Int. J. Nonlinear Anal. Appl. 4 (2013) No. 2, 45-52
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

Some new results using integration of arbitrary order

A. Anber[a], Z. Dahmani[b,*], B. Bendoukha[c]

[a]Department of Mathematics, USTO University of Oran, Algeria.
[b]LPAM, Faculty of SEI, UMAB, University of Mostaganem, Algeria.
[c]LPAM, Faculty of Exact Science and Informatics, UMAB, University of Mostaganem, Algeria.

Abstract

In this paper, we present recent results in integral inequality theory. Our results are based on the fractional integration in the sense of Riemann-Liouville

Keywords: Integral inequality, Riemann-Liouville integral operator.
2010 MSC: 26D10; 26A33.

1. Introduction

The integral inequalities involving functions of independent variables play a fundamental role in the theory of differential equations. Motivated by certain applications, many such new inequalities have been discovered in the past few years (see [2, 5, 13, 14, 15]). Moreover, the fractional type inequalities are of great importance. We refer the reader to [1, 16] for some applications. Let us now turn our attention to some results that have inspired our work. We consider the quantity

$$\begin{aligned} R_{a,b}(p,q,f,g) := \int_a^b pf^2(x)\,dx \int_a^b qg^2(x)\,dx + \int_a^b qf^2(x)\,dx \int_a^b pg^2(x)\,dx \\ -2\Big(\textstyle\int_a^b p|fg|(x)\,dx\Big)\Big(\textstyle\int_a^b q|fg|(x)\,dx\Big) - 2\Big(\textstyle\int_a^b p|fg|(x)\,dx\Big)\Big(\textstyle\int_a^b q|fg|(x)\,dx\Big), \end{aligned} \tag{1.1}$$

where f and g are two continuous functions on $[a,b]$ and p and q are two positive and continuous functions on $[a,b]$.
In the case, when $p = q$, S.S. Dragomir [10] proved the inequality:

$$0 < R_{1,\Omega}(p,f,g) := R_\Omega(p,p,f,g) \le \frac{(M-m)^2}{2mM}\Big(\int_\Omega p|fg|(x)\,d\mu(x)\Big), \tag{1.2}$$

*Corresponding author
Email addresses: ah.anber@yahoo.fr (A. Anber), zzdahmani@yahoo.fr (Z. Dahmani), b.bendoukha@yahoo.fr (B. Bendoukha)

Received: May 2012 *Revised:* January 2013

provided f and g are Lebesgue $\mu-$ measurable, pf^2, pg^2 are Lebesgue $\mu-$ integrable on Ω and $0 < m \leq |\frac{f(x)}{g(x)}| \leq M \leq \infty$, for μ $a.e.x \in \Omega$. For other results related to the Cauchy-Schwarz difference (1), in the case $p = q$, a number of valued extensions can be found in [3, 6, 7, 8, 9, 12, 18] and the references cited therein.
The main aim of this paper is to establish some new fractional integral inequalities of Cauchy-Schwarz type by giving an upper and a lower bound for the quantity (1.1) Some new fractional results related to Cassel's inequality [4], [17], [19] are also generated. For our results, some classical inequalities can be deduced as some special cases. Our results have some relationships with [3], [10].

2. Description of the fractional calculus

We introduce some definitions and properties which will be used in this paper:

Definition 2.1. *A real valued function f is said to be in the space $C_\mu([0,\infty[), \mu \in \mathbb{R}$ if there exists a real number $r > \mu$, such that $f(t) = t^r f_1(t)$, where $f_1 \in C([0,\infty))$.*

Definition 2.2. *A function f is said to be in the space $C_\mu^n([0,\infty[), n \in \mathbb{N}$, if $f^{(n)} \in C_\mu([0,\infty[)$.*

Definition 2.3. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu([0,\infty[), \mu \geq -1$, is defined as*

$$\begin{aligned} &J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0 \\ &J^0 f(t) = f(t). \end{aligned} \tag{2.1}$$

For the convenience of establishing the results, we give the following property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t). \tag{2.2}$$

For the expression (2.1), when $f(t) = t^\beta$ we get another expression that will be used later:

$$J^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \tag{2.3}$$

For more details, see [11, 16].

