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On the generalization of Trapezoid Inequality for functions of two variables with bounded variation and applications

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Abstract

In this paper, a generalization of trapezoid inequality for functions of two independent variables with bounded variation and some applications are given.

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1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whoose derivative $f' : (a, b) \to \mathbb{R}$ is baunded on (a, b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \tag{1.1}$$

for all $x \in [a, b]$[19]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [11], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

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Theorem 1.1. Let $f : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then

$$\left| \int_a^b f(t)dt - (b-a)\, f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f)$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

2. Preliminaries and Lemmas

In 1910, Fréchet [16] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x, y)$ and $\alpha(x, y)$ are defined over the rectangle $Q = [a, b] \times [c, d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x = x_i$, $y = y_i$,

$$a = x_0 < x_1 < ... < x_n = b, \text{ and } c = y_0 < y_1 < ... < y_m = d;$$

let ξ_i, η_j be any numbers satisfying $\xi_i \in [x_{i-1}, x_i]$, $\eta_j \in [y_{j-1}, y_j]$, $(i = 1, 2, ..., n;\ j = 1, 2, ..., m)$; and for all i, j let

$$\Delta_{11}\alpha(x_i, y_j) = \alpha(x_{i-1}, y_{j-1}) - \alpha(x_{i-1}, y_j) - \alpha(x_i, y_{j-1}) + \alpha(x_i, y_j).$$

Then if the sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_i, \eta_j)\, \Delta_{11}\alpha(x_i, y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to α is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b \int_c^d f(x, y) d_y d_x \alpha(x, y). \tag{2.1}$$

If in the above formulation S is replaced by the sum

$$S^* = \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_{ij}, \eta_{ij})\, \Delta_{11}\alpha(x_i, y_j),$$

where ξ_{ij}, η_{ij} are numbers satisfying $\xi_{ij} \in [x_{i-1}, x_i]$, $\eta_{ij} \in [y_{j-1}, y_j]$, we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$\int_a^b \int_c^d f(x, y) d_y d_x \alpha(x, y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson ([8]) has shown that the existence of (2.1) does not imply the existence of (2.2).

In [7], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

2.1. Definitions

The function $f(x, y)$ is assumed to be defined in rectangle $R(a \leq x \leq b, c \leq y \leq d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i(i = 0, 1, 2, ..., m),\ a = x_0 < x_1 < ... < x_m = b; \\ y &= y_j(j = 0, 1, 2, ..., n),\ c = y_0 < y_1 < ... < y_n = d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell*. We employ the notation

$$\Delta_{11} f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j),$$
$$\Delta f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_i, y_j).$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x}, y)$ $[f(x, \overline{y})]$ considered as a function of y $[x]$ alone in interval (c, d) $[(a, b)]$, or as $+\infty$ if $f(\overline{x}, y)$ $[f(x, \overline{y})]$ is of unbounded variation.

Definition 2.1. (Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x, y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets.

Definition 2.2. (Fréchet). The function $f(x, y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \epsilon_i \overline{\epsilon_j}\, |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets and all possible choices of $\epsilon_i = \pm 1$ and $\overline{\epsilon_j} = \pm 1$.

Definition 2.3. (Hardy-Krause). The function $f(x, y)$ is said tobe of bounded variation if it satisfies the condition of Definition 2.1 and if in addition $f(\overline{x}, y)$ is of bounded variation in y (i.e. $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x, \overline{y})$ is of bounded variation in y (i.e. $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.

