Approximately generalized additive functions in several variables via fixed point method

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Abstract

In this paper, we obtain the general solution and the generalized Hyers–Ulam–Rassias stability in random normed spaces, in non-Archimedean spaces and also in p-Banach spaces and finally the stability via fixed point method for a functional equation

$$D_f(x_1, .., x_m) := \sum_{k=2}^{m} (\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} ... \sum_{i_{m-k+1}=i_{m-k}+1}^{m}) f(\sum_{i=1, i \neq i_1,...,i_{m-k+1}}^{m} x_i - \sum_{r=1}^{m-k+1} x_{i_r})$$
$$+ f(\sum_{i=1}^{m} x_i) - 2^{m-1} f(x_1) = 0$$

where $m \geq 2$ is an integer number.

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1. Introduction and preliminaries

A basic question in the theory of functional equations is as follows: "when is it true that a function, which approximately satisfies a functional equation must be close to an exact solution of the equation?"

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If the problem accepts a solution, we say the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [60] in 1940 and affirmatively solved by Hyers [21]. The result of Hyers was generalized by Aoki [1] for approximate additive functions and by Rassias [43] for approximate linear functions by allowing the difference Cauchy equation $\|f(x+y) - f(x) - f(y)\|$ to be controlled by $\varepsilon(\| x \|^p + \| y \|^p)$. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called the Hyers–Ulam–Rassias stability. In 1994, a generalization of Rassias? theorem was obtained by Găvruta [17], who replaced $\varepsilon(\| x \|^p + \| y \|^p)$ by a general control function $\varphi(x, y)$ (see also [11]–[16], [22, 24, 25, 40, 41, 42] and [45]–[52]).

Baker [6] was the first author who applied the fixed point method in the study of Hyers–Ulam stability (see also [3]). A systematic study of fixed point theorems in nonlinear analysis is due to Isac and Rassias; cf. [23, 24]. Recently, Cădariu and Radu [10] applied the fixed point method to the investigation of the Cauchy additive functional equation [9, 42]. Using such a clever idea, they could present a short, simple proof for the Hyers-Ulam stability of Cauchy and Jensen functional equations (see also [14, 26, 35]).

We now introduce one of fundamental results of fixed point theory. For the proof, refer to [28, 55]. For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers, Isac and Rassias [22].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies:

(GM_1) $d(x, y) = 0$ if and only if $x = y$;

(GM_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(GM_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that

$$d(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator.

We recall the following theorem by Margolis and Diaz.

Theorem 1.1. [9, 28] Let (S, d) be a complete generalized metric space and let $J : S \to S$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in S$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $\Omega = \{y \in S \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Omega$.

By using the idea of Cădariu and Radu, we will prove the stability of the general n-dimensional additive functional equation (1.1).

It was shown by Rassias [44] that the norm defined over a real vector space X is induced by an inner product if and only if for a fixed integer $n \geq 2$,

$$n \left\| \frac{1}{n} \sum_{i=1}^{n} x_i \right\|^2 + \sum_{i=1}^{n} \left\| x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

for all $x_1, ..., x_n \in X$. In this paper, we consider the m-dimensional additive functional equation

$$\sum_{k=2}^{m} (\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} ... \sum_{i_{m-k+1}=i_{m-k}+1}^{m}) f(\sum_{i=1, i \neq i_1, .., i_{m-k+1}}^{m} x_i - \sum_{r=1}^{m-k+1} x_{i_r}) + f(\sum_{i=1}^{m} x_i) = 2^{m-1} f(x_1) \quad (1.1)$$

where $m \geq 2$ is an integer number. It is easy to see that the function $f(x) = ax$ is a solution of the functional equation (1.1).

As a special case, if $m = 2$ in (1.1), then the functional equation (1.1) reduces to

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1)$$

also by putting $m = 3$ in (1.1), we obtain

$$\sum_{i_1=2}^{2} \sum_{i_2=i_1+1}^{3} f(\sum_{i=1, i \neq i_1, i_2}^{3} x_i - \sum_{r=1}^{2} x_{i_r}) + \sum_{i_1=2}^{3} f(\sum_{i=1, i \neq i_1}^{3} x_i - x_{i_1}) + f(\sum_{i=1}^{3} x_i) = 2^2 f(x_1)$$

that is,

$$f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) = 4f(x_1).$$

The main purpose of this paper is to prove the stability for equation (1.1), in random normed spaces via fixed point method.