3. Main results

Our first result is the following theorem:

Theorem 3.1. *Suppose that f and g are two continuous functions on $[0,\infty[$ and p and q are two positive continuous function on $[0,\infty[$, such that $p|\frac{f}{g}|, p|\frac{g}{f}|, q|\frac{f}{g}|, q|\frac{g}{f}|, pf^2, pg^2, qf^2$ and qg^2 are integrable functions on $[0,\infty[$. If there exist m and M two positive real numbers, such that*

$$0 < m \leq |f(\tau)g(\tau)| \leq M; \tau \in [0,t], t > 0, \tag{3.1}$$

then we have

$$
\begin{aligned}
&m^2\Big(J^\alpha(q|\frac{f}{g}|)(t)J^\alpha(p|\frac{g}{f}|)(t)+J^\alpha(p|\frac{f}{g}|)(t)J^\alpha(|q\frac{g}{f}|)(t)-2J^\alpha p(t)J^\alpha q(t)\Big)\\
&\leq J^\alpha pf^2(t)J^\alpha qg^2(t)+J^\alpha qf^2(t)J^\alpha pg^2(t)-2J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t) \qquad (3.2)\\
&\leq M^2\Big(J^\alpha(p|\tfrac{f}{g}|)(t)J^\alpha(q|\tfrac{g}{f}|)(t)+J^\alpha(q|\tfrac{f}{g}|)(t)J^\alpha(p|\tfrac{g}{f}|)(t)-2J^\alpha p(t)J^\alpha q(t)\Big),
\end{aligned}
$$

for any $\alpha>0, t>0$.

Proof .

In the identity

$$\frac{u^2+v^2}{2}-uv=\frac{1}{2}uv\Big(\sqrt{\frac{u}{v}}-\sqrt{\frac{v}{u}}\Big)^2; u>0, v>0,$$

we take $u=|f(\tau)g(\rho)|$ and $v=|f(\rho)g(\tau)|, \tau,\rho\in[0,t], t>0$. Then we can write

$$
\begin{aligned}
&\frac{f^2(\tau)g^2(\rho)+f^2(\rho)g^2(\tau)}{2}-|f(\tau)g(\rho)||f(\tau)g(\rho)|\\
&=\tfrac{1}{2}|f(\tau)g(\tau)||f(\rho)g(\rho)|\Big(\sqrt{|\tfrac{f(\tau)}{g(\tau)}||\tfrac{g(\rho)}{f(\rho)}|}-\sqrt{|\tfrac{f(\rho)}{g(\rho)}||\tfrac{g(\tau)}{f(\tau)}|}\Big)^2.
\end{aligned} \qquad (3.3)
$$

On the other hand, we have

$$\Big(\sqrt{|\frac{f(\tau)}{g(\tau)}||\frac{g(\rho)}{f(\rho)}|}-\sqrt{|\frac{f(\rho)}{g(\rho)}||\frac{g(\tau)}{f(\tau)}|}\Big)^2=|\frac{f(\tau)}{g(\tau)}||\frac{g(\rho)}{f(\rho)}|+|\frac{f(\rho)}{g(\rho)}||\frac{g(\tau)}{f(\tau)}|-2. \qquad (3.4)$$

Using (3.4) and the condition (3.1) we can write

$$
\begin{aligned}
&\frac{m^2}{2}\Big(|\frac{f(\tau)}{g(\tau)}||\frac{g(\rho)}{f(\rho)}|+|\frac{f(\rho)}{g(\rho)}||\frac{g(\tau)}{f(\tau)}|-2\Big)\\
&\leq \tfrac{f^2(\tau)g^2(\rho)+f^2(\rho)g^2(\tau)}{2}-|f(\tau)g(\tau)||f(\rho)g(\rho)|\\
&\leq \tfrac{M^2}{2}\Big(|\tfrac{f(\tau)}{g(\tau)}||\tfrac{g(\rho)}{f(\rho)}|+|\tfrac{f(\rho)}{g(\rho)}||\tfrac{g(\tau)}{f(\tau)}|-2\Big).
\end{aligned} \qquad (3.5)
$$