Definition 2.4. (Arzelà). Let (x_i, y_i) $(i = 0, 1, 2, ..., m)$ be any set of points satisfiying the conditions

$$\begin{aligned} a &= x_0 < x_1 < ... < x_m = b; \\ c &= y_0 < y_1 < ... < y_m = d. \end{aligned}$$

Then $f(x, y)$ is said tobe of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i, y_i)|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of variables, as follows: Let f be of bounded variation on $Q=[a,b]\times[c,d]$, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}|\Delta_{11}f(x_i,y_j)|$ corresponding to the partition P of Q. The number

$$\bigvee_{Q}(f):=\bigvee_{c}^{d}\bigvee_{a}^{b}(f):=\sup\left\{\sum(P):P\in P(Q)\right\},$$

is called the total variation of f on Q. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [17], authors proved following Lemmas related double Riemann-Stieltjes integral:

Lemma 2.5. (Integrating by parts) If $f\in RS(\alpha)$ on Q, then $\alpha\in RS(f)$ on Q, and we have

$$\begin{aligned}&\int_{c}^{d}\int_{a}^{b}f(t,s)d_td_s\alpha(t,s)+\int_{c}^{d}\int_{a}^{b}\alpha(t,s)d_td_sf(t,s) \\ =\ & f(b,d)\alpha(b,d)-f(b,c)\alpha(b,c)-f(a,d)\alpha(a,d)+f(a,c)\alpha(a,c).\end{aligned} \tag{2.3}$$

Lemma 2.6. Assume that $g\in RS(\alpha)$ on Q and α is of bounded variation on Q, then

$$\left|\int_{c}^{d}\int_{a}^{b}g(x,y)d_xd_y\alpha(x,y)\right|\leq\sup_{(x,y)\in Q}|g(x,y)|\bigvee_{Q}(\alpha). \tag{2.4}$$

Moreover, Jawarneh and Noorani obtained following Ostrowski type inequality for functions of two variables with bounded variation in [17]:

Theorem 2.7. Let $f:Q\to\mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y)\in Q$, we have inequality

$$\begin{aligned}\left|(b-a)(d-c)f(x,y)-\int_{c}^{d}\int_{a}^{b}f(t,s)dtds\right| &\leq \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right] \\ &\quad\times\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]\bigvee_{Q}(f)\end{aligned}$$

where $\bigvee_{Q}(f)$ denotes the total (double) variation of f on Q.

In resent years the subject Ostrowski type inequalities for functions of bounded variation are studied by many authors. For more information and recent developments on inequalities for functions of bounded variation, please refer to ([1],[2] [4]-[6], [9]-[15],[18],[20]-[24]). But, There are a few works on inequalities functions of two variables with bounded variation (see [3],[17])

The aim of this paper is to establish the generalization of trapezoid inequality for functions of two independent variables with bounded variation and apply it for quadrature formula.

3. Main Results

First, we will give the following notations used in main Theorem.

$$\upsilon(h) := \max\{h_i|\ \ i = 0, ..., n-1\},\ h_i := x_{i+1} - x_i$$

$$\upsilon(l) := \max\{l_j|\ \ j = 0, ..., m-1\},\ l_j := y_{j+1} - l_j$$

$$\begin{aligned} J(x,y) &= (b-x)(d-y)f(b,d) + (b-x)(y-c)f(b,c) \\ &\quad + (x-a)(d-y)f(a,d) + (x-a)(y-c)f(a,c) - \int_a^b\int_c^d f(t,s)dsdt. \end{aligned}$$

Theorem 3.1. *Let the function $f : Q = [a, b] \times [c, d] \to R$ is of bounded variation on Q. Then we have the inequality*

$$|J(x,y)| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f) \tag{3.1}$$

for all $(x, y) \in Q$.

Proof . Using Lemma 2.5, we get

$$\begin{aligned} \int_a^b\int_c^d (x-t)(y-s)\, d_sd_tf(t,s) &= (b-x)(d-y)f(b,d) + (b-x)(y-c)f(b,c) \\ &\quad + (x-a)(d-y)f(a,d) + (x-a)(y-c)f(a,c) - \int_a^b\int_c^d f(t,s)dsdt \\ &= J(x,y) \end{aligned} \tag{3.2}$$

for all $(x, y) \in Q$.

