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Free Vibration Analysis of Composite Plates with Artificial Springs by Trigonometric Ritz Method

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ABSTRACT

In this paper free vibration analysis of two rectangular isotropic plates, which are connected to each other by two translational and rotational springs along the edges, are investigated. The equation of motion and associated boundary and continuity conditions are derived using the extended Hamilton principle. To solve the eigenvalue problem, the Ritz method is utilized. Numerical investigations are presented to show some applications of this method. In this research two types of problems are investigated: first, vibration of a continuous plate and second, free vibration of two hinged plates. This approach is usually referred to as the artificial spring method, which can be regarded as a variant of the classical penalty method. In order to validate the results, the achieved results are compared to results which are presented in literatures.

1. Introduction

The plate is one of the most common structural elements that are encountered by either scientific or technological interest. It's widely utilized in aerospace, marine, mechanical, electrical, nuclear and civil engineering structures. The vibration analysis of plates is one amongst the most important issues in coming up with this sort of structure. Vibration characteristics of plates were extensively studied by other researchers. Plates with completely different shapes, boundary conditions and complicating effects were thought of and also the frequency parameters were investigated in some monographs [1, 2], normal texts [3-5] and review papers [6, 7].

In engineering applications, plates with various complications were investigated. These effects include elastically restrained boundaries, presence of elastically or rigidly connected masses, point supports, variable thickness, anisotropic material, stiffeners[8, 9], interior openings[10] and line hinged [11-13], that are used to serve different purposes required in a structure. In Refs. [13-15] general studies on the vibration of plates with point supports have been deliberated and the vibration of plates with line supports have been studied in Refs. [13-19]. For instance, a line hinge in a plate can be used to expedite folding of gates, or the opening of doors and hatches [20]. The hinge can also be used to simulate a through crack prior to the edge misalignment.

Due to its conceptual simplicity, wide flexibility, high reliability and computational efficiency, the Ritz technique has been widely utilized to resolve the vibration problem of rectangular plates. The Ritz procedure consists in approximating the normal

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displacement variable through a linear combination of generally assumed functions, commonly known as admissible functions, trial functions or basis functions, each satisfying at least the geometrical boundary conditions of the plate. The unknown constant factors of the combination can be obtained by the minimization of the energy functions of the system. Convergence to the exact solution is assured if the admissible functions are linearly independent and form a mathematically complete set. The chosen functions are not required to satisfy the natural boundary conditions, although, if they do, better convergence and accuracy might be achieved. Moreover, the properties of trial functions have a significant effect on computational efficiency and numerical stability of the solution [21]. The rather weak conditions imposed by the Ritz method and the great sensitivity of the related solution to the choice of admissible functions have prompted many researchers in evaluating and developing suitable ways of constructing the trial set for Kirchhoff plates with arbitrary boundary conditions and complicating effects. There is a so huge amount of literature on the topic which prevents to provide a comprehensive review of all the available approaches here. Restricting the analysis to rectangular plates, notable solutions include the use of characteristic beam functions, simple and orthogonal polynomials, static beam functions, Fourier sine and cosine series or their appropriate combinations [21, 22]. Since the initial works of Young [23], Warburton [24]and Leissa [25], admissible functions as products of vibrating beam eigenfunctions have been presented by many researchers. They work well in most conditions. However, because of the occurrence of over-restraint at free edges, the results obtained for plates involving one or more free edges are much less accurate [26]. Characteristic beam functions are also clearly dependent on the boundary conditions. There exist 21 different cases for rectangular plates by considering all possible combinations of classical edge conditions [25]. Therefore, their use involves a boring solution process since a specific set is required for each type of boundary. Finally, numerical instability happens when the common expressions for the beam mode shape functions are evaluated at high orders [27].

In this paper, a more general case of free vibration analysis of assembled plates is considered for preparing a model that can approximate vibration behaviour of a hinged rectangular plate or a uniformed rectangular plate, only with determination of various stiffness's of translational and rotational springs that connect two plates to each other. In the process of solving the eigenvalue problem which is obtained from the variation method on the total system, the quasi analytical Ritz method is used. The trigonometric sets are used as admissible functions due to the fact that they are very effective from a computational point of view and their reliability and versatility for flexural vibration analysis of rectangular plates which may be subjected to various complicating factors. Finally some illustrating results are presented in tabular and graphical forms to validate the method.

2. The Determination of the Boundary Value Problem

Figure 1 represents an isotropic rectangular thin plate in the x-y plane. In this figure h is thickness, a and b are length and width of the plate, respectively. A slit that is parallel to the y-axis is located at x=c, as shown in Figure 1. The plate is considered to have two spans that are separated by the slit. The total domain of the plate A is divided into two sub-domains $A^{(1)}$ and $A^{(2)}$ as is shown in Figure 1. It is seen that these two sub-domains are separated by the line $L^{(c)}$ at x=c. $A^{(1)}$ and $A^{(2)}$ are connected to each other by using linear translational and rotational springs. k_t and k_r are constants of translational and rotational springs, respectively. It can be assumed that the thickness and deflection of each sub-domain are small compared with the wavelength of flexural vibration; consequently, thin plate theory is applicable. Throughout the remainder of the paper, the counter-clockwise four-letter symbolic notation introduced by Leissa [25] is used for describing classical boundary conditions. For instance, an SFSC plate has a simply supported (zero deflection and free rotation) left edge, a free (free deflection and rotation) bottom edge, a simply supported right edge and a clamped (zero deflection and rotation) top edge, respectively.

