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Lie ternary (σ, τ, ξ)-derivations on Banach ternary algebras

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Abstract

Let A be a Banach ternary algebra over a scalar field $\mathbb{R}$ or $\mathbb{C}$ and X be a ternary Banach A–module. Let σ, τ and ξ be linear mappings on A, a linear mapping $D : (A, [\ ]_A) \to (X, [\ ]_X)$ is called a Lie ternary (σ, τ, ξ)–derivation, if

$$D([a, b, c]) = [[D(a)bc]_X]_{(\sigma,\tau,\xi)} - [[D(c)ba]_X]_{(\sigma,\tau,\xi)}$$

for all $a, b, c \in A$, where $[abc]_{(\sigma,\tau,\xi)} = a\tau(b)\xi(c) - \sigma(c)\tau(b)a$ and $[a, b, c] = [abc]_A - [cba]_A$. In this paper, we prove the generalized Hyers–Ulam–Rassias stability of Lie ternary (σ, τ, ξ)–derivations on Banach ternary algebras and C^*–Lie ternary (σ, τ, ξ)–derivations on C^*–ternary algebras for the following Euler–Lagrange type additive mapping:

$$\sum_{i=1}^{n} f(\sum_{j=1}^{n} q(x_i - x_j)) + nf(\sum_{i=1}^{n} qx_i) = nq \sum_{i=1}^{n} f(x_i).$$

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1. Introduction

In the 19 th century, many mathematicians considered ternary algebraic operations and their generalizations. A. Cayley ([4]) introduced the notion of cubic matrix. It was later generalized by Kapranov,

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Gelfand and Zelevinskii in 1990 ([11]). Below, a composition rule includes a simple example of such non–trivial ternary operation:

$$\{a,b,c\}_{ijk}=\sum_{l,m,n}a_{nil}b_{ljm}c_{mkn},\quad i,j,k,l,m,n=1,2,\ldots,N.$$

There are a lot of hopes that ternary structures and their generalization will have certain possible applications in physics. Some of these applications are (see [8], [2], [9], [12]–[6]). A ternary (associative) algebra $(A,[\ ])$ is a linear space A over a scalar field $\mathbb{F}=(\mathbb{R}$ or $\mathbb{C})$ equipped with a linear mapping, the so-called ternary product, $[\ ]$: $A\times A\times A\to A$ such that $[[abc]de]=[a[bcd]e]=[ab[cde]]$ for all $a,b,c,d,e\in A$. This notion is a natural generalization of the binary case. Indeed if $(A,\odot)$ is a usual (binary) algebra then $[abc]:=(a\odot b)\odot c$ induced a ternary product making A into a ternary algebra which will be called trivial. It is known that unital ternary algebras are trivial and finitely generated ternary algebras are ternary subalgebras of trivial ternary algebras [3]. There are other types of ternary algebras in which one may consider other versions of associativity. Some examples of ternary algebras are (i) "cubic matrices" introduced by Cayley [4] which were in turn generalized by Kapranov, Gelfand and Zelevinskii [11]; (ii) the ternary algebra of polynomials of odd degrees in one variable equipped with the ternary operation $[p_1p_2p_3]=p_1\odot p_2\odot p_3$, where $\odot$ denotes the usual multiplication of polynomials.

By a Banach ternary algebra we mean a ternary algebra equipped with a complete norm $\|.\|$ such that $\|[abc]\|\leq\|a\|\|b\|\|c\|$. If a ternary algebra $(A,[\ ])$ has an identity, i.e. an element e such that $a=[aee]=[eae]=[eea]$ for all $a\in A$, then $a\odot b:=[aeb]$ is a binary product for which we have

$$(a\odot b)\odot c=[[aeb]ec]=[ae[bec]]=a\odot(b\odot c)$$

and

$$a\odot e=[aee]=a=[eea]=e\odot a$$

for all $a,b,c\in A$ and so $(A,[\ ])$ may be considered as a (binary) algebra. Conversely, if $(A,\odot)$ is any (binary) algebra, then $[abc]:=a\odot b\odot c$ makes A into a ternary algebra with the unit e such that $a\odot b=[aeb]$.