Taking modulus and using Lemma 2.6 in (3.2), we obtain

$$\begin{aligned} |J(x,y)| &= \left|\int_a^b\int_c^d (x-t)(y-s)\, d_sd_tf(t,s)\right| \\ &\le \sup_{\substack{t\in[a,b]\\ s\in[c,d]}} |x-t|\,|y-s| \bigvee_a^b\bigvee_c^d(f) \\ &= \max\{x-a, b-x\}\max\{y-c, b-y\}\bigvee_a^b\bigvee_c^d(f) \\ &= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f) \end{aligned}$$

which is the required result. □

Remark 3.2. Under assumption Theorem 3.1 with $x = b$ and $y = d$,then we have the "left rectangle inequality"

$$\left|(b-a)(d-c)f(a,c) - \int_a^b \int_c^d f(t,s)dsdt\right| \leq (b-a)(d-c)\bigvee_a^b \bigvee_c^d (f).$$

Remark 3.3. If we take $x = a$ and $y = c$ in Theorem 3.1, then we have the "right rectangle inequality"

$$\left|(b-a)(d-c)f(b,d) - \int_a^b \int_c^d f(t,s)dsdt\right| \leq (b-a)(d-c)\bigvee_a^b \bigvee_c^d (f).$$

Remark 3.4. Under assumption Theorem 3.1 with $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, then we get the "trapezoid inequality"

$$\left|\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4} - \frac{4}{(b-a)(d-c)}\int_a^b \int_c^d f(t,s)dsdt\right| \leq \frac{1}{4}\bigvee_a^b \bigvee_c^d (f)$$

which was given by Jawarneh and Noorani in [17]. The constant $\frac{1}{4}$ is the best possible. For a simple proof of sharpness of constant see [3].

4. Applications to quadrature formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$. We introduce intermediate points $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$, and $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, 1, ..., m-1)$, and define

$$\begin{aligned} T(f, I_n, J_m, \xi, \eta) :=& \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} (x_{i+1} - \xi_i)(y_{j+1} - \eta_j) f(x_{i+1}, y_{j+1}) \\ &+ \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} (x_{i+1} - \xi_i)(\eta_j - y_j) f(x_{i+1}, y_j) \\ &+ \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} (\xi_i - x_i)(y_{j+1} - \eta_j) f(x_i, y_{j+1}) \\ &+ \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} (\xi_i - x_i)(\eta_j - y_j) f(x_i, y_j). \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let the function $f : Q = [a, b] \times [c, d] \to R$ is of bounded variation on Q. Then*

$$\int_a^b \int_c^d f(t,s)dsdt = T(f, I_n, J_m, \xi, \eta) + R(f, I_n, J_m, \xi, \eta)$$

where $T(f, I_n, J_m, \xi, \eta)$ *defined in* (4.1) *and the remainder term* $R(f, I_n, J_m, \xi, \eta)$ *satisfies*

$$\begin{aligned} |R(f, I_n, J_m, \xi, \eta)| &\leq \left[\frac{1}{2}\upsilon(h) + \max_{0\leq i<n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \\ &\times \left[\frac{1}{2}\upsilon(l) + \max_{0\leq j<m}\left|\eta_j - \frac{y_j + y_{j+1}}{2}\right|\right] \bigvee_a^b \bigvee \\ &\leq \upsilon(h)\upsilon(l) \bigvee_a^b \bigvee_c^d (f). \end{aligned} \tag{4.2}$$

Proof . Applying Theorem 3.1 to interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we have

$$\begin{aligned} &\Big| (x_{i+1} - \xi_i)(y_{j+1} - \eta_j) f(x_{i+1}, y_{j+1}) + (x_{i+1} - \xi_i)(\eta_j - y_j) f(x_{i+1}, y_j) \\ &+ (\xi_i - x_i)(y_{j+1} - \eta_j) f(x_i, y_{j+1}) + (\xi_i - x_i)(\eta_j - y_j) f(x_i, y_j) \\ &- \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t,s) ds dt \Big| \\ &\leq \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \left[\frac{1}{2}l_j + \left|\eta_j - \frac{y_j + y_{j+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \end{aligned} \tag{4.3}$$

for $i \in \{0, 1, ..., n-1\}$and $j \in \{0, 1, ..., m-1\}$.