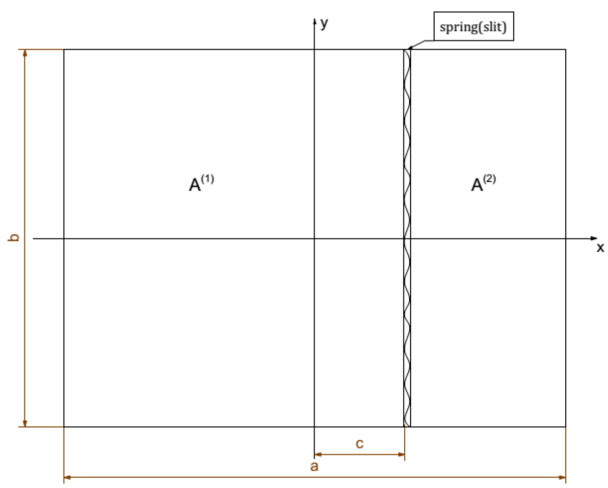


Figure 1. A schematic diagram of hinged plates in x-y plane

In order to analyse the transverse displacements of the system, it can be assumed that the vertical position of the k-th (k=1, 2) plate at any time t, is defined by the function $w^{(k)}=w^{(k)}(x,y,t)$. The total kinetic energy of the system is [28]:

$$T_p(w)=\frac{1}{2}\sum_{k=1}^{2}\iint_{A^{(k)}}\rho h_k\left(\frac{\partial w_k(x,y,t)}{\partial t}\right)^2 dxdy \quad (1)$$

Where ρh_k is the mass density of the k-th plate and $(x,y)\in A$ such that A is constituted of both sub-domains $A^{(1)}$, $A^{(2)}$ and each respective boundary $\partial A^{(k)}$.

The total potential energy due to the elastic deformation of the plate is [28]:

$$\begin{aligned}V_p(w)=\frac{1}{2}\sum_{k=1}^{2}\Bigg\{&\int_{A^{(k)}}\Bigg[D^{(k)}\left(\left(\frac{\partial^2 w_k}{\partial x^2}+\frac{\partial^2 w_k}{\partial y^2}\right)^2-\right.\\&\left.-2(1-\nu)\left(\frac{\partial^2 w_k}{\partial x^2}\frac{\partial^2 w_k}{\partial y^2}-\left(\frac{\partial^2 w_k}{\partial x\,\partial y}\right)^2\right)\right)\Bigg]+\\&2q^{(k)}w_k(x,y,t)dxdy\Bigg\}\\&+\frac{1}{2}\int_{L^{(c)}}\Bigg[k_t\left(\Delta w(c,y,t)\right)^2+\\&k_r\left(\Delta\frac{\partial w}{\partial n}(c,y,t)\right)^2\Bigg]ds\end{aligned} \quad (2)$$

where $D^{(k)}$ is the rigidity of the *k-th* plate, ν is the Poisson's ratio, $q^{(k)}$ is the external load, $\Delta w(c,y,t)$ is the difference in the lateral displacement of the two plates along the slit, $\Delta\partial w/\partial n$ is the difference between the normal slopes and $\partial w/\partial n$ denotes the directional derivative of w with respect to the outward normal unit vector to the curve $L^{(c)}$ and s is the coordinate along the line of slit.

To derive the equations of motion, the extended Hamilton principle is used as:

$$\int_{t_0}^{t_1}\delta F(w)=0 \quad (3)$$

where:

$$\begin{aligned}F(w)=\frac{1}{2}\int_{t_0}^{t_1}\sum_{k=1}^{2}\Bigg\{&\iint_{A^{(k)}}\rho h_k\left(\frac{\partial w_k(x,y,t)}{\partial t}\right)^2-\\&D^{(k)}\left(\left(\frac{\partial^2 w_k}{\partial x^2}+\frac{\partial^2 w_k}{\partial y^2}\right)^2-\right.\\&\left.2(1-\nu)\left(\frac{\partial^2 w_k}{\partial x^2}\frac{\partial^2 w_k}{\partial y^2}-\left(\frac{\partial^2 w_k}{\partial x\,\partial y}\right)^2\right)\right)-\\&2q^{(k)}w_k(x,y,t)dxdy-\frac{1}{2}\int_{L^{(c)}}\Big[k_t\times\\&\left(\Delta w(c,y,t)\right)^2+k_r\left(\Delta\frac{\partial w}{\partial n}(c,y,t)\right)^2\Big]ds\Bigg\}dt\end{aligned} \quad (4)$$

and δ is variational operator.

Consequently, by using the Hamilton principle, the equations of motion for free vibration analysis of the coupled plates are obtained as follows:

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}\left(D^{(k)}\frac{\partial^2 w_k}{\partial x^2}+\nu D^{(k)}\frac{\partial^2 w_k}{\partial y^2}\right)+\\&\frac{\partial^2}{\partial y^2}\left(D^{(k)}\frac{\partial^2 w_k}{\partial y^2}+\nu D^{(k)}\frac{\partial^2 w_k}{\partial x^2}\right)+\\&2(1-\nu)\frac{\partial^2}{\partial x\,\partial y}\left(D^{(k)}\frac{\partial^2 w_k}{\partial x\,\partial y}\right)+\\&\rho h_k\left(\frac{\partial^2 w_k(x,y,t)}{\partial t^2}\right)=0\end{aligned} \quad (5)$$

The above equations represent the dynamical behaviour of the vibrating plates.