Hence we get,

$$
\begin{aligned}
&\frac{m^2}{2}\Big(|\frac{g(\rho)}{f(\rho)}|J^\alpha(p|\frac{f}{g}|)(t)+|\frac{f(\rho)}{g(\rho)}|J^\alpha(p|\frac{g}{f}|)(t)-2J^\alpha p(t)\Big)\\
&\leq \tfrac{g^2(\rho)J^\alpha pf^2(t)+f^2(\rho)J^\alpha pg^2(t)}{2}-|f(\rho)g(\rho)|J^\alpha(p|fg|)(t)\\
&\leq \tfrac{M^2}{2}\Big(|\tfrac{g(\rho)}{f(\rho)}|J^\alpha(p|\tfrac{f}{g}|)(t)+|\tfrac{f(\rho)}{g(\rho)}|J^\alpha(p|\tfrac{g}{f}|)(t)-2J^\alpha p(t)\Big).
\end{aligned} \qquad (3.6)
$$

Multiplying both sides of (3.6) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}q(\rho)$, then integrating the resulting inequalities with respect to ρ over $[0,t]$, we obtain

$$\frac{m^2}{2}\Big(J^\alpha(q|\frac{f}{g}|)(t)J^\alpha(p|\frac{g}{f}|)(t)+J^\alpha(p|\frac{f}{g}|)(t)J^\alpha(q|\frac{g}{f}|)(t)-2J^\alpha p(t)J^\alpha q(t)\Big)$$
$$\leq \tfrac{J^\alpha pf^2(t)J^\alpha qg^2(t)+J^\alpha qf^2(t)J^\alpha pg^2(t)}{2}-J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t) \tag{3.7}$$
$$\leq \tfrac{M^2}{2}\Big(J^\alpha(p|\tfrac{f}{g}|)(t)J^\alpha(q|\tfrac{g}{f}|)(t)+J^\alpha(q|\tfrac{f}{g}|)(t)J^\alpha(p|\tfrac{g}{f}|)(t)-2J^\alpha p(t)J^\alpha q(t)\Big).$$

Theorem 3.1 is thus proved. □

Remark 3.2. *Applying Theorem 3.1 for $p=q,\alpha=1,d\mu(\tau)=d\tau$, we obtain Theorem 1 of [3] on $[0,t]=\Omega$.*

The previous result can be generalized to the following:

Theorem 3.3. *Suppose that f and g are two continuous functions on $[0,\infty[$ and let p and q be two positive continuous functions on $[0,\infty[$, such that $p|\frac{f}{g}|,p|\frac{g}{f}|,q|\frac{f}{g}|,q|\frac{g}{f}|,pf^2,qf^2,pg^2$ and qg^2 are integrable functions on $[0,\infty[$. If there exist m and M two positive real numbers, such that*

$$0<m\leq |f(\tau)g(\tau)|\leq M;\tau\in[0,t],t>0, \tag{3.8}$$

then the inequalities

$$m^2\Big(J^\alpha(p|\frac{f}{g}|)(t)J^\beta(q|\frac{g}{f}|)(t)+J^\beta(q|\frac{f}{g}|)(t)J^\alpha(p|\frac{g}{f}|)(t)-2J^\alpha p(t)J^\beta q(t)\Big)$$
$$\leq J^\alpha pf^2(t)J^\beta qg^2(t)+J^\beta qf^2(t)J^\alpha pg^2(t)-2J^\alpha(p|fg|)(t)J^\beta(q|fg|)(t) \tag{3.9}$$
$$\leq M^2\Big(J^\alpha(p|\tfrac{f}{g}|)(t)J^\beta(q|\tfrac{g}{f}|)(t)+J^\beta(q|\tfrac{f}{g}|)(t)J^\alpha(p|\tfrac{g}{f}|)(t)-2J^\alpha p(t)J^\beta q(t)\Big)$$

are valid for any $\alpha>0,\beta>,t>0$.

