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A determinant inequality and log-majorisation for operators

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Abstract

Let $\lambda_1, \ldots, \lambda_n$ be positive real numbers such that $\sum_{k=1}^n \lambda_k = 1$. In this paper, we prove that for any positive operators $a_1, a_2, \ldots, a_n$ in semifinite von Neumann algebra M with faithful normal trace that $\mathrm{tr}\,(1) < \infty$,

$$\prod_{k=1}^n (\det a_k)^{\lambda_k} \le \det(\sum_{k=1}^n \lambda_k a_k),$$

where $\det a = exp(\int_0^{\mathrm{tr}\,(1)} \mu_a(t)\,dt)$. If furthermore $\mathrm{tr}\,(a_i) < \infty$ for every $1 \le i \le n$ and $\prod_{k=1}^n (\det a_k)^{\lambda_k} \neq 0$, then equality holds if and only if $a_1 = a_2 = \cdots = a_n$. A log-majorisation version of Young inequality are given as well.

Keywords: Singular values; Semifinite trace; Majorisation; log-majorisation.
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1. Introduction

It was shown in [2, 13] that for any positive real numbers $\lambda_1, \ldots, \lambda_n$ that $\sum_{k=1}^n \lambda_k = 1$ and for any positive semidefinite matrices $a_1, a_2, \ldots, a_n$ in $M_n(\mathbb{C})$

$$\prod_{k=1}^n (\det a_k)^{\lambda_k} \le \det(\sum_{k=1}^n \lambda_k a_k)\,. \tag{1.1}$$

In this paper we first introduce the singular values and determinant for operator in semifinite von Neumann algebras and then give an extension of (1.1) in the context of semifinite von Neumann algebras which is one of our main results.

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For a real vector $X = (x_1\,, x_2\,, \ldots\,, x_n)$, $X^{\downarrow} = (x_1^{\downarrow}\,, x_2^{\downarrow}\,, \ldots\,, x_n^{\downarrow})$ is the decreasing rearrangement of X. Let $X,\ Y$ be two vectors in $\mathbb{R}^n$. Then we say X is (weakly) **submajorised** by Y, in symbols $X \prec_w Y$ if

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}\,, \qquad 1 \leq k \leq n\,,$$

and X is **majorised** by Y in symbols $X \prec Y$, if X is submajorised by Y and

$$\sum_{j=1}^{n} x_j^{\downarrow} = \sum_{j=1}^{n} y_j^{\downarrow}\,.$$

For any two vectors X and Y in $\mathbb{R}^n$ with nonnegative component if

$$\prod_{i=1}^{k} x_i^{\downarrow} \leq \prod_{i=1}^{k} y_i^{\downarrow}, \qquad\qquad k = 1, 2, \ldots, n,$$

then we say that X **weakly log-majorised** by Y and denote $X \prec_{w-log} Y$. If $X \prec_{w-log} Y$ and $\prod_{i=1}^{n} x_i^{\downarrow} = \prod_{i=1}^{n} y_i^{\downarrow}$, then we say that X **log-majorised** by Y and denote $X \prec_{log} Y$.

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ matrices. The singular values of $A \in M_n(\mathbb{C})$, denoted by $s_j(A),\ j = 1, 2, \cdots, n$, are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$ arrange in decreasing order and repeated according to multiplicity. $S(A)$ will denote the vector that its component are singular values of A. The matrix A is weakly log-majorised or log-majorised by matrix B and denoted by $A \prec_{w-\log} B$ or $A \prec_{\log} B$ if $S(A) \prec_{w-\log} S(B)$ or $S(A) \prec_{\log} S(B)$.

We shall introduce a majorization among operators acting on an infinite dimensional Hilbert space in a similar to majorization for matrices [1]. In our approach the existence of a normal faithful trace is essential. The second main result of this paper is the log-majorisation version of the Young's inequality as follows:

$$a^{\lambda_1} b^{\lambda_2} \prec_{log} \lambda_1 a + \lambda_2 b\,, \tag{1.2}$$

where a and b are positive elements in semi-finite von Neumann algebra M with faithful trace, λ_1 and λ_2 are positive real numbers that $\lambda_1 + \lambda_2 = 1$.