2.1. Classical Boundary Conditions

In this study, plates may take any classical boundary conditions, including free, simply supported and clamped. The boundary conditions along the edges $x=-a/2$ and $x=a/2$ are satisfied by the following relations [17]:

(a) For a free edge:

$$M_{xx}=0\ ,\quad K_x=0 \quad (6)$$

(b) For a simply supported edge:

$$M_{xx}=0\ ,\quad w=0 \quad (7)$$

(c) For a clamped edge:

$$w=0\ ,\quad \frac{\partial w}{\partial x}=0 \quad (8)$$

Where M_{xx} is the bending moment on the edges $x=-a/2$ and $x=a/2$, and K_x is the Kirchhoff equivalent force, one can write:

At $x=-a/2$:

$$M_{xx}=D^{(1)}\left(\frac{\partial^2 w_1}{\partial x^2}+\nu\frac{\partial^2 w_1}{\partial y^2}\right)\left(-\frac{a}{2},y,t\right) \quad (9)$$

$$K_x = \left[\frac{\partial}{\partial x} D^{(1)} \left(\frac{\partial^2 w_1}{\partial x^2} + \nu \frac{\partial^2 w_1}{\partial y^2} \right) + \frac{\partial}{\partial y} \left(2(1-\nu) D^{(1)} \frac{\partial^2 w_1}{\partial x \, \partial y} \right) \right] \left(\frac{a}{2}, y, t\right) \tag{10}$$

And similarly it could be obtained for the other edges. Applying the states before boundary conditions and continuity conditions, a set of the homogeneous equations would be obtained.

2.2. Continuity conditions for connection of two regions

At the slit location ($x=c$), continuity conditions along the slit line can be written as:

$$\begin{aligned} & -D^{(1)} \left(\frac{\partial^2 w_1}{\partial x^2} + \nu \frac{\partial^2 w_1}{\partial y^2} \right) \frac{\partial (\delta w_1)}{\partial x} \\ & + \frac{\partial}{\partial x} \left(D^{(1)} \frac{\partial^2 w_1}{\partial x^2} + \nu \frac{\partial^2 w_1}{\partial y^2} \right) (\delta w_1) \\ & + (1-\nu) \frac{\partial}{\partial y} \left(D^{(1)} \frac{\partial^2 w_1}{\partial x \, \partial y} \right) (\delta w_1) - \\ & k_t (w_1 - w_2)(\delta w_1) + \\ & k_r \left(\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial x} \right) \frac{\partial (\delta w_1)}{\partial x} = 0 \end{aligned} \tag{11}$$

$$\begin{aligned} & -D^{(2)} \left(\frac{\partial^2 w_2}{\partial x^2} + \nu \frac{\partial^2 w_2}{\partial y^2} \right) \frac{\partial (\delta w_2)}{\partial x} + \\ & \frac{\partial}{\partial x} \left(D^{(2)} \frac{\partial^2 w_2}{\partial x^2} + \nu \frac{\partial^2 w_2}{\partial y^2} \right) (\delta w_2) + \\ & (1-\nu) \frac{\partial}{\partial y} \left(D^{(2)} \frac{\partial^2 w_2}{\partial x \, \partial y} \right) (\delta w_2) + \\ & k_t (w_1 - w_2)(\delta w_2) - \\ & k_r \left(\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial x} \right) \frac{\partial (\delta w_2)}{\partial x} = 0 \end{aligned} \tag{12}$$

According to the variational principle, the coefficients of $\bar{C} = \delta w / \partial n$ and δw should be zero. Consequently, the following equations are obtained:

$$-D^{(1)} \left(\frac{\partial^2 w_1}{\partial x^2} + \nu \frac{\partial^2 w_1}{\partial y^2} \right) + k_r \left(\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial x} \right) = 0 \tag{13}$$

$$-D^{(2)} \left(\frac{\partial^2 w_2}{\partial x^2} + \nu \frac{\partial^2 w_2}{\partial y^2} \right) - k_r \left(\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial x} \right) = 0 \tag{14}$$

$$D^{(1)} \frac{\partial}{\partial x} \left(\frac{\partial^2 w_1}{\partial x^2} + \nu \frac{\partial^2 w_1}{\partial y^2} \right) + 2D^{(1)} (1-\nu) \frac{\partial}{\partial y} \left(\frac{\partial^2 w_1}{\partial x \, \partial y} \right) - k_t (w_1 - w_2) = 0 \tag{15}$$

$$D^{(2)} \frac{\partial}{\partial x} \left(\frac{\partial^2 w_2}{\partial x^2} + \nu \frac{\partial^2 w_2}{\partial y^2} \right) + 2D^{(2)} (1-\nu) \frac{\partial}{\partial y} \left(\frac{\partial^2 w_2}{\partial x \, \partial y} \right) + k_t (w_1 - w_2) = 0 \tag{16}$$

where all of the above equations would be evaluated at $x=c$. Eqs. (11) and (12) express an equal moment with the opposite signs, which are applied to the edges $x=c$ and indicate the Kirchhoff equivalent force on the common edge between two regions.