2. Approximately additive functions in random normed spaces via fixed point method

The aim of this section is to investigate the stability of the given general m-dimensional additive functional equation (1.1), in random normed spaces.

In the sequel we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [8, 29, 30, 57, 58]. Throughout this paper, let Δ^+ is the space of distribution functions that is,

$$\Delta^+ := \{F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1] | \ F \ is \ left \ continuous, nondecreasing \ on \ \mathbb{R}$$
$$, F(0) = 0 \ and \ F(+\infty) = 1\}$$

and the subset $D^+ \subseteq \Delta^+$ is the set,

$$D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$$

where, $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.1. ([57]) A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous triangular norm (briefly, a t–norm) if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t–norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Łukasiewicz t-norm).

Recall (see [18], [19]) that if T is a t–norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^n x_i$ is defined recurrently by

$$T_{i=1}^n x_i = \begin{cases} x_1, & \text{if } n = 1, \\ T(T_{i=1}^{n-1} x_i, x_n), & \text{if } n \geq 2. \end{cases}$$

$T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$.

It is known ([19]) that for the Łukasiewicz t-norm the following implication holds:

$$\lim_{n \to \infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \iff \sum_{n=1}^\infty (1 - x_n) < \infty. \tag{2.1}$$

Definition 2.2. ([58]) A *Random Normed space* (briefly, RN-space) is a triple (X, Λ, T), where X is a vector space, T is a continuous t–norm, and Λ is a mapping from X into D^+ such that, the following conditions hold:

(RN1) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\Lambda_{\alpha x}(t) = \Lambda_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;

(RN3) $\Lambda_{x+y}(t+s) \geq T(\Lambda_x(t), \Lambda_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 2.3. Let (X, Λ, T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\Lambda_{x_n-x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\Lambda_{x_n-x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) A RN-space (X, Λ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X. A complete RN-space is said to be random Banach space.

Theorem 2.4. *([57])* If (X, Λ, T) is a RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n \to \infty} \Lambda_{x_n}(t) = \Lambda_x(t)$ almost everywhere.

The theory of random normed spaces (RN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces, RN-spaces and fuzzy normed spaces has been recently studied in, Alsina [2], Mirmostafaee, Mirzavaziri and Moslehian [35]-[38], Miheţ and Radu [29]-[32], Miheţ, Saadati and Vaezpour [33, 34], Baktash et. al [5] and Saadati et. al. [56].

We start our work with a general solution for equation (1.1).

Lemma 2.5. [15] Let X and Y be real vector spaces. A function $f : X \to Y$ with $f(0) = 0$ satisfies (1.1) if and only if $f : X \to Y$ is additive.

From now on, let X be a linear space and (Y, Λ, T_M) be a complete RN-space. For convenience, we use the following abbreviation for a given function $f : X \to Y$:

$$D_f(x_1, .., x_m) = \sum_{k=2}^{m}(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} ... \sum_{i_{m-k+1}=i_{m-k}+1}^{m}) f(\sum_{i=1, i \neq i_1,...,i_{m-k+1}}^{m} x_i - \sum_{r=1}^{m-k+1} x_{i_r})$$
$$+ f(\sum_{i=1}^{m} x_i) - 2^{m-1} f(x_1)$$

for all $x_1, ..., x_m \in X$, where $m \geq 2$ is an integer number.