2. Singular values and determinanat inequality

In 1912, Schmidt (working with Hilbert) initiated the study of the singular values of a compact operator acting on a separable Hilbert space. The notion of singular value in the context of semifinite von Neumann algebras underwent formal study in 1982, beginning with a paper of Fack [4]. However, the ideas and many of the fundamental properties of singular values were already sketched in the seminal paper of Murray and von Neumann [14] in 1936.

This section is a brief exposition of those parts of the theory of singular values that are used in the study of operator inequalities herein. The results are mainly taken from the papers of Fack [4] and Petz [15].

Throughout this paper, M shall denote a semifinite von Neumann algebra and $\mathrm{tr}\,(\cdot)$ will denote a semifinite, faithful, normal trace on M.

Definition 2.1. The singular value function $\mu(x) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ of $x \in M$ is defined by the equation

$$\mu_t(x) \;=\; \inf\,\{\|x\,e\| \,:\, e \in \mathcal{P}(M),\ \mathrm{tr}\,(1-e) \leq t\}\ . \tag{2.1}$$

Elements in the range of the function $\mu(x)$ are called the `singular values` of x.

Note that $\mu(x)$ is a decreasing function of t and that $\mu_0(x) = \|x\|$ and $\mu_t(x) = 0$ for all $t \geq \mathrm{tr}\,(1)$. Further properties of the singular value function are described in Theorem 2.5.

Another useful expression for the singular values is given by the following proposition. Recall that $p^{|x|}$ denotes the spectral resolution of the identity of $|x|$.

Theorem 2.2. If $x \in M$, then for any $t \in \mathbb{R}_0^+$,

$$\mu_t(x) = \min\left\{s \in \mathbb{R}_0^+ \,:\, \mathrm{tr}\,\left(p^{|x|}(s\,,\,\infty)\right) \leq t\right\}\,. \tag{2.2}$$

Example 2.3. Let $\mathrm{tr}\,(\cdot)$ be the canonical trace on the semifinite factor$\mathcal{B}(\mathfrak{H})$: namely, if $\{\phi_k\}_{k\in\mathbb{Z}^+}$ is a fixed orthonormal basis of $\mathfrak{H}$, then

$$\mathrm{tr}\,(x) \;=\; \sum_{k=1}^{\infty}\langle x\,\phi_k\,,\,\phi_k\rangle\,, \quad x \in \mathcal{B}(\mathfrak{H})\,.$$

Then for any projection $p \in \mathcal{B}(\mathfrak{H})$,

$$\mathrm{tr}\,(p) \;=\; \sum_{k=1}^{\infty}\langle p\,\phi_k\,,\,\phi_k\rangle \;=\; \dim p(\mathfrak{H})\,,$$

which is a nonnegative integer or ∞. Thus, by Proposition 2.2, the values of $\mu_t(x)$ change only at integer values of t. Specifically,

$$\mu_t(x) \;=\; \mu_n(x) \quad \text{if} \quad n \leq t \;<\; n+1\,.$$

In particular, if x is a compact operator, then

$$\mu_t(x) \;=\; \lambda_n(|x|) \quad \text{if} \;\; t \in [n-1, n)\,,$$

where $\lambda_1(|x|) \geq \lambda_{2|}(|x|) \geq \ldots$ are the usual singular values of x (the eigenvalues of $|x|$) in descending order with multiplicities counted.

Example 2.4. A min-max–type characterisation of singular values. If $a \in M$ is positive, then

$$\mu_t(a) \;=\; \inf_{\substack{e\in\mathcal{P}(M)\\ \mathrm{tr}\,(1-e)\leq t}} \left[\; \sup_{\substack{\xi\in e(\mathfrak{H})\\ \|\xi\|\,=\,1}} \langle a\,\xi\,,\,\xi\rangle \;\right]\,. \tag{2.3}$$

The (relatively few) properties of singular values that some of them are used to establish the operator inequalities in this paper are collected in the following theorem.