3. Eigenvalue Problem in Ritz Method

For free vibrations of the plate, the displacements can be written as:

$$w_k(x, y, t) = W_k(x, y) e^{i\omega t} \tag{17}$$

where ω is the circular frequency of the plate. Substituting Eq. (17) into Eqs. (1) and (2), the maximum kinetic energy $T_p^{\max}$ and the maximum potential energy $V_p^{\max}$ are obtained. For the sake of simplicity the following dimensionless parameters are used:

$$\xi = \frac{2x}{a} \quad , \ \eta = \frac{2y}{b} \tag{18}$$

Therefore, it can be written:

$$\begin{aligned} & V_p^{\max}(w) = \\ & \frac{1}{2} ab \int_{-1}^{1} \int_{-1}^{\bar{c}} \left[4D \left(\frac{1}{a^2} \frac{\partial^2 W_1}{\partial \xi^2} + \frac{1}{b^2} \frac{\partial^2 W_1}{\partial \eta^2} \right)^2 - \frac{8D(1-\nu)}{a^2 b^2} \left(\frac{\partial^2 W_1}{\partial \xi^2} \frac{\partial^2 W_1}{\partial \eta^2} - \left(\frac{\partial^2 W_1}{\partial \xi \partial \eta} \right)^2 \right) \right] d\xi d\eta + \\ & \frac{1}{2} ab \int_{-1}^{1} \int_{\bar{c}}^{1} \left[4D \left(\frac{1}{a^2} \frac{\partial^2 W_2}{\partial \xi^2} + \frac{1}{b^2} \frac{\partial^2 W_2}{\partial \eta^2} \right)^2 - \frac{8D(1-\nu)}{a^2 b^2} \left(\frac{\partial^2 W_2}{\partial \xi^2} \frac{\partial^2 W_2}{\partial \eta^2} - \left(\frac{\partial^2 W_2}{\partial \xi \partial \eta} \right)^2 \right) \right] d\xi d\eta + \\ & \frac{1}{2} b \int_{-1}^{1} \left[\frac{1}{2} k_t (W_1 - W_2)^2 + \frac{2}{a^2} k_r \left(\frac{\partial W_1}{\partial \xi} - \frac{\partial W_2}{\partial \xi} \right)^2 \right] d\eta \end{aligned} \tag{19}$$

$$T_p^{\max}(w)=\frac{1}{2}ab\int_{-1}^{1}\int_{-1}^{\bar{c}}\frac{1}{4}\rho hW_1^{2}d\xi d\eta+\frac{1}{2}ab\int_{-1}^{1}\int_{\bar{c}}^{1}\frac{1}{4}\rho hW_2^{2}d\xi d\eta \tag{20}$$

Where $\bar{C}=2c/a$, $D^{(k)}$ and $h^{(k)}$ are assumed to be constant for two areas i.e.

$$D^{(1)}=D^{(2)}=D\ ,\ h^{(1)}=h^{(2)}=h \tag{21}$$

Due to classical boundary conditions, the functional energy of the system is expressed as:

$$F^{\max}(w)=T_p^{\max}(w)-V_p^{\max}(w) \tag{22}$$

The Ritz approximation is:

$$W_k(\xi,\eta)=\sum_{i=1}^{M}\sum_{j=1}^{N}a_{ij}^{(k)}\phi_i^{(k)}(\xi)\phi_j^{(k)}(\eta) \tag{23}$$

Where the superscript k denotes the *k-th* subdomain, a_{ij} are unknown coefficients and $\phi_i(\xi)$ and $\phi_j(\eta)$ are appropriate admissible functions satisfying at least the geometrical boundary conditions of the problem. After substituting Eqs. (19) and (20) into Eq. (22), the coefficients a_{ij} can be obtained by finding extremum of the functional energy $F(w)$ as follows:

$$\frac{\partial F_{\max}(w)}{\partial a_{ij}^{(k)}}=0 \tag{24}$$

Consequently the following eigenvalue equation is obtained:

$$\left(\left[K\right]-\Omega^2\left[M\right]\right)\left\{a\right\}=\left\{0\right\} \tag{25}$$

where $\Omega=\omega b^2\sqrt{\rho h/D}$ is the dimensionless frequency.

The stiffness and mass matrices K and M are presented respectively as:

$$\left[K\right]=\begin{bmatrix}\left[K^{(1,1)}\right] & \left[K^{(1,2)}\right]\\ \left[K^{(1,2)}\right] & \left[K^{(2,2)}\right]\end{bmatrix} \tag{26}$$

$$\left[M\right]=\begin{bmatrix}\left[M^{(1,1)}\right] & \left[0\right]\\ \left[0\right] & \left[M^{(2,2)}\right]\end{bmatrix} \tag{27}$$

where,

$$K_{ijmn}^{(k,k)}=r^4I_{im}^{(2,2,k)}I_{jn}^{(0,0,k)}+I_{im}^{(0,0,k)}I_{jn}^{(2,2,k)}+2r^2(1-\nu)I_{im}^{(1,1,k)}I_{jn}^{(1,1,k)}+r^2\nu\left(I_{im}^{(2,0,k)}I_{jn}^{(0,2,k)}+I_{im}^{(0,2,k)}I_{jn}^{(2,0,k)}\right)+r^4K_TI_c^{(0,0)}I_{jn}^{(0,0,k)}+r^4K_RI_c^{(1,1)}I_{jn}^{(0,0,k)} \tag{28}$$

$$K_{ijmn}^{(1,2)}=-\frac{1}{2}r^4K_TR_1^{(0,0)}R_2^{(0,0)}-\frac{1}{8}r^4K_RR_1^{(1,1)}R_2^{(0,0)} \tag{29}$$

$$M_{ijmn}^{(k,k)}=\left(\frac{1}{16}\right)I_{im}^{(0,0,k)}I_{jn}^{(0,0,k)} \tag{30}$$