Theorem 2.6. *Let $\Phi : \underbrace{X \times X \times ... \times X}_{m-terms} \to D^+$ be a function ($\Phi(x_1, ..., x_m)$ is denoted by $\Phi_{x_1,...,x_m}$) such that, for some $0 < \alpha < 2$,*

$$\Phi_{2x_1,...,2x_m}(\alpha t) \geq \Phi_{x_1,...,x_m}(t) \tag{2.2}$$

for all $x_1, ..., x_m \in X$ and all $t > 0$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\Lambda_{D_f(x_1,...,x_m)}(t) \geq \Phi_{x_1,...,x_m}(t) \tag{2.3}$$

for all $x_1, ..., x_m \in X$ and all $t > 0$. Then there exists a unique additive function $A : X \to Y$ such that

$$\Lambda_{A(x)-f(x)}(t) \geq T_{\ell=1}^{\infty}(\Phi_{2^{\ell-1}x,2^{\ell-1}x,0,...,0}(2^{m-1}t)) \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof . Letting $x_i = 0 \quad (i = 3, ..., m)$ in (2.3), we get

$$\Lambda_{(1+\sum_{\ell=1}^{m-2}\binom{m-2}{\ell})(f(x_1+x_2)+f(x_1-x_2))-2^{m-1}f(x_1)}(t) \geq \Phi_{x_1,x_2,0,...,0}(t) \tag{2.5}$$

for all $x_1, x_2 \in X$ and all $t > 0$. Setting $x_1 = x_2 = x$ in (2.5). On the other hand, we have the relation

$$1 + \sum_{\ell=1}^{m-\jmath}\binom{m-\jmath}{\ell} = \sum_{\ell=0}^{m-\jmath}\binom{m-\jmath}{\ell} = 2^{m-\jmath} \tag{2.6}$$

for all $m > \jmath$. Hence we obtain from (2.6) and $f(0) = 0$ that

$$\Lambda_{2^{m-2}f(2x)-2^{m-1}f(x)}(t) \geq \Phi_{x,x,0,...,0}(t)$$

for all $x \in X$ and all $t > 0$, or

$$\Lambda_{\frac{f(2x)}{2}-f(x)}(t) \geq \Phi_{x,x,0,...,0}(2^{m-1}t) \tag{2.7}$$

for all $x \in X$ and all $t > 0$. Let S be the set of all functions $h : X \to Y$ with $h(0) = 0$ and introduce a generalized metric on S as follows:

$$d(h, k) = \inf \left\{ u \in \mathbb{R}^+ : \Lambda_{h(x)-k(x)}(ut) \geq \Phi_{x,x,0,...,0}(t), \quad \forall x \in X, \forall t > 0 \right\}$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d) is a generalized complete metric space [10, 29].

Now we consider the function $J : S \to S$ defined by

$$Jh(x) := \frac{h(2x)}{2}$$

for all $h \in S$ and $x \in X$.

Now let $g, f \in S$ such that $d(f, g) < \varepsilon$. Then

$$\Lambda_{Jg(x)-Jf(x)}(\frac{\alpha u}{2}t) = \Lambda_{g(2x)-f(2x)}(\alpha u t) \geq \Phi_{2x,2x,0,\dots,0}(\alpha t) \geq \Phi_{x,x,0,\dots,0}(t)$$

that is, if $d(f, g) < \varepsilon$ we have $d(Jf, Jg) < \frac{\alpha}{2}\varepsilon$. This means that

$$d(Jf, Jg) \leq \frac{\alpha}{2}d(f, g)$$

for all $f, g \in S$, that is, J is a strictly contractive self-function on S with the Lipschitz constant $\frac{\alpha}{2}$.

It follows from (2.7) that

$$\Lambda_{Jf(x)-f(x)}(\frac{t}{2^{m-1}}) \geq \Phi_{x,x,0,\dots,0}(t)$$

for all $x \in X$ and all $t > 0$, which implies that $d(Jf, f) \leq \frac{1}{2^{m-1}}$.

Due to Theorem 1.1, there exists a function $A : X \to Y$ such that A is a unique fixed point of J, i.e., $A(2x) = 2A(x)$ for all $x \in X$.