Theorem 2.5. Assume that $x, y \in M$, $a, b \in M^+$, $\alpha \in \mathbb{C}$, and $r, s, t \in \mathbb{R}_0^+$.

1. The function $\mu(x) : \mathbb{R}_0^+ \to \mathbb{R}$ is non-increasing and continuous from the right.
2. $\mu_t(x) = \mu_t(|x|) = \mu_t(x^*)$ and $\mu_t(\alpha\,x) = |\alpha|\,\mu_t(x)$.
3. If $a \leq b$, then $\mu_t(a) \;\leq\; \mu_t(b)$.
4. $\mu_t(a^r) = (\mu_t(a))^r$.
5. $\mu_{t+s}(x+y) \;\leq\; \mu_t(x) \;+\; \mu_s(y)$.
6. $\mu_{t+s}(xy) \;\leq\; \mu_t(x).\mu_s(y)$.
7. $|\mu_t(x) \;-\; \mu_t(y)| \;\leq\; \|x \;-\; y\|$.

8. $\mu_t(xy) \le \|x\|\, \mu_t(y)$.
9. $\mu_t(yxy^*) \le \|y\|^2\, \mu_t(x)$.
10. $\mu_t(f(a)) = f(\mu_t(a))$ for any increasing continuous function $f : [0, \|a\|] \longrightarrow \mathbb{R}^+$ such that $f(0) = 0$.
11. $\int_0^t f(\mu_s(xy))ds \le \int_0^t f(\mu_s(x)\mu_s(y))ds$ for any increasing function f on $\mathbb{R}^+$ such that $f(e^t)$ is convex.
12. $\int_0^t f(\mu_s(x+y))ds \le \int_0^t f(\mu_s(x) + \mu_s(y))ds$ for any convex continuous increasing function f on $\mathbb{R}^+$.
13. $\int_0^t g(\mu_s(x+y))ds \le \int_0^t g(\mu_s(x))ds + \int_0^t g(\mu_s(y))ds$ for any increasing concave function g on $\mathbb{R}^+$ which is operator concave and $g(0) = 0$.

Remark 2.6. Part (13) was recently strengthened by P. G. Dodds and F. A. Sokochev [3].

The second type of useful result present in the following theorem, which shows that the trace of a positive operator can be recovered from the operator's singular values.

Theorem 2.7. If $a \in M^+$, then

$$\mathrm{tr}\,(a) = \int_0^\infty \mu_t(a)dt\,.$$

The following theorems are needed in the proof of first main result [7].

Theorem 2.8. Assume that M is a semifinite von Neumann algebra and that $\mathrm{tr}\,(\cdot)$ is a semifinite, faithful, normal trace on M. If $x, y \in M$ and $t \in \mathbb{R}_0^+$, then for positive real numbers p and q that $\frac{1}{p} + \frac{1}{q} = 1$;

$$\mu_t(|xy^*|) \le \mu_t(p^{-1}|x|^p + q^{-1}|y|^q)\,. \tag{2.4}$$

Inequality(2.4) is known as Young's inequality in singular values.

Theorem 2.9. Assume that M is a semifinite von Neumann algebra and that $\mathrm{tr}\,(\cdot)$ is a semifinite, faithful, normal trace on M. If $x, y \in M$ are such that $\mathrm{tr}\,(|x|) < \infty$ and $\mathrm{tr}\,(|y|) < \infty$, then

$$\mu_t(|xy^*|) = \mu_t(p^{-1}|x|^p + q^{-1}|y|^q)\,, \quad \text{for all } \ t \in \mathbb{R}_0^+\,,$$

if and only if $|y|^q = |x|^p$.