Proof . Multiplying both sides of (3.6) by $\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}q(\rho)$, then integrating the resulting inequalities with respect to ρ over $[0,t]$, we obtain:

$$\frac{m^2}{2}\Big(J^\alpha(p|\frac{f}{g}|)(t)J^\beta(q|\frac{g}{f}|)(t)+J^\beta(q|\frac{f}{g}|)(t)J^\alpha(p|\frac{g}{f}|)(t)-2J^\alpha p(t)J^\beta q(t)\Big)$$
$$\leq \tfrac{J^\alpha pf^2(t)J^\beta qg^2(t)+J^\beta qf^2(t)J^\alpha pg^2(t)}{2}-J^\alpha(p|fg|)(t)J^\beta(q|fg|)(t) \tag{3.10}$$
$$\leq \tfrac{M^2}{2}\Big(J^\alpha(p|\tfrac{f}{g}|)(t)J^\beta(q|\tfrac{g}{f}|)(t)+J^\beta(q|\tfrac{f}{g}|)(t)J^\alpha(p|\tfrac{g}{f}|)(t)-2J^\alpha p(t)J^\beta q(t)\Big).$$

The proof of Theorem 3.3 is thus achieved. □

Remark 3.4. *It is clear that Theorem 3.1 would follow as a special case of of Theorem 3.3 when $\alpha=\beta$.*

Now, we shall propose a new generalization of Cassel's inequality. We have:

Theorem 3.5. *Let f, g be two continuous functions on $[0,\infty[$ and let p and q be two positive continuous functions on $[0,\infty[$, such that pf^2, qf^2, pg^2 and qg^2 are integrable on $[0,\infty[$. If there exist m and M two positive real numbers, such that*

$$0 < m \leq |\frac{f(\tau)}{g(\tau)}| \leq M; \tau \in [0,t], t > 0, \tag{3.11}$$

then we have

$$\begin{aligned} &J^\alpha pf^2(t)J^\alpha qg^2(t) - J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t) \\ &\leq \tfrac{(M-m)^2}{4mM}J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t), \end{aligned} \tag{3.12}$$

for any $\alpha > 0, t > 0$.

Proof . From the condition $|\frac{f(\tau)}{g(\tau)}| \leq M; \tau \in [0,t], t > 0$, we have

$$f^2(\tau) \leq M|f(\tau)g(\tau)|; \tau \in [0,t], t > 0. \tag{3.13}$$

Therefore,

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} p(\tau)f^2(\tau)d\tau \leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} p(\tau)|f(\tau)g(\tau)|d\tau. \tag{3.14}$$

Consequently,

$$J^\alpha pf^2(t) \leq MJ^\alpha(p|fg|)(t). \tag{3.15}$$

Now, using the condition $m \leq |\frac{f(\tau)}{g(\tau)}|; \tau \in [0,t], t > 0$, we can write

$$mJ^\alpha qg^2(t) \leq J^\alpha(q|fg|)(t). \tag{3.16}$$

Multiplying (3.15) and (3.16) we obtain

$$J^\alpha pf^2(t)J^\alpha qg^2(t) \leq \frac{M}{m}J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t). \tag{3.17}$$

Consequently, we get

$$\begin{aligned} &J^\alpha pf^2(t)J^\alpha qg^2(t) - J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t) \\ &\leq \tfrac{M-m}{m}J^\alpha(p|fg|)(t)J^\alpha(q|fg|)(t), \end{aligned} \tag{3.18}$$

which implies (3.12) Theorem 3.5 is thus proved. □

Remark 3.6. *If we take $\alpha = 1, p = q$, then we obtain Cassel's inequality [10],[19] on $[0,t]$.*

Also, with the same assumptions as before, we get the following generalization of Theorem 3.5:

Theorem 3.7. *Let f, g be two continuous functions on $[0,\infty[$ and let p and q be two positive continuous functions on $[0,\infty[$, such that pf^2, qf^2, pg^2 and qg^2 are integrable on $[0,\infty[$. If there exist m, M positive real numbers, such that*

$$0 < m \leq |\frac{f(\tau)}{g(\tau)}| \leq M, \tau \in [0,t], t > 0, \tag{3.19}$$

then, for any $\alpha > 0, \beta > 0, t > 0$, the inequality

$$J^\alpha pf^2\,(t)\,J^\beta qg^2\,(t) - J^\alpha(p|fg|)(t)J^\beta(q|fg|)(t) \leq \frac{(M-m)^2}{4mM} J^\alpha(p|fg|)\,(t)\,J^\beta(q|fg|)(t) \tag{3.20}$$

is valid.