In the above equations the integral statements are defined as:

$$I_{im}^{(\alpha,\beta,1)}=\int_{-1}^{\bar{c}}\frac{d^{\alpha}\phi_i^{(1)}}{d\xi^{\alpha}}\frac{d^{\beta}\phi_m^{(1)}}{d\xi^{\beta}}d\xi \tag{31}$$

$$I_{im}^{(\alpha,\beta,2)}=\int_{\bar{c}}^{1}\frac{d^{\alpha}\phi_i^{(2)}}{d\xi^{\alpha}}\frac{d^{\beta}\phi_m^{(2)}}{d\xi^{\beta}}d\xi \tag{32}$$

$$I_{jn}^{(\alpha,\beta,k)}=\int_{-1}^{1}\frac{d^{\alpha}\phi_j^{(k)}}{d\eta^{\alpha}}\frac{d^{\beta}\phi_n^{(k)}}{d\eta^{\beta}}d\eta \tag{33}$$

$$I_c^{(\alpha,\beta)}=\frac{d^{\alpha}\phi_i^{(k)}}{d\xi^{\alpha}}\frac{d^{\beta}\phi_m^{(k)}}{d\xi^{\beta}}(\bar{c}) \tag{34}$$

$$R_1^{(\alpha,\beta)}=\frac{d^{\alpha}\phi_i^{(1)}}{d\xi^{\alpha}}\frac{d^{\beta}\phi_m^{(2)}}{d\xi^{\beta}}(\bar{c}) \tag{35}$$

$$R_2^{(\alpha,\beta)}=\int_{0}^{1}\frac{d^{\alpha}\phi_j^{(1)}}{d\eta^{\alpha}}\frac{d^{\beta}\phi_n^{(2)}}{d\eta^{\beta}}d\eta \tag{36}$$

where α and β denote the order of derivatives.

It is noted that $r=b/a$, $K_T=k_ta^3/D$ and $K_R=k_ra/D$ are dimensionless coefficients. The spring stiffnesses K_T and K_R are selected either as the actual connecting stiffnesses, if flexible joints are represented, or as very high values compared with the adjoining plates, therefore approximating rigid connections. (For example, at a free edge, K_T and K_R are taken as zero, while, to approximate a hinged boundary or connection between two adjacent plates, K_T is given some very high value and K_R is taken as zero.)[21]

In the present study the admissible functions $\phi_i^{(k)}(\xi)$ and $\phi_j^{(k)}(\eta)$ are defined by means of the trigonometric set. The following trial functions which are used by Beslin and Nicolas [29] for flexural vibration of Kirchhoff plates are presented:

$$\phi_i^{(k)}(\xi)=\sin(a_i\xi+b_i)\sin(c_i\xi+d_i) \tag{37}$$

where the coefficients a_i , b_i , c_i and d_i are listed in Table 1.

The function $\phi_i^{(k)}(\xi)$ is defined according to Eq. (30), where ξ and m are replaced by η and n, respectively. A subset of $\phi_i^{(k)}(\xi)$ is plotted in Figure 2 where functions of increasing order are arranged in a matrix form [21].

It is seen that the first and third functions $\phi_1^{(k)}(\xi)$ and $\phi_3^{(k)}(\xi)$ have a non-zero displacement at $\xi=-1$ and $\xi=1$, respectively. The second and fourth trigonometric functions $\phi_2^{(k)}(\xi)$ and $\phi_4^{(k)}(\xi)$ have a free slope at the same edges at $\xi=\pm1$, respectively. The functions $\phi_1^{(k)}(\eta)$ and $\phi_4^{(k)}(\eta)$ are arranged in a similar style for $\eta=\pm1$.

As it is, the first four functions $\phi_1^{(k)}$ to $\phi_4^{(k)}$ enable one to easily satisfy any classical boundary condition by selecting a suitable combination among them. For example, the analysis of a completely free plate (FFFF) will keep all these four functions in the final sequence. If a simple support condition is imposed on the edge $\eta=1$ (FFFS plate), the function $\phi_3^{(k)}(\eta)$ will be eliminated from the set. For a fully clamped plate (CCCC) all the couples of these functions both in ξ and in η direction will be removed. The nine combinations of classical boundary conditions are reported in Table 2 where a bullet denotes that the corresponding function must be kept in the final set. The first letter in the table refers to the edge at $\xi=-1$ or $\eta=-1$, whereas the second letter refers to that at $\xi=1$ or $\eta=1$ [22].

Table 1. Coefficients of the trigonometric set

i	a_i	b_i	c_i	d_i
1	$\pi/4$	$3\pi/4$	$\pi/4$	$3\pi/4$
2	$\pi/4$	$3\pi/4$	$-\pi/2$	$-3\pi/2$
3	$\pi/4$	$-3\pi/4$	$\pi/4$	$-3\pi/4$
4	$\pi/4$	$-3\pi/4$	$\pi/2$	$-3\pi/2$
>4	$\pi/2(i-4)$	$\pi/2(i-4)$	$\pi/2$	$\pi/2$

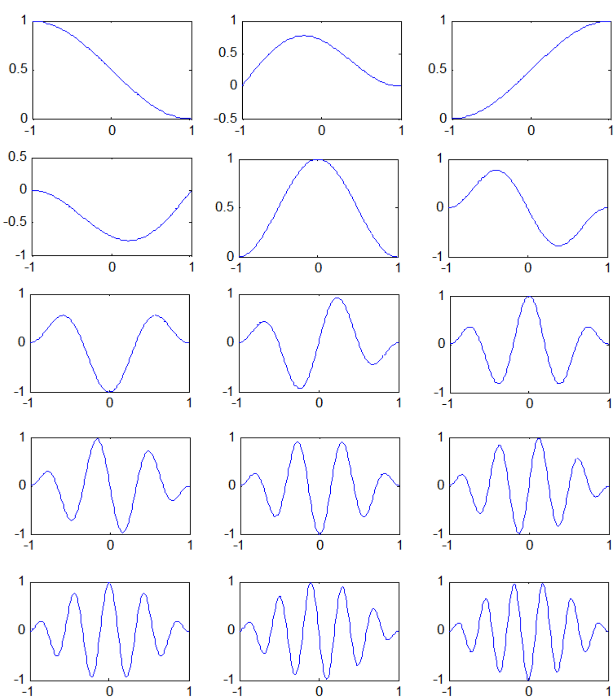


Figure 2. The first 15 functions of the trigonometric set

Table 2. Combination of the first four functions in the trigonometric set to satisfy the related boundary conditions.