Definition 2.10. If $a \in M^+$, then $\Lambda_a : (0, \mathrm{tr}(1)) \to \mathbb{R}^+$ denotes the determinant-like function

$$\Lambda_s(a) = \exp\left(\int_0^s \log \mu_t(a)\, dt\right)\,.$$

$\Lambda_{\mathrm{tr}\,(1)}(a)$ is called determinant of a related to $\mathrm{tr}\,(\cdot)$ on M and will denoted by $\det(a)$, that is

$$det(a) = \Lambda_{\mathrm{tr}\,(1)}(a) = \exp\left(\int_0^{\mathrm{tr}\,(1)} \log \mu_t(a)\, dt\right)\,.$$

Example 2.11. Let M be the semifinite factor $\mathcal{B}(\mathfrak{H})$ and $c \in M$ be a compact operator. By example 2.3, we have

$$\Lambda_{n+1}(c) = \mu_0(c)\mu_1(c)\cdots\mu_n(c)\,.$$

Remark 2.12. This determinant is called Fugede-Kadison determinant and denoted by $\Delta_t(a)$ [5, 8].

Basic properties of $\Lambda_a(\cdot)$ are collected in the following theorem. For the proof see [4] and [6].

Theorem 2.13. Assume that $x, y \in M$, $a, b \in M^+$, $\alpha \in \mathbb{C}$, and $r, t \in \mathbb{R}_0^+$.

1. $\Lambda_t(x) = \Lambda_t(x^*) = \Lambda_t(|x|)$.
2. $\Lambda_t(\alpha x) = |\alpha|^t \Lambda_t(x)$.
3. $\Lambda_t(a^r) = \Lambda_t(a)^r$.
4. $\Lambda_t(|ab|) \leq \Lambda_t(a)\Lambda_t(b)$.
5. If $\mathrm{tr}\,(1) < \infty$, then $\Lambda_{\mathrm{tr}\,(1)}(|ab|) = \Lambda_{\mathrm{tr}\,(1)}(a)\Lambda_{\mathrm{tr}\,(1)}(b)$ that is $\det(|ab|) = \det a \cdot \det b$.
6. $(\Lambda_t(|ab|))^{\frac{1}{t}} \leq (\Lambda_t(a))^{\frac{1}{t}}(\Lambda_t(b))^{\frac{1}{t}}$.
7. $\Lambda_t((1 + |x + y|)) \leq \Lambda_t(1 + x)\Lambda_t(1 + y)$.

Remark 2.14. The result in part (5) does not hold If $\mathrm{tr}\,(1) = \infty$. Consider the case $M = \mathcal{B}(\mathfrak{H})$. When p is an orthogonal projection on $\mathfrak{H}$ such that $\dim p\mathfrak{H} = \dim(1-p)\mathfrak{H} = \infty$, we have $p(1-p) = 0$ so that $\det(|p(1-p)|) = 0$ while $\det p = \det(1-p) = 1$. What we have for the Fulglede-Kadison determinant when $\mathrm{tr}\,(1) = \infty$ is only an inequality as given in [5], Theorem 1.6 (also [6], Theorem 4.2).

Now we are in the position to present our first main result.

Theorem 2.15. *Let $\lambda_1, \ldots, \lambda_n$ be positive real numbers such that $\sum_{k=1}^n \lambda_k = 1$. For any positive operators $a_1, a_2, \ldots, a_n$ in semifinite von Neumann algebra M with faithful normal trace that $tr\,(1) < \infty$, we have*

$$rcl \prod_{k=1}^{n} (\det a_k)^{\lambda_k} = \det(\prod_{k=1}^{n} a_k^{\lambda_k}) \leq \det(\sum_{k=1}^{n} \lambda_k a_k). \tag{2.5}$$

If $tr\,(a_i) < \infty$ for every $1 \leq i \leq n$ and $\prod_{k=1}^n (\det a_k))^{\lambda_k} \neq 0$, then equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Proof . We use induction. First let $n = 2$. Then by Theorem 2.8, we have

$$\mu_t(|a_1^{\lambda_1} a_2^{\lambda_2}|) \leq \mu_t(\lambda_1 a_1 + \lambda_2 a_2) .$$