Let $(A,[\ ]_A)$ be a Banach ternary algebra and $(X,[\ ]_X)$ be a Banach space. Then X is called a ternary Banach A–module, if module operations $A\times A\times X\to X$, $A\times X\times A\to X$, and $X\times A\times A\to X$ are $\mathbb{C}$–linear in every variable. Moreover satisfy:

$$[[abc]_A\ dx]_X=[a[bcd]_A\ x]_X=[ab[cdx]_X]_X,$$

$$[abc]_A\ xd]_X=[a[bcx]_X\ d]_X=[ab[cxd]_X]_X,$$

$$[[xab]_X\ cd]_X=[x[abc]_A\ d]_X=[xa[bcd]_A]_X,$$

$$[[axb]_X\ cd]_X=[a[xbc]_X\ d]_X=[ax[bcd]_A]_X,$$

$$[[abx]_X\ cd]_X=[a[bxc]_X\ d]_X=[ab[xcd]_X]_X$$

for all $x\in X$ and all $a,b,c,d\in A$, and

$$\max\{\|[xab]_X\|,\|[axb]_X\|,\|[abx]_X\|\}\leq\|a\|\|b\|\|x\|$$

for all $x\in X$ and all $a,b\in A$.

Let A be a normed algebra, σ and τ two mappings on A and X be an A–bimodule. A linear mapping $L:A\to X$ is called a Lie (σ,τ)–derivation, if

$$L([a,b])=[L(a),b]_{\sigma,\tau}-[L(b),a]_{\sigma,\tau}$$

for all $a,b\in A$, where $[a,b]_{\sigma,\tau}$ is $a\tau(b)-\sigma(b)a$ and $[a,b]$ is the commutator $ab-ba$ of elements a,b. Now, Let $(A,[\,]_A)$ be a Banach ternary algebra over a scalar field $\mathbb{R}$ or $\mathbb{C}$ and $(X,[\,]_X)$ be a ternary Banach A–module. Let σ,τ and ξ be linear mappings on A. A linear mapping $D:(A,[\,]_A)\to(X,[\,]_X)$ is called a Lie ternary (σ,τ,ξ)–derivation, if

$$D([a,b,c])=[[D(a)bc]_X]_{(\sigma,\tau,\xi)}-[[D(c)ba]_X]_{(\sigma,\tau,\xi)} \tag{1.1}$$

for all $a,b,c\in A$, where $[abc]_{(\sigma,\tau,\xi)}=a\tau(b)\xi(c)-\sigma(c)\tau(b)a$ and $[a,b,c]$ is the commutator $[abc]_A-[cba]_A$ of elements a,b,c.
If a Banach ternary algebra A has an identity e such that $\|e\|=1$, as we said above, A may be considered as a (binary) algebra. Now let X be a ternary Banach A–module, then X may be considered as a Banach A–module by following module product:

$$a.x=[aex]_X \qquad x.a=[xea]_X$$

for all $a\in A, x\in X$.
Let A be a unital Banach ternary algebra and X be a ternary Banach A–module. If $D:A\to X$ is a Lie ternary (σ,τ,ξ)–derivation such that σ,τ and ξ are linear mappings on A, additionally, $\tau(e)=e$, then it is easy to prove that D is a Lie (σ,ξ)–derivation.

The stability of functional equations was started in 1940 with a problem raised by S. M. Ulam [20]. In 1941 Hyers affirmatively solved the problem of S. M. Ulam in the context of Banach spaces. In 1950 T. Aoki [1] extended the Hyers' theorem. in 1978, Th.M. Rassias [17] formulated and proved the following Theorem:
Theorem A. Assume that E_1 and E_2 are real normed spaces with E_2 complete, $f:E_1\to E_2$ is a mapping such that for each fixed $x\in E_1$ the mapping $t\to f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\epsilon\geq 0$ and $p\in[0,1)$ such that $\|f(x+y)-f(x)-f(y)\|\leq\epsilon(\|x\|^p+\|y\|^p)$ for all $x,y\in E_1$. Then there exists a unique linear mapping $T:E_1\in E_2$ such that $\|f(x)-T(x)\|\leq\epsilon\frac{\|x\|^p}{(1-2^p)}$ for all $x\in E_1$.

The equality $\|f(x+y)-f(x)-f(y)\|\leq\epsilon(\|x\|^p+\|y\|^p)$ has provided extensive influence in the development of of what we now call Hyers–Ulam–Rassias stability of functional equations [5, 10, 6, 18, 19]. In 1994, a generalization of Rassias' theorem was obtained by Gavruta [7], in which he replaced the bound $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function.