Also, $d(J^n g, A) \to 0$ as $n \to \infty$, implies the equality

$$\lim_{n \to \infty} \frac{f(2^n x)}{2^n} = A(x)$$

for all $x \in X$. If we replace $x_1, \dots, x_m$ with $2^n x_1, \dots, 2^n x_m$ in (2.3), respectively, and divide by 2^n, then it follows from (2.2) that

$$\Lambda_{\frac{D_f(2^n x_1,\dots,2^n x_n)}{2^n}}(t) \geq \Phi_{2^n x_1,\dots,2^n x_m}(2^n t) = \Phi_{2^n x_1,\dots,2^n x_m}(\alpha^n(\frac{2}{\alpha})^n t) \geq \Phi_{x_1,\dots,x_m}((\frac{2}{\alpha})^n t) \qquad (2.8)$$

for all $x_1, \dots, x_m \in X$ and all $t > 0$. By letting $n \to \infty$ in (2.8), we find that $\Lambda_{D_A(x_1,\dots,x_m)}(t) = 1$ for all $t > 0$, which implies $D_A(x_1, \dots, x_n) = 0$ thus A satisfies (1.1). Hence by Lemma 2.5, the function $A : X \to Y$ is additive.

It follows from (2.7) that

$$\Lambda_{\frac{f(2^{\ell+1}x)}{2^{\ell+1}} - \frac{f(2^\ell x)}{2^\ell}}(t) \geq \Phi_{2^\ell x, 2^\ell x, 0, \dots, 0}(2^{m+\ell-1}t) \qquad (2.9)$$

for all $x \in X$ and $t > 0$, which by using (RN_3) implies that

$$\Lambda_{\frac{f(2^2 x)}{2^2} - f(x)}(t) \geq T(\Lambda_{\frac{f(2^2 x)}{2^2} - \frac{f(2x)}{2}}(\frac{t}{2}), \Lambda_{\frac{f(2x)}{2} - f(x)}(\frac{t}{2}))$$
$$\geq T(\Phi_{2x,2x,0,\dots,0}(2^m t), \Phi_{x,x,0,\dots,0}(2^{m-1}t))$$
$$\geq T(\Phi_{2x,2x,0,\dots,0}(2^{m-1}t), \Phi_{x,x,0,\dots,0}(2^{m-1}t))$$

for all $x \in X$ and $t > 0$. Thus

$$\Lambda_{\frac{f(2^n x)}{2^n} - f(x)}(t) \geq T_{\ell=1}^n(\Phi_{2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}(2^{m-1}t)) \qquad (2.10)$$

for all $x \in X$ and $t > 0$. By taking n to approach infinity in (2.10), we obtain (2.4). This completes the proof. $\square$

3. Approximately additive functions in non-Archimedean spaces

In 1897, Hensel [20] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [27, 61, 53, 62].

A non-Archimedean field is a field K equipped with a function (valuation) $|\,.\,|$ from K into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in N$. An example of a non-Archimedean valuation is the function $|\,.\,|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial.

Definition 3.1. Let X be a vector space over a scalar field K with a non–Archimedean non-trivial valuation $|\,.\,|$. A function $\|\,.\,\| : X \to \mathbb{R}$ is a non–Archimedean norm (valuation) if it satisfies the following conditions:

(NA1) $\|x\| = 0$ if and only if $x = 0$;
(NA2) $\|rx\| = |r|\|x\|$ for all $r \in K$ and $x \in X$;
(NA3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$ (the strong triangle inequality).
Then $(X, \|\,.\,\|)$ is called a non–Archimedean space.

Remark 3.2. Thanks to the inequality

$$\|x_m - x_l\| \leq \max\{\|x_{\jmath+1} - x_\jmath\| : l \leq \jmath \leq m - 1\} \quad (m > l)$$

a sequence $\{x_m\}$ is Cauchy if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non–Archimedean space. By a complete non–Archimedean space we mean one in which every Cauchy sequence is convergent.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: "for $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$."

Example 3.3. Let p be a prime number. For any nonzero rational number $x = \frac{a}{b}p^{n_x}$ such that a and b are integers not divisible by p, define the p-adic absolute value $|x|_p := p^{-n_x}$. Then $|\,.\,|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\,.\,|$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field.

Note that if $p > 3$, then $|2^n| = 1$ in for each integer n.

