^{2\pi} \left|p(e^{i\theta})\right|^q d\theta.
\end{aligned} \tag{3.5}$$

Now for every real $\alpha$ and $t \ge r \ge 1$, from the fact that $|t + e^{i\alpha}| \ge |r + e^{i\alpha}|$, one obtains

$$\int_0^{2\pi} \left|t + e^{i\alpha}\right|^q d\alpha \ge \int_0^{2\pi} \left|r + e^{i\alpha}\right|^q d\alpha.$$

If $f(\theta) \ne 0$, taking $t = \frac{|g(\theta)|}{|f(\theta)|}$, by (3.1) we have $t \ge \Lambda_{k,s} \ge 1$. It yields

$$\begin{aligned}
\int_0^{2\pi} \left|f(\theta) + e^{i\alpha} g(\theta)\right|^q d\alpha &= |f(\theta)|^q \int_0^{2\pi} \left|1 + e^{i\alpha}\frac{g(\theta)}{f(\theta)}\right|^q d\alpha \\
&= |f(\theta)|^q \int_0^{2\pi} \left|\frac{g(\theta)}{f(\theta)} + e^{i\alpha}\right|^q d\alpha \\
&= |f(\theta)|^q \int_0^{2\pi} \left|\left|\frac{g(\theta)}{f(\theta)}\right| + e^{i\alpha}\right|^q d\alpha \\
&\ge |f(\theta)|^q \int_0^{2\pi} \left|\Lambda_{k,s} + e^{i\alpha}\right|^q d\alpha.
\end{aligned} \tag{3.6}$$

For $f(\theta)=0$, the inequality (3.6) is obvious.

Combining inequalities (3.5) and (3.6) and substituting $f(\theta)$ from (3.4) we reach at

$$\int_0^{2\pi} \left|\Lambda_{k,s}+e^{i\alpha}\right|^q d\alpha \int_0^{2\pi} \left\{\left|p^{(s)}(e^{i\theta})\right| + \frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}} m\right\}^q d\theta$$
$$\leq 2\pi(n-s+1)^q(n-s+2)^q\cdots(n-1)^q n^q \int_0^{2\pi} \left|p(e^{i\theta})\right|^q d\theta.$$

This gives for every $\beta\in\mathbb{C}$ with $|\beta|\leq 1,\ q\geq 1$ and $\alpha$ real, that

$$\int_0^{2\pi} \left|\Lambda_{k,s}+e^{i\alpha}\right|^q d\alpha \int_0^{2\pi} \left|p^{(s)}(e^{i\theta}) + \beta\frac{n(n-1)\cdots(n-s+1)}{1+\Lambda_{k,s}} m\right|^q d\theta$$
$$\leq 2\pi(n-s+1)^q(n-s+2)^q\cdots(n-1)^q n^q \int_0^{2\pi} \left|p(e^{i\theta})\right|^q d\theta.$$

This completes the proof of Theorem 1.1. □


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