Proof . From the condition $m \leq |\frac{f(\tau)}{g(\tau)}|; \tau \in [0,t], t > 0$, we can write

$$mJ^\beta qg^2(t) \leq J^\beta(q|fg|)(t). \tag{3.21}$$

Thanks to (3.16) and (3.21) we obtain

$$J^\alpha pf^2(t)J^\beta qg^2(t) \leq \frac{M}{m} J^\alpha(p|fg|)(t)J^\beta(q|fg|)(t). \tag{3.22}$$

Therefore,

$$\begin{aligned} &J^\alpha pf^2(t)J^\beta qg^2(t) - J^\alpha(p|fg|)(t)J^\beta(q|fg|)(t) \\ &\leq \tfrac{M-m}{m} J^\alpha(p|fg|)(t)J^\beta(q|fg|)(t). \end{aligned} \tag{3.23}$$

Hence, we deduce the desired inequality (3.20). □
We give also the following corollaries:

Corollary 3.8. *Let F, G be two continuous functions on $[0,\infty[$ and let p and q be two positive continuous functions on $[0,\infty[$, such that $p|\frac{F}{G}|, p|\frac{G}{F}|, q|\frac{F}{G}|, q|\frac{G}{F}|, pF^2, pG^2, qF^2$ and qG^2 are integrable functions on $[0,\infty[$. If there exist n, N, M positive real numbers, such that $|F(\tau)G(\tau)| \leq M$ and*

$$0 < n \leq |\frac{F(\tau)}{G(\tau)}| \leq N, \tau \in [0,t], t > 0, \tag{3.24}$$

then, for any $\alpha > 0, t > 0$, the inequality

$$\begin{aligned} &J^\alpha pF^2(t)J^\alpha qG^2(t) + J^\alpha qF^2(t)J^\alpha pG^2(t) - 2J^\alpha(p|FG|)(t)J^\alpha(q|FG|)(t) \\ &\leq \tfrac{M^2(N-n)^2}{2nN} J^\alpha pJ^\alpha q(t) \end{aligned} \tag{3.25}$$

is valid.

Proof . In Theorem 3.5, we take $f := \sqrt{|\frac{F}{G}|}, g := \sqrt{|\frac{G}{F}|}$. We constat that $n \leq \frac{f(\tau)}{g(\tau)} \leq N; \tau \in [0,t], t > 0$, and then

$$\begin{aligned} &J^\alpha(p|\frac{F}{G}|)(t)J^\alpha(q|\frac{G}{F}|)(t) - J^\alpha p(t)J^\alpha q(t) \\ &\leq \tfrac{(N-n)^2}{4nN} J^\alpha p(t)J^\alpha q(t). \end{aligned} \tag{3.26}$$

We have also

$$
\begin{aligned}
&J^\alpha(q|\frac{F}{G}|)(t)J^\alpha(p|\frac{G}{F}|)(t) - J^\alpha p(t)J^\alpha q(t)\\
&\leq \tfrac{(N-n)^2}{4nN} J^\alpha p(t) J^\alpha q(t).
\end{aligned}
\tag{3.27}
$$

Combining (3.26) and (3.27), we obtain

$$
\begin{aligned}
&J^\alpha(p|\frac{F}{G}|)(t)J^\alpha(q|\frac{G}{F}|)(t) + J^\alpha(q|\frac{F}{G}|)(t)J^\alpha(p|\frac{G}{F}|)(t) - 2J^\alpha p(t)J^\alpha q(t)\\
&\leq \tfrac{(N-n)^2}{2nN} J^\alpha p(t) J^\alpha q(t).
\end{aligned}
\tag{3.28}
$$

Since $|F(\tau)G(\tau)| \leq M; \tau \in [0,t], t > 0$, then thanks to the second inequality of (3.2) (Theorem 3.1), we claim that

$$
\begin{aligned}
&J^\alpha pF^2(t)J^\alpha qG^2(t) + J^\alpha qF^2(t)J^\alpha pG^2(t) - 2J^\alpha(p|FG|)(t)J^\alpha(q|FG|)(t)\\
&\leq M^2\Big(J^\alpha(p|\tfrac{F}{G}|)(t)J^\alpha(q|\tfrac{G}{F}|)(t) + J^\alpha(q|\tfrac{F}{G}|)(t)J^\alpha(p|\tfrac{G}{F}|)(t) - 2J^\alpha p(t)J^\alpha q(t)\Big).
\end{aligned}
\tag{3.29}
$$