Thus,

$$\log \mu_t(|a_1^{\lambda_1} a_2^{\lambda_2}|) \leq \log \mu_t(\lambda_1 a_1 + \lambda_2 a_2) , \qquad \forall t > 0 ,$$

which implies

$$\begin{aligned} det(|a_1^{\lambda_1} a_2^{\lambda_2}|) &= \exp\left(\int_0^{\mathrm{tr}\,(1)} \log \mu_t(|a_1^{\lambda_1} a_2^{\lambda_2}|)\, dt\right) \leq \exp\left(\int_0^{\mathrm{tr}\,(1)} \log \mu_t(\lambda_1 a_1 + \lambda_2 a_2)\, dt\right) \\ &= \det(\lambda_1 a_1 + \lambda_2 a_2). \end{aligned}$$

By Theorem 2.13 part (5) we get

$$\det(a_1^{\lambda_1}) \det(a_2^{\lambda_2}) = \det(|a_1^{\lambda_1} a_2^{\lambda_2}|) \leq \det(\lambda_1 a_1 + \lambda_2 a_2).$$

If equality holds, then we have equality in related singular values by the definition of determinanat and Theorem 2.5 part(1). Therefore $a_1 = a_2$, by Theorem 2.9.

Now, suppose that the last inequality holds for $a_1, a_2, \ldots, a_{n-1}$. Then

$$\begin{aligned}
\det(\sum_{i=1}^{n} \lambda_i a_i) &= \det((\sum_{i=1}^{n-1} \lambda_i a_i) \ + \ \lambda_n a_n) \\
&= \det((\sum_{i=1}^{n-1} \lambda_i)(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} a_i) \ + \ \lambda_n a_n) \\
&\geq \det\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} a_i\right)^{(\sum_{i=1}^{n-1} \lambda_i)} \cdot \det(a_n^{\lambda_n}) \\
&\geq (\prod_{i=1}^{n-1} (\det a_i)^{\lambda_i}) \cdot \det(a_n^{\lambda_n}) \\
&= \prod_{i=1}^{n} (\det a_i)^{\lambda_i}.
\end{aligned}$$

If equality holds, then by above inequalities

$$\begin{aligned}
\det((\textstyle\sum_{i=1}^{n-1} \lambda_i)(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} a_i) \ + \ \lambda_n a_n) &= \det\left(\textstyle\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} a_i\right)^{(\sum_{i=1}^{n-1} \lambda_i)} \cdot \det(a_n^{\lambda_n}) \\
&= \textstyle\prod_{i=1}^{n} (\det a_i)^{\lambda_i}.
\end{aligned}$$

Thus, by the case $n = 2$, we get

$$\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} a_i = a_n \, . \tag{2.6}$$

On the other hand, by deleting $\det(a_n^{\lambda_n})$ from both sides of the inequality we get

$$\det\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} a_i\right)^{(\sum_{i=1}^{n-1} \lambda_i)} = \prod_{i=1}^{n-1} (\det a_i)^{\lambda_i},$$

which implies

$$\det\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} a_i\right) = \prod_{i=1}^{n-1} (\det a_i)^{\frac{\lambda_i}{(\sum_{i=1}^{n-1} \lambda_i)}}.$$

By induction assumption, $a_1 = a_2 = \cdots = a_{n-1}$. From substituting in equation (2.6), it follows that $a_1 = a_2 = \cdots = a_{n-1} = a_n$. It is clear that if $a_1 = a_2 = \cdots = a_{n-1} = a_n$, then

$$\det(\sum_{k=1}^{n} \lambda_k a_k) \geq \prod_{k=1}^{n} (\det a_k)^{\lambda_k}$$

and the proof is completed. $\square$

Remark 2.16. Inequality (2.5) is not true when $\operatorname{tr}(1) = \infty$. Consider $M = \mathcal{B}(\mathfrak{H})$, then there are two projections p and q on $\mathfrak{H}$ such that $\dim p\mathfrak{H} = \dim q\mathfrak{H} = \infty$ and $\frac{p+q}{2} < 1$. We then have $(\det p)^{\frac{1}{2}}(\det q)^{\frac{1}{2}} = 1$ but $\det(\frac{p+q}{2}) = 0$.