2. The main results

In this section, let $(X,[\ ]_X)$ be a ternary Banach $(A,[\ ]_A)$–module. Our aim is to establish the Hyers–Ulam–Rassias stability of Lie ternary (σ,τ,ξ)–derivations.

Theorem 2.1. *Suppose $f:A\to X$ is a mapping with $f(0)=0$ for which there exist mappings $g,h,k:A\to A$ with $g(0)=h(0)=k(0)=0$ and a function $\varphi:A\times A\times A\times A\times A\to[0,\infty]$ such that*

(i)

$$\widetilde{\varphi}(x,y,u,v,w)=\frac{1}{2}\sum_{n=0}^{\infty}\varphi(2^nx,2^ny,2^nu,2^nv,2^nw)<\infty; \tag{2.1}$$

(ii)

$$\|f(\lambda x+\lambda y+[u,v,w])-\lambda f(x)-\lambda f(y)-[[f(u)vw]_X]_{(g,h,k)}+[[f(w)vu]_X]_{(g,h,k)}\| \leq\varphi(x,y,u,v,w); \tag{2.2}$$

(iii)
$$\|g(\lambda x+\lambda y)-\lambda g(x)-\lambda g(y)\| \le \varphi(x,y,0,0,0);$$
(iv)
$$\|h(\lambda x+\lambda y)-\lambda h(x)-\lambda h(y)\| \le \varphi(x,y,0,0,0);$$
(v)
$$\|k(\lambda x+\lambda y)-\lambda k(x)-\lambda k(y)\| \le \varphi(x,y,0,0,0)$$

for all $\lambda \in \mathbb{T}^1(:=\{\lambda\in\mathbb{C}\ ;|\lambda|=1\})$ and for all $x,y,u,v,w\in A$. Then there exist unique linear mappings σ,τ and ξ from A to A satisfying

$$\|g(x)-\sigma(x)\| \le \widetilde{\varphi}(x,x,0,0,0), \tag{2.3}$$
$$\|h(x)-\tau(x)\| \le \widetilde{\varphi}(x,x,0,0,0), \tag{2.4}$$
$$\|k(x)-\xi(x)\| \le \widetilde{\varphi}(x,x,0,0,0) \tag{2.5}$$

and there exist a unique Lie ternary (σ,τ,ξ)–derivation on $D: A\to X$ such that

$$\|f(x)-D(x)\| \le \widetilde{\varphi}(x,x,0,0,0) \tag{2.6}$$

for all $x\in A$.

Proof . One can show that the limits

$$\sigma(x):=\lim_n \frac{1}{2^n}g(2^nx),$$
$$\tau(x):=\lim_n \frac{1}{2^n}h(2^nx),$$
$$\xi(x):=\lim_n \frac{1}{2^n}k(2^nx),$$

exist for all $x\in A$, also σ,τ and ξ are unique linear mappings which satisfy (2.3), (2.4) and (2.5), respectively (see [18]). Put $\lambda=1$ and $u=v=w=0$ in (2.2) to obtain

$$\|f(x+y)-f(x)-f(y)\| \le \phi(x,y,0,0,0) \qquad (x,y\in A). \tag{2.7}$$

Fix $x\in A$. Replace y by x in (2.7) to get

$$\|f(2x)-2f(x)\| \le \varphi(x,x,0,0,0).$$

One can use the induction to show that

$$\left\|\frac{f(2^px)}{2^p}-\frac{f(2^qx)}{2^q}\right\| \le \frac{1}{2}\sum_{k=q}^{p-1}\varphi(2^kx,2^kx,0,0,0) \tag{2.8}$$

for all $x\in A$, and all $p>q\ge 0$. It follows from the convergence of series (2.1) that the sequence $\left\{\frac{f(2^nx)}{2^n}\right\}$ is Cauchy. By the completeness of X, this sequence is convergent. Set

$$D(x)=\lim_{n\to\infty}\frac{f(2^nx)}{2^n},$$

for all $x \in A$. Putting $u = v = w = 0$ and replacing x, y by $2^n x$ and $2^n y$ in (2.2), respectively and divide the both sides of the inequality by 2^n we get

$$\|2^{-n} f(2^n(\lambda x + \lambda y)) - 2^{-n}\lambda f(2^n x) - 2^{-n}\lambda f(2^n y)\| \leq \frac{1}{2^n}\varphi(2^n x, 2^n x, 0, 0, 0).$$