Using (3.28) and (3.29), we obtain the desired inequality (3.25). □

Remark 3.9. *If we take $p = q, \alpha = 1, d\mu(\tau) = d\tau$, then we obtain Corollary 3.8 on Ω provided that $\Omega = [0,t]$.*

Corollary 3.10. *Let F, G, p and q satisfy the conditions of Corollary 3.8. Then, for any $\alpha > 0, \beta > 0, t > 0$, we have*

$$
\begin{aligned}
&J^\alpha pF^2(t)J^\beta qG^2(t) + J^\beta qF^2(t)J^\alpha pG^2(t) - 2J^\alpha(p|FG|)(t)J^\beta(q|FG|)(t)\\
&\leq \tfrac{M^2(N-n)^2}{2nN} J^\alpha p J^\beta q(t).
\end{aligned}
\tag{3.30}
$$

Proof . We apply Theorem 3.5 and Theorem **??**. □

Remark 3.11. *If we take $\alpha = \beta$, then we obtain Corollary 3.8.*

References

[1] G.A. Anastassiou, *Fractional Differentiation Inequalities.* Springer, (2009).

[2] D. Bainov and P. Simeonov, *Integral Inequalities and Applications*, Kluwer Academic Dordrecht, (1992).

[3] N.S. Barnett, S.S. Dragomir and I. Gomm, *On some integral inequalities related to the Cauchy-Bunyakovsky-Schwarz inequality*, Appl. Math. Letters, 23 (2010), 1008-1012.

[4] P. Cerone and S.S. Dragomir, *New bounds for the Chebyshev functional*, ajmaa.org/RGMIA/papers/v6n2/NICF.pdf, 2003.

[5] P. Cerone and S.S. Dragomir, *A refinement of the Gruss inequality and applications*, Tankang J. Math., 38 (1), (2007), 37-49.

[6] Z. Dahmani and L. Tabharit, *New fractional bounds for some classical inequalities.* Submitted paper.

[7] S.S. Dragomir, *A survey on Cauchy-Bunyakovsky-Schwarz type discrete inequalities*, Journal of Inequalities in Pure and Applied Mathematics, 4 (3) (2003), Art. 63.

[8] S.S. Dragomir and N.T. Diamond, *Integral inequalities of Gruss type via Polya-Szego and Shisha-Mond results*, East Asian Mathematical Journal, 19 (1), (2003), 27-39.

[9] S.S. Dragomir, *Reverses of Schwarz,Triangle and Bessel inequalities in inner product spaces*, J. Inequal. Pure Appl. Math., 5 (3), (2004) Art. 76.
[10] S.S. Dragomir, *Advances in Inequalities of the Schwarz, Gruss and Bessel Type in Inner Product Spaces*, Nova Science Publishers Inc., New York, (2005).
[11] R. Gorenflo and F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien, (1997), 223-276.
[12] W. Greub and W. Rheinboldt, *On a generalisation of an inequality of L.V. Kantorovich*, Proc. Amer. Math. Soc., 10 (1959), 407-415.
[13] S. Mazouzi and F. Qi, *On an open problem regarding an integral inequality.* JIPAM. J. Inequal. Pure Appl. Math., 4 (2) (2003), Art. 31.
[14] D.S. Mitrinovic, J.E. Pecaric and A.M. Fink, *Classical and new inequalities in Analysis.* Kluwer Academic Publishers, Dordrecht, (1993).
[15] B.G.Pachpatte, *Inequalities for Differential and Integral Equations* , Academic Press, New York, (1998).
[16] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, (1999).
[17] G. Polya and G. Szego, *Aufgaben and lahrsotse ans der Analysis*, Berlin, (1), (1925), 213-214.
[18] M.Z. Sarikaya, N. Aktan and H. Yildirim, *On weighted Chebyshev-Gruss like inequalities on time scales*, J. Math. Inequal., 2 (2) (2008), 185-195.
[19] G.S. Watson, *Serial correlation in regression analysis*, I, Biometrika, 42 (1955), 327-341.