Remark 2.17. There is another proof for the inequality (2.5) by the concavity of the Fuglede-Kadison determinant [11]. Indeed, we may assume that $\mathrm{tr}(1) = 1$. Then Proposition 13 in [11] says that for every concave function $f : [0, \infty) \to [0, \infty)$, the function that $a \to \det(f(a))$ for every $a \in M^+$, is concave. In particular, $a \to \det(a)$ is concave. This implies that

$$\det(\sum_{k=1}^{n} \lambda_k a_k) \geq \sum_{k=1}^{n} \lambda_k \det(a_k) \geq \prod_{k=1}^{n} (\det a_k)^{\lambda_k},$$

where the latter inequality is the numerical arithmetic-geometric inequality.

Remark 2.18. Young's inequality in singular value was the key point in the proof of last theorem. By using the Young's inequality in numbers, we have the following version of determinant inequality:

$$\det \prod_{k=1}^{n} (a_k)^{\lambda_k} = \prod_{k=1}^{n} (\det a_k)^{\lambda_k} \leq \sum_{k=1}^{n} \lambda_k \det a_k. \tag{2.7}$$

Equality holds if and only if $\det a_1 = \det a_2 = \cdots = \det a_n$.

We write an extension of Theorem 2.13 part (7) in the following corollary.

Corollary 2.19. *If $\lambda_1, \ldots, \lambda_n$ are positive real numbers such that $\sum_{k=1}^{n} \lambda_k = 1$ and $a_1, a_2, \ldots, a_n$ are positive operators in semifinite von Neumann algebra M with faithful normal trace, then*

$$\Lambda_t(\sum_{k=1}^{n} \lambda_k(1 + a_k)) \leq \prod_{k=1}^{n} \Lambda_t(1 + \lambda_k a_k), \qquad \forall\, t \geq 0.$$

Proof . By induction, we get from Theorem 2.13 part (7) that

$$\Lambda_t(1 + \sum_{k=1}^{n} a_k) \leq \prod_{k=1}^{n} \Lambda_t(1 + a_k), \qquad \forall\, t \geq 0.$$

We have

$$\Lambda_t(\textstyle\sum_{k=1}^{n} \lambda_k(1 + a_k)) = \Lambda_t(1 + \sum_{k=1}^{n} \lambda_k a_k) \leq \prod_{k=1}^{n} \Lambda_t(1 + \lambda_k a_k), \qquad \forall\, t \geq 0.$$

□

Corollary 2.20. *Assume that $\lambda_1, \ldots, \lambda_n$ are positive real numbers such that $\sum_{k=1}^{n} \lambda_k = 1$ and $a_1, a_2, \ldots, a_n$ are positive operators in semifinite von Neumann algebra M with faithful normal trace. Then*

$$\det(\textstyle\prod_{k=1}^{n} (1 + a_k)^{\lambda_k}) = \prod_{k=1}^{n} \det(1 + a_k)^{\lambda_k} \leq \prod_{k=1}^{n} \det(1 + \lambda_k a_k) = \det(\prod_{k=1}^{n} (1 + \lambda_k a_k)).$$

If $tr(1) < \infty$ and $\prod_{k=1}^{n} (\det a_k)^{\lambda_k} \neq 0$, then equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Proof . It is direct result of Crollary 2.19 and Theorem 2.13. □

3. Majorization in Semifinite von Neumann Algebras

Definition 3.1. If $x, y \in M^+$, then we write $x \prec_w y$ to denoted that x is **weakly majorised** by y if

$$\int_0^s \mu_t(x)\,dt \;\leq\; \int_0^s \mu_t(y)\,dt\,, \qquad \text{for all } s \in \mathbb{R}_0^+\,.$$

If $x \prec_w y$ and

$$\mathrm{tr}\,(x) \;=\; \int_0^\infty \mu_t(x)\,dt \;=\; \int_0^\infty \mu_t(y)\,dt \;=\; \mathrm{tr}\,(y)\,,$$

then we say that x is **majorised** by y in symbol $x \prec y$.