Passing to the limit as $n \to \infty$ we obtain $D(\lambda x + \lambda y) = \lambda D(x) + \lambda D(y)$. Put $q = 0$ in (2.8) to get

$$\left\| \frac{f(2^p x)}{2^p} - f(x) \right\| \leq \frac{1}{2} \sum_{k=0}^{p-1} \varphi(2^k x, 2^k x, 0, 0, 0)$$

for all $x \in A$. Taking the limit as $p \to \infty$ we infer that

$$\|f(x) - D(x)\| \leq \widetilde{\varphi}(x, x, 0, 0, 0)$$

for all $x \in A$. Next, let $\gamma \in \mathbb{C}(\gamma \neq 0)$ and let N be a positive integer number greater than $|\gamma|$. It is shown that there exist two numbers $\lambda_1, \lambda_2 \in \mathbb{T}$ such that $2\frac{\gamma}{N} = \lambda_1 + \lambda_2$. since D is a additive, we have $D(\frac{1}{2}x) = \frac{1}{2}D(x)$ for all $x \in A$. Hence

$$\begin{aligned} D(\gamma x) &= D(\frac{N}{2}.2.\frac{\gamma}{N}x) = ND(\frac{1}{2}.2.\frac{\gamma}{N}x) = \frac{N}{2}D(2.\frac{\gamma}{N}x) \\ &= \frac{N}{2}D(\lambda_1 x + \lambda_2 x) = \frac{N}{2}(D(\lambda_1 x) + D(\lambda_2 x)) \\ &= \frac{N}{2}(\lambda_1 + \lambda_2)D(x) = (\frac{N}{2}.2.\frac{\gamma}{N})D(x) = \gamma D(x) \end{aligned}$$

for all $x \in A$. Thus D is linear. Suppose that there exists another ternary (σ, τ, ξ)-derivation $D' : A \to X$ satisfying (2.6). Since $D'(x) = \frac{1}{2^n}D'(2^n x)$, we see that

$$\begin{aligned} \|D(x) - D'(x)\| &= \frac{1}{2^n}\|D(2^n x) - D'(2^n x)\| \\ &\leq \frac{1}{2^n}(\|f(2^n x) - D(2^n x)\| + \|f(2^n x) - D'(2^n x)\|) \\ &\leq 4\theta \frac{2^p}{2 - 2^p} 2^{n(p-1)} \|x\|^p , \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. Therefore $D' = D$ as claimed. Similarly one can use (2.3), (2.4) and (2.5) to show that there exist unique linear mappings σ, τ and ξ defined by $\lim_{n\to\infty} \frac{g(2^n x)}{2^n}$, $\lim_{n\to\infty} \frac{h(2^n x)}{2^n}$ and $\lim_{n\to\infty} \frac{k(2^n x)}{2^n}$, respectively. Putting $x = y = 0$ and replacing u, v, w by $2^n u, 2^n v$ and $2^n w$ in (2.2) respectively, we obtain

$$\|f([2^{3n}u, v, w]) - [[f(2^n u)2^{2n}vw]_X]_{(g,h,k)} + [[f(2^n w)2^{2n}vu]_X]_{(g,h,k)}\| \leq \varphi(0, 0, 2^n u, 2^n v, 2^n w).$$

Then

$$\begin{aligned} \frac{1}{2^{3n}}\|f([2^{3n}u, v, w]) - [[f(2^n u)2^{2n}vw]_X]_{(g,h,k)} + [[f(2^n w)2^{2n}vu]_X]_{(g,h,k)}\| \\ \leq \frac{1}{2^{3n}}\varphi(0, 0, 2^n u, 2^n v, 2^n w) \end{aligned}$$

for all $u, v, w \in A$,. Hence

$$\lim_{n\to\infty} \frac{1}{2^{3n}} \|f([2^{3n}u, v, w]) - [[f(2^n u)2^{2n} vw]_X]_{(g,h,k)} + [[f(2^n w)2^{2n} vu]_X]_{(g,h,k)}\| \leq \lim_{n\to\infty} \frac{1}{2^{3n}} \varphi(0, 0, 2^n u, 2^n v, 2^n w) = 0,$$