Arriola and Beyer [4] investigated stability of approximate additive functions $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous function for which there exists a fixed ϵ :

$$|f(x + y) - f(x) - f(y)| \leq \epsilon$$

for all $x, y \in Q_p$, then there exists a unique additive function $T : \mathbb{Q}_p \to \mathbb{R}$ such that

$$|f(x) - T(x)| \leq \epsilon$$

for all $x \in \mathbb{Q}_p$. Additionally in 2007, Moslehian and Rassias [39] proved the generalized Hyers-Ulam stability of the Cauchy functional equation and the quadratic functional equation in non–Archimedean normed spaces.

Theorem 3.4. *Let G is an additive group, X is a complete non–Archimedean space and $\psi : G^n \to [0, \infty)$ be a function such that, for some $0 < \alpha < 2$*

$$\psi(2x_1, 2x_2, ..., 2x_n) \leq \alpha \psi(x_1, x_2, ..., x_n) \tag{3.1}$$

for all $x_1, ..., x_n \in G$, and

$$\tilde{\psi}(x) := \lim_{m \to \infty} \max\{\frac{1}{2^\ell}\psi(2^\ell x, 2^\ell x, 0, ..., 0) : \quad 0 \leq \ell < m\} \tag{3.2}$$

for each $x \in G$, exists. Suppose that a function $f : G \to X$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x_1, ..., x_n)\| \leq \psi(x_1, ..., x_n) \tag{3.3}$$

for all $x_1, ..., x_n \in G$. Then there exists a unique additive function $A : G \to X$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2^{n-1}}\tilde{\psi}(x) \tag{3.4}$$

for all $x \in G$.

Proof . Putting $x_1 = x_2 = x$ and $x_i = 0 \quad (i = 3, ..., n)$ in (3.3), we get

$$\|2^{n-2}f(2x) - 2^{n-1}f(x)\| \leq \psi(x, x, 0, ..., 0)$$

for all $x \in G$, or

$$\|\frac{f(2x)}{2} - f(x)\| \leq \frac{1}{2^{n-1}}\psi(x, x, 0, ..., 0) \tag{3.5}$$

for all $x \in G$. Let S be the set of all functions $h : G \to X$ with $h(0) = 0$ and introduce a generalized metric on S as follows:

$$d(h, k) = \inf\left\{u \in \mathbb{R}^+ : \|h(x) - k(x)\| \leq u\psi(x, x, 0, ..., 0), \quad \forall x \in G\right\}$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d) is a generalized complete metric space [10, 29].

Now we consider the function $J : S \to S$ defined by

$$Jh(x) := \frac{h(2x)}{2}$$

for all $h \in S$ and $x \in G$.

Now let $g, f \in S$ such that $d(f, g) < \varepsilon$. Then

$$\frac{2}{\alpha u}\|Jg(x) - Jf(x)\| = \frac{1}{\alpha u}\|g(2x) - f(2x)\| \leq \frac{1}{\alpha}\psi(2x, 2x, 0, ..., 0) \leq \psi(x, x, 0, ..., 0)$$

that is, if $d(f, g) < \varepsilon$ we have $d(Jf, Jg) < \frac{\alpha}{2}\varepsilon$. This means that

$$d(Jf, Jg) \leq \frac{\alpha}{2}d(f, g)$$

for all $f, g \in S$, that is, J is a strictly contractive self-function on S with the Lipschitz constant $\frac{\alpha}{2}$.

It follows from (3.5) that

$$\|Jf(x) - f(x)\| \leq \frac{1}{2^{n-1}}\psi(x, x, 0, ..., 0)$$

for all $x \in G$, which implies that $d(Jf, f) \leq \frac{1}{2^{n-1}}$.

Due to Theorem 1.1, there exists a function $A : G \to X$ such that A is a unique fixed point of J, i.e., $A(2x) = 2A(x)$ for all $x \in G$.