Boundary condition	$\phi_1^{(k)}$	$\phi_2^{(k)}$	$\phi_3^{(k)}$	$\phi_4^{(k)}$
FF	•	•	•	•
FS	•	•	-	•
FC	•	•	-	-
SF	-	•	•	•
SS	-	•	-	•
SC	-	•	-	-
CF	-	-	•	•
CS	-	-	-	•
CC	-	-	-	-

Due to the zero determinant of the coefficient matrix in Eq. (21), the problem has a nontrivial solution. The stated determinant gives the natural frequencies of the system. It is important to note that the nontrivial solution of the system gives the mode shapes of the plate.

4. Numerical Results and Discussions

In order to examine the accuracy and applicability of the approach developed and discussed in the previous sections, numerical results were calculated for a number of different problems for which comparison values were available in the literature and also convergence studies have been carried out in the graphical and tabular form. All calculations have been performed with Poisson's ratio $\nu=0.3$.

4.1. Rectangular Plate with Different Boundary Conditions

The discussed method is applied to rectangular isotropic plates with four different combinations of classical boundary conditions. To obtain the frequency parameters corresponding to a continuous rectangular plate without slit, it is necessary that the translational and rotational stiffnesses approach infinity. For this purpose in this section dimensionless translational stiffness per unit length K_T is assumed to be equal to the dimensionless rotational stiffness K_R and it can be assumed to be equal to a high value ($>10^6$). The problems are solved for two cases which have well-known closed form solutions [25], i.e., SSSS and SFSF plates; and, last, a cantilever plate (CFFF) that there is no exact solution for this case [30]. Table 3 shows the convergence of the dimensionless frequency parameter $\Omega=\omega b^2\sqrt{\rho h/D}$ for the first eight modes.

Results are obtained by using a square selection method, i.e., the similar number of terms *M=N* is adopted in the series expansion, with no regard to symmetry. It is seen that the solutions monotonically decrease as the number of terms in the set increases. Table 3 also shows that the higher frequency convergent values to four digits are obtained with a different number of terms for each mode and case.

Moreover, this number does not necessarily increase as the mode number increases. It is illustrated from Table 3 that the frequency parameters which are obtained by the presented method in this paper in comparison to reference [21] are higher about 0.2% at worst state. This difference is due to the discontinuity in the plate and the values of K_T and K_R which are assumed to be a finite value.

4.2. Rectangular Plate with an Internal Line Hinge

The problem is the transverse vibration of two rectangular plates which hinged together along the *y* coordinate at *x=c*. There are some literatures about this problem. As it was referred to previously, to approximate a hinged connection between two adjacent plates, K_T is given some very high value and K_R is taken as zero. In this case some results of a convergence study of the values of the frequency $\Omega = \omega b^2 \sqrt{\rho h / D}$ are presented in Table 4 and Table 5. The first eight values of Ω are presented for a square SSSS plate with an internal line hinge located at two different positions, namely $\bar{c} = -0.1$ and $\bar{c} = 0$. It can be observed that with an increase in iteration number the frequency parameters converge monotonically. From Table 4 it can be observed that the results are in a good agreement with exact frequency parameters for a SSSS plate with an internal line hinge presented in Ref. [28].

Similar results for plates with different boundary conditions, aspect ratios and position of line hinge are presented in Table 5. A convergence study is shown in Figure 3 and Figure 4 for fundamental frequency in terms of translational and rotational frequencies that vary from 0 to 10^8 .

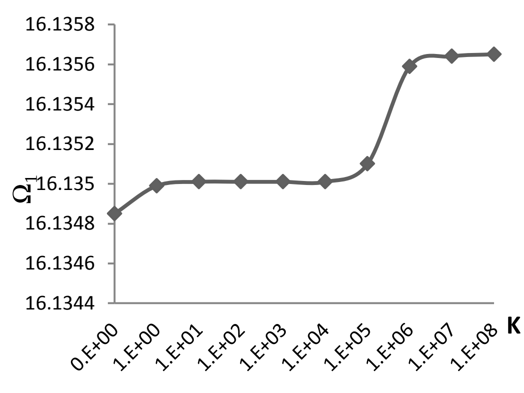
These figures are presented in two cases in terms of boundary conditions; Case 1: symmetric boundary conditions (SSSS, SFSC) and Case 2: asymmetric boundary conditions (CCFF, CCSS).

Table 3. Convergence study of the first eight frequency parameters $\Omega = \omega b^2 \sqrt{\rho h / D}$ for square isotropic plates with different classical boundary conditions.