therefore

$$\begin{aligned} D([u, v, w]) &= \lim_{n\to\infty} \frac{f(2^{3n}[u, v, w])}{2^{3n}} = \lim_{n\to\infty} \frac{f([2^n u, 2^n v, 2^n w])}{2^{3n}} \\ &= \lim_{n\to\infty} \left(\frac{[[f(2^n u)2^n v 2^n w]_X]_{(g,h,k)} - [[f(2^n w)2^n v 2^n u]_X]_{(g,h,k)}}{2^{3n}} \right) \\ &= \lim_{n\to\infty} \left(\frac{f(2^n u)h(2^n v)k(2^n w) - g(2^n w)h(2^n v)f(2^n u)}{2^{3n}} \right. \\ &\quad \left. - \frac{f(2^n w)h(2^n v)k(2^n u) - g(2^n u)h(2^n v)f(2^n w)}{2^{3n}} \right) \\ &= (D(u)\tau(v)\xi(w) - \sigma(w)\tau(v)D(u)) - (D(w)\tau(v)\xi(u) - \sigma(u)\tau(v)D(w)) \\ &= [[D(u)vw]_X]_{(\sigma,\tau,\xi)} - [[D(w)vu]_X]_{(\sigma,\tau,\xi)} \end{aligned}$$

for each $u, v, w \in A$. Hence, the linear mapping D is a Lie ternary (σ, τ, ξ)–derivation. □

Corollary 2.2. *Suppose $f : A \to X$ is a mapping with $f(0) = 0$ for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ and there exists $\theta \geq 0$ and $p \in [0, 1)$ such that*

(i)

$$\|f(\lambda x + \lambda y + [u, v, w]) - \lambda f(x) - \lambda f(y) - [[f(u)vw]_X]_{(g,h,k)} + [[f(w)vu]_X]_{(g,h,k)}\| \leq \theta(\|x\|^p + \|y\|^p + \|u\|^p + \|v\|^p + \|w\|^p),$$

(ii)

$$\|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

(iii)

$$\|h(\lambda x + \lambda y) - \lambda h(x) - \lambda h(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

(iv)

$$\|k(\lambda x + \lambda y) - \lambda k(x) - \lambda k(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $\lambda \in \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and for all $x, y \in A$. Then there exist unique linear mappings σ, τ and ξ from A to A satisfying $\|g(x) - \sigma(x)\| \leq \frac{\theta\|x\|^p}{1-2^{p-1}}, \|h(x) - \tau(x)\| \leq \frac{\theta\|x\|^p}{1-2^{p-1}}$ and $\|k(x) - \xi(x)\| \leq \frac{\theta\|x\|^p}{1-2^{p-1}}$, and there exists a unique Lie ternary (σ, τ, ξ)–derivation $D : A \to X$ such that

$$\|f(x) - D(x)\| \leq \frac{\theta\|x\|^p}{1 - 2^{p-1}} \tag{2.9}$$

for all $x \in A$.

Proof . Put $\varphi(x, y, u, v, w) = \theta(\|x\|^p + \|y\|^p + \|u\|^p + \|v\|^p + \|w\|^p)$ in Theorem 2.1. □

A C^*–ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \to [xyz]$ of A^3 into A, which is $\mathbb{C}$–linear in the outer variables, conjugate $\mathbb{C}$–linear in the middle variable, and associative in the sense that $[xy[zwv]] = [x[wzy]v] = [[xyz]wv]$, and satisfies $\|[xyz]\| \leq$

$\|x\|.\|y\|.\|z\|$ and $\|[xxx]\| = \|x\|^3$ (see [8], [21]). Every left Hilbert C^*–module is a C^*–ternary algebra via the ternary product $[xyz] := \langle x, y\rangle z$.

If a C^*–ternary algebra $(A, [\ ])$ has an identity, i.e. an element $e \in A$ such that $x = [xee] = [eex]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [xey]$ and $x^* := [exe]$, is a unital C^*–algebra. Conversely, if $(A, \circ)$ is a unital C^*–algebra, then $[xyz] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra [14].

A Lie (σ,τ,ξ)–ternary derivation $L : A \to A$ on a C^*–ternary algebra A is called a C^*–Lie ternary (σ,τ,ξ)–derivation.

Throughout this section, assume that A is a C^*–ternary with norm $\|.\|_A$. Let q be a positive rational number. For a given mapping $f : A \to A$ and a given $\mu \in \mathbb{C}$, we define $D_\mu f : A^n \to A$ by

$$D_\mu f(x_1,\dots,x_n) := \sum_{i=1}^{n} f\Big(\sum_{j=1}^{n} q\mu(x_i - x_j)\Big) + nf\Big(\sum_{i=1}^{n} q\mu x_i\Big) - nq\mu \sum_{i=1}^{n} f(x_i)$$

for all $x_1,\dots,x_n \in A$.
In this section our aim is to establish the Hyers–Ulam stability of C^*–Lie ternary (σ,τ,ξ)–derivations in C^*–ternary algebras for the Euler–Lagrange type additive mapping.