Also, $d(J^n g, A) \to 0$ as $n \to \infty$, implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x)$$

for all $x \in G$. It follows from (3.5) by using induction that

$$\left\| f(x) - \frac{1}{2^n}f(2^n x) \right\| \leq \frac{1}{2^{n-1}}\max\left\{\frac{1}{2^{\jmath}}\psi(2^{\jmath}x, 2^{\jmath}x, 0, ..., 0) : \quad 0 \leq \jmath < n\right\} \tag{3.6}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (3.6) and using (3.2), we obtain (3.4). By (3.1) and (3.3), we get

$$\|D_A(x_1, ..., x_n)\| = \lim_{m\to\infty} \frac{1}{2^m}\|D_f(2^m x_1, ..., 2^m x_n)\|$$

$$\leq \lim_{m\to\infty} \frac{1}{2^m}\psi(2^m x_1, ..., 2^m x_n)$$

$$\leq \lim_{m\to\infty} \left(\frac{\alpha}{2}\right)^m \psi(x_1, x_2, ..., x_n) = 0$$

for all $x_1, ..., x_n \in G$. Therefore the function $A : G \to X$ satisfies (1.1). By lemma 2.5, the function $A : G \to X$ is additive. $\square$

Corollary 3.5. *Let $\eta : [0, \infty) \to [0, \infty)$ be a function satisfying*

    (i) $\eta(2t) \leq \eta(2)\eta(t)$ for all $t \geq 0$;

    (ii) $\eta(2) < 2$.

Suppose that $\varepsilon > 0$ and G be a normed space and let $f : G \to X$ satisfying

$$\|D_f(x_1, ..., x_n)\| \leq \varepsilon \sum_{i=1}^{n} \eta(\|x_i\|)$$

for all $x_1, ..., x_n \in G$. Then there exists a unique additive function $A : G \to X$ such that

$$\|f(x) - A(x)\| \leq \frac{\varepsilon}{2^{n-2}} \eta(\|x\|)$$

for all $x \in G$.

Proof . Defining $\psi : G^n \to [0, \infty)$ by $\psi(x_1, ..., x_n) := \varepsilon \sum_{i=1}^{n} \eta(\|x_i\|)$, we have

$$\psi(2x_1, ..., 2x_n) \leq \varepsilon \sum_{i=1}^{n} \eta(2)\eta(\|x_i\|) \leq \eta(2)\psi(x_1, ..., x_n)$$

for all $x_1, ..., x_n \in G$. We have

$$\tilde{\psi}(x) := \lim_{m\to\infty} \max\left\{\frac{1}{2^{\ell}}\psi(2^{\ell}x, 2^{\ell}x, 0, ..., 0) : \quad 0 \leq \ell < m\right\} = \psi(x, x, 0, ..., 0)$$

for all $x \in G$. $\square$

Remark 3.6. The classical example of the function η is the function $\eta(t) = t^p$ for all $t \in [0, \infty)$, where $p < 1$.

4. Approximately additive functions in p–Banach spaces

We consider some basic concepts concerning p–normed spaces.

Definition 4.1. (See [7, 54]) Let X be a real linear space. A function $\| \, . \, \| : X \to \mathbb{R}$ is a quasi-norm (valuation) if it satisfies the following conditions:

(QN1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

(QN2) $\|\lambda . \, x\| = |\lambda| . \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;

(QN3) There is a constant $M \geq 1$: $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

Then $(X, \| \, . \, \|)$ is called a quasi-normed space.

The smallest possible M is called the modulus of concavity of $\| \, . \, \|$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\| \, . \, \|$ is called a p–norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

By the Aoki-Rolewicz Theorem [54], each quasi-norm is equivalent to some p-norm (see also [7]) Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms. Moreover in [59], J. Tabor has investigated a version of Hyers-Rassias-Gajda Theorem (see [16, 48]) in quasi-Banach spaces.

Our main result in this section is the following:

Theorem 4.2. *Let $\ell \in \{-1, 1\}$ be fixed, X be a p-normed space, Y be a p-Banach space and $\varphi : X^n \to [0, \infty)$ be a function such that, for some $0 < \alpha < 2$*

$$\varphi(2x_1, 2x_2, ..., 2x_n) \leq \alpha \varphi(x_1, x_2, ..., x_n) \tag{4.1}$$

for all $x_1, ..., x_n \in X$, and

$$\widetilde{\varphi}(x) := \sum_{j=\frac{1-\ell}{2}}^{\infty} \frac{1}{2^{\ell j p}} \varphi^p(2^{\ell j}x, 2^{\ell j}x, 0, ..., 0) < \infty \tag{4.2}$$

for all $x \in X$ (denoted $(\varphi(x_1, ..., x_n))^p$ by $\varphi^p(x_1, ..., x_n)$). Suppose that $f : X \to Y$ is a function with $f(0) = 0$ that satisfies

$$\|D_f(x_1, ..., x_n)\| \leq \varphi(x_1, ..., x_n) \tag{4.3}$$

for all $x_1, ..., x_n \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2^{n-1}} [\widetilde{\varphi}(\frac{x}{2^{\frac{1-\ell}{2}}})]^{\frac{1}{p}} \tag{4.4}$$

for all $x \in X$.