To clarify the meaning of majorisation and weakly majorisation, we present their characterisation in the next Theorems. For the proof one can see [9] and [12].

Theorem 3.2. For every $a\,, b \in M^+$, the following conditions are equivalent:

1. $a \prec_w b$.
2. $\mathrm{tr}\,((a-r)^+) \leq \mathrm{tr}\,((b-r)^+)$ for all $r > 0$ where x^+ is the positive part of x.
3. $\mathrm{tr}\,(f(a)) \leq \mathrm{tr}\,(f(b))$ for all non-decreasing covex continuous function f on $\mathbb{R}_0^+$ with $f(0) \geq 0$.
4. $f(a) \prec_w f(b)$ for all f in part (3).

Theorem 3.3. If $\mathrm{tr}\,(I) < \infty$, then for every $x\,, y \in M^{sa}$, the following conditions are equivalent:

1. $x \prec y$.
2. $\mathrm{tr}\,(x) = \mathrm{tr}\,(y)$ and $\mathrm{tr}\,((x-r)^+) \leq \mathrm{tr}\,((y-r)^+)$ for all $r \in \mathbb{R}$.
3. $\mathrm{tr}\,(|x-r|) \leq \mathrm{tr}\,(|y-r|)$ for all $r \in \mathbb{R}$.
4. $\mathrm{tr}\,(f(x)) \leq \mathrm{tr}\,(f(y))$ for all convex function f on $\mathbb{R}$.
5. $f(x) \prec_w f(y)$ for all f in part (3).

For the function $f(t) = t$ in part (11) Theorem 2.5 we get

$$\int_0^t \mu_s(xy)\,ds \;\leq\; \int_0^t \mu_s(x)\mu_s(y)\,ds.$$

By the Young's inequality for real numbers, we have

$$\int_0^t \mu_s(xy)\,ds \;\leq\; \int_0^t \mu_s(x)\mu_s(y)ds \leq \frac{1}{p}\int_0^t \mu_s(x)^p\,ds \;+\; \frac{1}{q}\int_0^t \mu_s(y)^q\,ds\,.$$

The following corollary is direct result of this fact with Theorems 2.8 and 2.9.

Corollary 3.4. The following statements are equivalent for $a, b \in M^+$ satisfy $\mathrm{tr}\,(a) < \infty$ and $\mathrm{tr}\,(b) < \infty$:

1. $\mathrm{tr}\,(|ab|) = [\mathrm{tr}\,(a^p)]^{\frac{1}{p}}\,[\mathrm{tr}\,(b^q)]^{\frac{1}{q}}$;
2. $\mathrm{tr}\,(|ab|) = \frac{1}{p}\mathrm{tr}\,(a^p) + \frac{1}{q}\mathrm{tr}\,(b^q)$;
3. $|ab| \prec \frac{1}{p}\,a^p + \frac{1}{q}\,b^q$;
4. $b^q = a^p$.

Part (3) is known as a majorisation version of Young's inequality. Theorem 2.8 has important rule in this result. Since still we do not have such a result for more than two positive operators in M, we do not know how to prove same majorisation version of Young's inequality for more than two elements even in $M_n(\mathbb{C})$.

Definition 3.5. If $x, y \in M$, then we write $x \prec_{w-\log} y$ to denoted that x is **weakly log-majorised** by y if

$$\Lambda_t(x) \leq \Lambda_t(y) \qquad \text{for every} \in \mathbb{R}^+ .$$

If $x \prec_{w-\log} y$ and $\det(x) = \det(y)$, then we say that x is **log- majorised** by y in symbol $x \prec_{\log} y$.

By using Theorem 2.8, we have the log-majorisation version of Young's inequality in the following corollary for two elements in M.

Corollary 3.6. Let λ_1, λ_2 be two positive real numbers that $\lambda_1 + \lambda_2 = 1$ and a, b be two positive elements in M. Then

$$|a^{\lambda_1} b^{\lambda_2}| \prec_{w-\log} \lambda_1 a_1 + \lambda_2 a_2 . \tag{3.1}$$

If $\det(\lambda_1 a_1 + \lambda_2 a_2) \neq 0$, then $|a^{\lambda_1} b^{\lambda_2}| \prec_{\log} \lambda_1 a_1 + \lambda_2 a_2$ if and only if $a = b$.