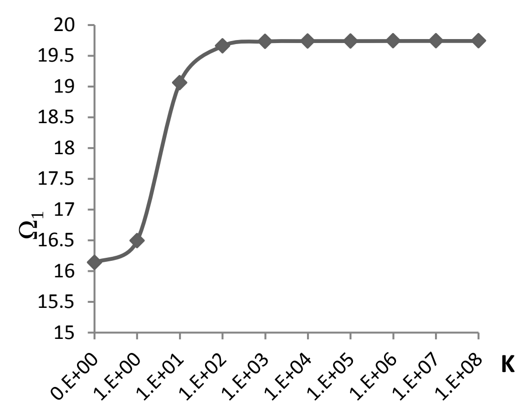
BCs	N	Mode sequence			
		1	2	3	4
SSSS	6	19.7438	49.3666	49.3772	78.9872
	8	19.7408	49.3544	49.3596	78.9682
	10	19.7398	49.3515	49.3537	78.9627
	12	19.7392	49.3501	49.3510	78.9602
	Ref.[30]	19.7392	49.3480	49.3480	78.9569
SFSF	6	9.6506	16.2247	37.1184	39.0561
	8	9.6362	16.1626	36.8367	38.9784
	10	9.6332	16.1466	36.7684	38.9594
	12	9.6321	16.1407	36.7454	38.9523
	Ref.[30]	9.6314	16.1348	36.7256	38.9451
CFFF	6	3.4893	8.5887	21.3913	27.4601
	8	3.4778	8.5354	21.3250	27.2712
	10	3.4745	8.5209	21.3058	27.2279
	12	3.4723	8.5108	21.2917	27.2038
	Ref.[21]	3.47108	8.5066	21.2848	27.1990

Table 4. convergence study of the first eight values of the Ω for a SSSS plate with an internal line hinge.

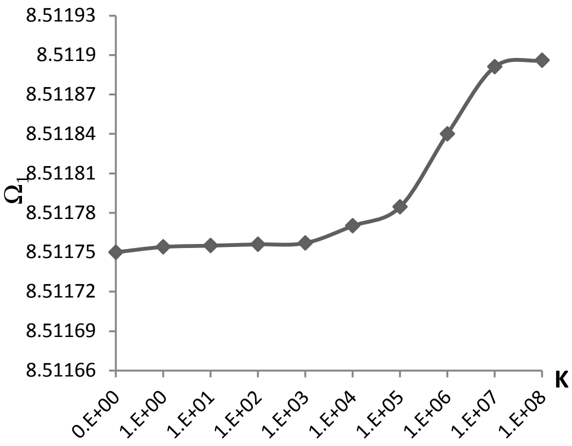
$\bar{c}$	N	Mode sequence			
		1	2	3	4
-0.1	6	16.8401	39.1790	47.4875	72.1332
	8	16.7937	39.0915	47.4346	72.0232
	10	16.7904	39.0859	47.4268	72.0129
	12	16.7890	39.0851	47.4248	72.0106
	Ref.[28]	16.7891	39.0862	47.4207	72.0098
0	6	16.1427	46.7697	49.3666	75.3799
	8	16.1372	46.7504	49.3545	75.3063
	10	16.1360	46.7443	49.3517	75.2949
	12	16.1353	46.7416	49.3499	75.2836
	Ref.[28]	16.1347	46.7381	49.3480	75.2833



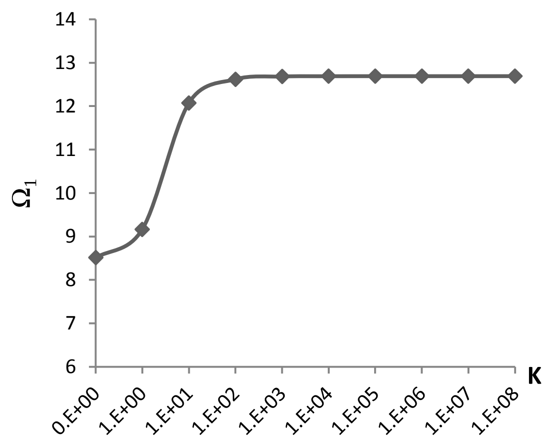
(a1)



(a2)



(b1)



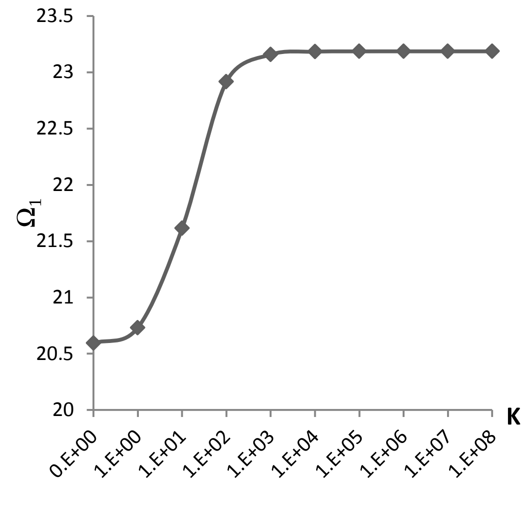
(b2)

Figure 1 Fundamental frequency parameter $\Omega = \omega b^2 \sqrt{\rho h / D}$ for symmetric boundary conditions as (1) translational and (2) both stiffnesses equally vary from 0 to 10^8 for (a)SSSS plates and (b)SFSC plates

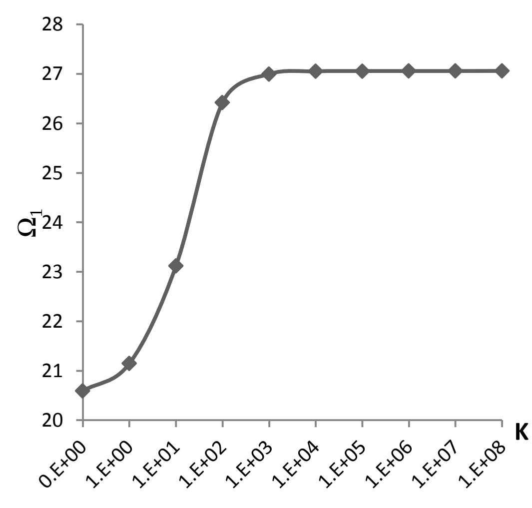
It should be noted that the first natural frequency of vibration is named as fundamental frequency and its magnitude is less than all of the other frequencies. In Figure 3 and Figure 4, the "*a*" and "*b*" letters introduce the types of boundary conditions and their indices are used to show the variation of the translational and rotational stiffnesses which are specified in each figure. As it can be seen, the fundamental frequency Ω_1 oscillates in a range which is very small when only translational frequency K_T varies, but when both the translational and rotational stiffnesses vary, the fundamental frequency converges monotonically to a specific value. This result is due to the symmetry in boundary conditions which yields this fact that in the low values of K_T a good accuracy of Ω_1 can be obtained.