Proof . for $\ell = 1$, putting $x_1 = x_2 = x$ and $x_i = 0$ $(i = 3, ..., n)$ in (4.3), we get

$$\|2^{n-2}f(2x) - 2^{n-1}f(x)\| \leq \varphi(x, x, 0, ..., 0) \tag{4.5}$$

for all $x \in X$, or

$$\|\frac{f(2x)}{2} - f(x)\| \le \frac{1}{2^{n-1}} \varphi(x, x, 0, ..., 0) \tag{4.6}$$

for all $x \in X$. The same proof of theorem (3.4), there exists a unique additive function $A : X \to Y$ such that

$$\lim_{n \to \infty} \frac{f(2^n x)}{2^n} = A(x) \tag{4.7}$$

for all $x \in X$. Replacing x by $2x$ in (4.6) and dividing by 2 and summing the resulting inequality with (4.6), we get

$$\|f(x) - \frac{f(2^2 x)}{2^2}\|^p \le \frac{1}{2^{(n-1)p}} (\varphi^p(x, x, 0, ..., 0) + \frac{\varphi^p(2x, 2x, 0, ..., 0)}{2^p}) \tag{4.8}$$

for all $x \in X$. Hence

$$\|\frac{f(2^l x)}{2^l} - \frac{f(2^m x)}{2^m}\|^p \le \frac{1}{2^{(n-1)p}} \sum_{j=l}^{m-1} \frac{1}{2^{jp}} \varphi^p(2^j x, 2^j x, 0, ..., 0) \tag{4.9}$$

for all nonnegative integers m and l with $m > l$ and for all $x \in X$. By $l = 0$ and taking m to approach infinity in (4.9) for $\ell = 1$ with regard to the (4.7), we obtain (4.4).

Also, for $\ell = -1$, it follows from (4.5) with replacing x by $\frac{x}{2}$ that

$$\|f(x) - 2f(\frac{x}{2})\| \le \frac{1}{2^{n-1}} \varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0)$$

for all $x \in X$. Hence

$$\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\|^p \le \frac{1}{2^{(n-1)p}} \sum_{j=l}^{m-1} 2^{jp} \varphi^p(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0, ..., 0) \tag{4.10}$$

for all nonnegative integers m and l with $m > l$ and for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.10), we get the inequality (4.4) for $\ell = -1$. This completes the proof. $\square$

5. Approximately additive functions by using alternative fixed point

By using the idea of Cădariu and Radu, we will prove the stability of the general n-dimensional additive functional equation (1.1).

Theorem 5.1. *Let X be a real vector space and Y be a real Banach space. Suppose that $\ell \in \{-1, 1\}$ be fixed and $f : X \to Y$ a function for which there exists a function $\varphi : X^n \to [0, \infty)$ that satisfying (4.1) and (4.3) for all $x_1, ..., x_n \in X$. If there exists $0 < L = L(\ell) < 1$ such that the function $x \mapsto \psi(x) = \varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0)$ has the property*

$$\psi(x) \le L \cdot 2^\ell \cdot \psi(\frac{x}{2^\ell}) \tag{5.1}$$

for all $x \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{L^{\frac{\ell+1}{2}}}{2^{n-2}(1-L)} \psi(x) \tag{5.2}$$

for all $x \in X$.

Proof . Let Ω be the set of all functions $g : X \to Y$ and introduce a generalized metric on Ω as follows:

$$d(g,h) = d_\psi(g,h) = \inf\{K \in (0,\infty) : \|g(x) - h(x)\| \le K\psi(x), \ x \in X\}$$

It is easy to show that (Ω, d) is a generalized complete metric space [10].