Theorem 3.7. *Let $\lambda_1, \ldots, \lambda_n$ be positive real numbers such that $\sum_{k=1}^n \lambda_k = 1$ and $a_1, a_2, \ldots, a_n$ be positive operators in M. If $tr(1) < \infty$, then*

$$|\prod_{k=1}^{n} a_k^{\lambda_k}| \prec_{w-\log} \sum_{k=1}^{n} \lambda_k a_k. \tag{3.2}$$

If $tr(a_i) < \infty$ for every $1 \leq i \leq n$ and $\prod_{k=1}^n (\det a_k)^{\lambda_k} \neq 0$, then

$$|\prod_{k=1}^{n} a_k^{\lambda_k}| \prec_{\log} \sum_{k=1}^{n} \lambda_k a_k, \tag{3.3}$$

if and only if $a_1 = a_2 = \cdots = a_n$.

Proof . By similar argument in Theorem 2.15 and using induction we get

$$\begin{aligned}
\Lambda_t(\textstyle\sum_{k=1}^n \lambda_k a_k) &= \Lambda_t((\textstyle\sum_{k=1}^{n-1} \lambda_k a_k) + \lambda_n a_n) \\
&\geq \Lambda_t((\textstyle\sum_{i=1}^{n-1} \lambda_i)(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} a_i) + \lambda_n a_n) \\
&\geq \Lambda_t\left(\textstyle\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i} a_i\right)^{(\sum_{i=1}^{n-1} \lambda_i)} \cdot \Lambda_t(a_n^{\lambda_n}) \\
&\geq \Lambda_t\left(\textstyle\prod_{i=1}^{n-1} a_i^{\frac{\lambda_i}{\sum_{i=1}^{n-1} \lambda_i}}\right)^{(\sum_{i=1}^{n-1} \lambda_i)} \cdot \Lambda_t(a_n^{\lambda_n}) \\
&= \Lambda_t\left(\textstyle\prod_{i=1}^{n-1} a_i^{\lambda_i}\right) \cdot \Lambda_t(a_n^{\lambda_n}) \\
&\geq \Lambda_t\left(\textstyle\prod_{i=1}^{n} a_i^{\lambda_i}\right) . \qquad (\text{ by Theorem 2.13 part (4) })
\end{aligned}$$

This proves that

$$|\prod_{k=1}^{n} a_k^{\lambda_k}| \prec_{w-\log} \sum_{k=1}^{n} \lambda_k a_k \,.$$

By Theorem 2.15,

$$\det(|\prod_{k=1}^{n} a_k^{\lambda_k}|) = \det(\sum_{k=1}^{n} \lambda_k a_k\,)$$

if and only if $a_1 = a_2 = \cdots = a_n$, so that the theorem is proved. $\square$

Corollary 3.8. *Let $\lambda_1, \ldots, \lambda_n$ be positive real numbers such that $\sum_{k=1}^{n} \lambda_k = 1$ and $a_1, a_2, \ldots, a_n$ be positive operators in M. If $tr(1) < \infty$, $tr(a_i) < \infty$ for every $1 \leq i \leq n$ and $\prod_{k=1}^{n} (\det 1 + a_k)^{\lambda_k} \neq 0$, then*

$$|\prod_{k=1}^{n} (1 + a_k)^{\lambda_k}| \prec_{\log} (1 + \sum_{k=1}^{n} \lambda_k a_k), \tag{3.4}$$

if and only if $a_1 = a_2 = \cdots = a_n$.

Proof . By Theorem 3.7,

$$|\prod_{k=1}^{n} (1 + a_k)^{\lambda_k}| \prec_{\log} \sum_{k=1}^{n} \lambda_k (1 + a_k) = 1 + \sum_{k=1}^{n} \lambda_k a_k.$$

$\square$

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