By the generalized triangle inequality and summing the inequality (4.3) over i from 0 to $n-1$ and j from 0 to $m-1$,

$$\begin{aligned} &|R(f, I_n, J_m, \xi, \eta)| \\ \leq\ &\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \Big| (x_{i+1} - \xi_i)(y_{j+1} - \eta_j) f(x_{i+1}, y_{j+1}) + (x_{i+1} - \xi_i)(\eta_j - y_j) f(x_{i+1}, y_j) \\ &+ (\xi_i - x_i)(y_{j+1} - \eta_j) f(x_i, y_{j+1}) + (\xi_i - x_i)(\eta_j - y_j) f(x_i, y_j) - \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t,s) ds dt \Big| \\ \leq\ &\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \left[\frac{1}{2}l_j + \left|\eta_j - \frac{y_j + y_{j+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \\ \leq\ &\max_{0\leq i<n} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \max_{0\leq j<m} \left[\frac{1}{2}l_j + \left|\eta_j - \frac{y_j + y_{j+1}}{2}\right|\right] \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f) \\ \leq\ &\left[\frac{1}{2}\upsilon(h) + \max_{0\leq i<n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \left[\frac{1}{2}\upsilon(l) + \max_{0\leq j<m}\left|\eta_j - \frac{y_j + y_{j+1}}{2}\right|\right] \bigvee_a^b \bigvee_c^d (f). \end{aligned}$$

This completes the proof of first inequality in (4.2).

In last inequality, if we observe that

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2}h_i \text{ and } \max_{i\in[0,\dots,n-1]}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2}\upsilon(h), \tag{4.4}$$

and similarly,

$$\max_{j\in[0,\ldots,m-1]}\left|\eta_j-\frac{y_j+y_{j+1}}{2}\right|\leq\frac{1}{2}\upsilon(l) \tag{4.5}$$

we have the inequality

$$|R(f,I_n,J_m,\xi,\eta)|\leq\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).$$

This completes the proof of theorem. □

Remark 4.2. If we take $\xi_i=x_{i+1}$ and $\eta_j=y_{j+1,}$ then

$$\int_a^b\int_c^d f(t,s)dsdt=D_L(f,I_n,J_m)+R_L(f,I_n,J_m)$$

where $D_L(f,I_n,J_m)$ is built from the left rectangle rule

$$D_L(f,I_n,J_m)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f(x_i,y_j)h_il_j$$

and remainder term $R_L(f,I_n,J_m)$ satisfies

$$|R_L(f,I_n,J_m)|\leq\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).$$

Remark 4.3. If we take $\xi_i=x_i$ and $\eta_j=y_{j,}$ then

$$\int_a^b\int_c^d f(t,s)dsdt=D_R(f,I_n,J_m)+R_R(f,I_n,J_m)$$

where $D_R(f,I_n,J_m)$ is constructed from the right rectangle rule

$$D_R(f,I_n,J_m)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f(x_{i+1},y_{j+1})h_il_j$$

and remainder satisfies

$$|R_R(f,I_n,J_m)|\leq\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).$$

Remark 4.4. If we take $\xi_i=\frac{x_i+x_{i+1}}{2}$ and $\eta_j=\frac{y_{j,}+y_{j+1}}{2}$ then

$$\int_a^b\int_c^d f(t,s)dsdt=T(f,I_n,J_m)+R(f,I_n,J_m)$$

where $T(f,I_n,J_m)$ is constructed from the trapezoid rule

$$T(f,I_n,J_m)=\frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_{i+1},y_{j+1})+f(x_{i+1},y_j)+f(x_i,y_{j+1})+f(x_i,y_j)\right]h_il_j$$

and remainder satisfies

$$|R(f, I_n, J_m)| \leq \frac{1}{4}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).$$

The constant $\frac{1}{4}$ is the best possible.

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