Now we define a function $J : \Omega \to \Omega$ by $Jg(x) = \frac{1}{2^\ell} \ g(2^\ell x)$ for all $x \in X$.

Note that for all $\ g, h \in \Omega$,

$$
\begin{aligned}
d(g,h) < K \ &\Rightarrow \ \|g(x) - h(x)\| \le K\psi(x), && for \ all \ \ x \in X, \\
&\Rightarrow \ \|\frac{1}{2^\ell}g(2^\ell x) - \frac{1}{2^\ell}h(2^\ell x)\| \le \frac{1}{2^\ell} \ K \ \psi(2^\ell x), && for \ all \ \ x \in X, \\
&\Rightarrow \ \|\frac{1}{2^\ell}g(2^\ell x) - \frac{1}{2^\ell}h(2^\ell x)\| \le L \ K \ \psi(x), && for \ all \ \ x \in X, \\
&\Rightarrow \ d(Jg, Jh) \le L \ K.
\end{aligned}
$$

Hence we see that $d(Jg, Jh) \le L \ d(g, h)$ for all $g, h \in \Omega$, that is, J is a strictly self-function of Ω with the Lipschitz constant L.

Putting $x_1 = x_2 = x$ and $x_i = 0 \ (i = 3, ..., n)$ in (4.3), we have (4.5) for all $x \in X$, thus, by using (5.1) with the case $\ell = 1$, we obtain that

$$\|f(x) - \frac{f(2x)}{2}\| \le \frac{1}{2^{n-1}} \ \varphi(x, x, 0, ..., 0) = \frac{1}{2^{n-1}} \ \psi(2x) \le \frac{L}{2^{n-2}} \ \psi(x)$$

for all $x \in X$, that is, $d(f, Jf) \le \frac{L}{2^{n-2}} < \infty$.

Also, if we substitute $x = \frac{x}{2}$ in (4.5) and use (5.1) with the case $\ell = -1$, then we see that

$$\|f(x) - 2f(\frac{x}{2})\| \le \frac{1}{2^{n-2}} \ \psi(x)$$

for all $x \in X$, that is, $d(f, Jf) \le \frac{1}{2^{n-2}} < \infty$.

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point A of J in Ω such that

$$A(x) = \lim_{m \to \infty} \frac{f(2^{m\ell}x)}{2^{m\ell}}$$

for all $x \in X$, since $\lim_{m \to \infty} d(J^m f, A) = 0$.

Also, if we replace $x_1, ..., x_n$ with $2^{m\ell}x_1, ..., 2^{m\ell}x_n$ in (4.3), respectively, and divide by $2^{m\ell}$, then it follows from (4.1) that

$$
\begin{aligned}
\|D_A(x_1, ..., x_n)\| &= \lim_{m \to \infty} \frac{1}{2^m} \|D_f(2^m x_1, ..., 2^m x_n)\| \\
&\le \lim_{m \to \infty} \frac{1}{2^m} \varphi(2^m x_1, ..., 2^m x_n) \\
&\le \lim_{m \to \infty} (\frac{\alpha}{2})^m \varphi(x_1, ..., x_n) = 0
\end{aligned}
$$

for all $x_1, ..., x_n \in X$, so $D_A(x_1, ..., x_n) = 0$. Thus the function A is additive.

According to the fixed point alterative, since A is the unique fixed point of J in the set $\Lambda = \{g \in \Omega : d(f, g) < \infty\}$, A is the unique function such that

$$\|f(x) - A(x)\| \le K \ \psi(x)$$

for all $x \in X$ and $K > 0$. Again using the fixed point alterative, gives

$$d(f, A) \leq \frac{1}{1 - L} d(f, Jf) \leq \frac{L^{\frac{\ell+1}{2}}}{2^{n-2}(1 - L)}$$

so we conclude that

$$\|f(x) - A(x)\| \leq \frac{L^{\frac{\ell+1}{2}}}{2^{n-2}(1 - L)} \, \psi(x)$$

for all $x \in X$. This completes the proof. $\square$

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