Table 5. The first ten values of the Ω for a rectangular plate with different boundary conditions and aspect ratios

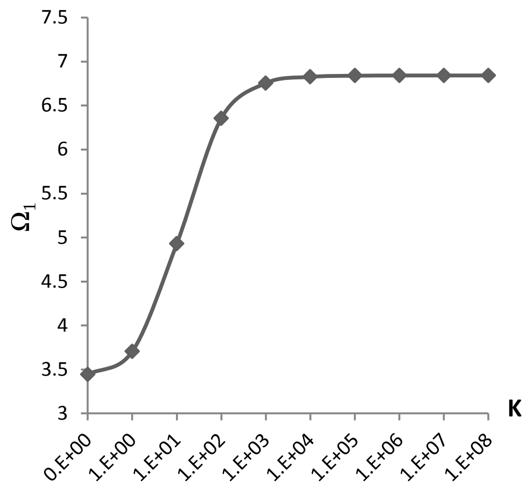
BSc	*b/a*	$\bar{C}$	Mode sequence			
			1	2	3	4
SSSS	1/2	-1/3	3.7002	4.2606	11.0746	11.4277
		0	2.4345	3.6877	11.3615	15.0393
		1/3	1.5930	3.6902	10.4245	11.9999
	1/3	-1/3	2.3530	4.9276	7.2380	12.9054
		0	1.0761	2.3474	6.6699	7.2363
		1/3	0.7038	2.3549	4.6153	7.3915
SFSF	1/2	-1/3	3.5258	7.6768	8.2681	16.5416
		0	2.2370	7.4994	11.3763	18.0312
		1/3	1.5060	7.5078	10.0619	17.0707
	1/3	-1/3	1.5587	3.6584	4.7732	10.0594
		0	0.9886	4.7513	5.0260	8.5270
		1/3	0.6654	4.4529	4.7517	10.1410
CFFF	1/2	-1/3	6.6122	10.4308	14.1414	21.8162
		0	6.6445	13.4687	14.8999	19.5932
	1/3	-1/3	4.3687	4.6372	9.1419	12.8157
		0	4.3766	6.6159	8.9322	9.5142



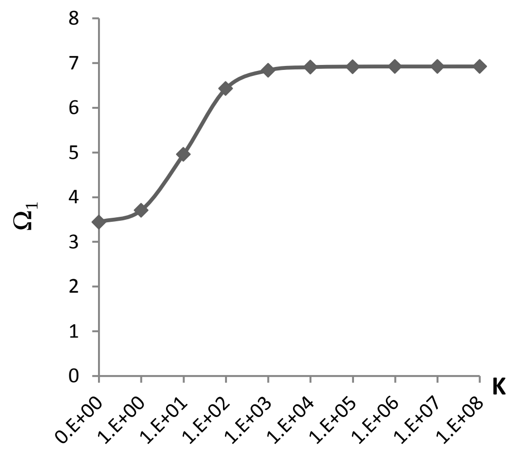
(a1)



(a2)



(b1)



(b2)

Figure 2 Fundamental frequency parameter $\Omega = \omega b^2 \sqrt{\rho h/D}$ for asymmetric boundary conditions as (1) translational and (2) both stiffnesses equally vary from 0 to 10^8 for (a)CCFF plates and (b)CCSS plates

On the other hand, for asymmetric boundary conditions such as CCFF and CCSS, for the cases in which only K_T varies or both K_T and K_R vary, the fundamental frequency parameter Ω_1 converges monotonically to a specific value as is shown in Figure 2.

5. Conclusions

This paper presents the formulation of an analytical model for the dynamic behaviour of rectangular isotropic plates, with an arbitrarily located slit and classical boundaries. The equations of motion and associated boundary and compatibility conditions are derived by using the extended Hamilton principle. An approach has been presented to solve the free vibration of the previously mentioned plates in a direct variational and numerical way. A not complicated, computationally efficient and accurate method has been developed for the determination of natural frequencies and modal shapes. The approach is the trigonometric Ritz method which is based on a simple, stable and computationally efficient set of admissible trigonometric functions and has been presented for free vibration analysis of rectangular Kirchhoff plates. The versatility and reliability of the present approach have been shown in various states of the slotted plate with an arbitrarily selected subset of complicating factors. Very accurate and stable solutions have been obtained for all cases with lesser computational effort in comparison with the other similar methods. Consequently, the present analysis shows that the trigonometric Ritz method is a valuable way for solving transverse free vibrations of thin rectangular plates and is easily applicable to a wide class of problems with complicating effects. To investigate the effect of the line slit and its location on the vibration behaviour, parametric studies have been performed. It is valuable to note that by using a modified version of this method, the static deflection problems and buckling can be analysed. On the other hand, this method can be easily generalized for analysing problems that include anisotropic plates.

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