Theorem 2.3. *Assume that $r > 3$ if $nq > 1$ and that $0 < r < 1$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying*

(i)
$$\|D_\mu f(x_1,\dots,x_n)\| \le \theta \sum_{j=1}^{n} \|x_j\|^r, \tag{2.10}$$

(ii)
$$\|g(q\mu x_1 + \dots + q\mu x_n) - q\mu g(x_1) - \dots - q\mu g(x_n)\| \le \theta(\|x_1\|^r + \dots + \|x_n\|^r), \tag{2.11}$$

(iii)
$$\|h(q\mu x_1 + \dots + q\mu x_n) - q\mu h(x_1) - \dots - q\mu h(x_n)\| \le \theta(\|x_1\|^r + \dots + \|x_n\|^r), \tag{2.12}$$

(iv)
$$\|k(q\mu x_1 + \dots + q\mu x_n) - q\mu k(x_1) - \dots - q\mu k(x_n)\| \le \theta(\|x_1\|^r + \dots + \|x_n\|^r), \tag{2.13}$$

such that
$$\|f([x,y,z]) - [f(x)yz]_{(g,h,k)} + [f(z)yx]_{(g,h,k)}\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{2.14}$$

for all $x, y, z \in A$. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*–Lie ternary (σ,τ,ξ)–derivation $L : A \to A$ satisfying

(i)
$$\|g(x) - \sigma(x)\| \le \frac{n\theta}{(nq)^r - nq}\|x\|^r, \tag{2.15}$$

(ii)
$$\|h(x) - \tau(x)\| \le \frac{n\theta}{(nq)^r - nq}\|x\|^r, \tag{2.16}$$

(iii)
$$\|k(x) - \xi(x)\| \le \frac{n\theta}{(nq)^r - nq}\|x\|^r, \tag{2.17}$$

such that

$$\|f(x) - L(x)\| \leq \frac{\theta}{(nq)^r - nq}\|x\|^r. \tag{2.18}$$

Proof . Letting $\mu = 1$ and $x_1 = \cdots = x_n = x$ in (2.10), we get

$$\|nf(nqx) - n^2qf(x)\| \leq n\theta\|x\|^r$$

for all $x \in A$. So

$$\|f(x) - nqf(\frac{x}{nq})\| \leq \frac{\theta}{(nq)^r}\|x\|^r$$

for all $x \in A$. So

$$\|(nq)^l f(\frac{x}{(nq)^l}) - (nq)^{l+m} f(\frac{x}{(nq)^{l+m}})\| \tag{2.19}$$

$$\leq \sum_{j=l}^{l+m-1} \|(nq)^j f(\frac{x}{(nq)^j}) - (nq)^{j+1} f(\frac{x}{(nq)^{j+1}})\| \tag{2.20}$$

$$\leq \frac{\theta}{(nq)^r} \sum_{j=l}^{l+m-1} \frac{(nq)^j}{(nq)^{rj}}\|x\|^r \tag{2.21}$$

for all nonnegative integers m and l with $x \in A$. It follows from (2.19) that the sequence $\{(nq)^m f(\frac{x}{(nq)^m})\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{(nq)^m f(\frac{x}{(nq)^m})\}$ converges. So one can define the mapping $L : A \to A$ by

$$L(x) := \lim_{m\to\infty} (nq)^m f(\frac{x}{(nq)^m})$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.19), we get

$$\|f(x) - L(x)\| \leq \frac{\theta}{(nq)^r} \sum_{j=0}^{m-1} \frac{(nq)^j}{(nq)^{rj}}\|x\|^r$$

for all $x \in A$. So (2.18) holds for all $x \in A$. It follows from (2.10) that

$$\begin{aligned}\|D_1L(x_1, \ldots, x_n)\| &= \lim_{m\to\infty} (nq)^m \|D_1 f(\frac{x_1}{(nq)^m}, \ldots, \frac{x_n}{(nq)^m})\| \\ &\leq \lim_{m\to\infty} \frac{(nq)^m\theta}{(nq)^{mr}} \sum_{j=1}^{n} \|x_j\|^r\end{aligned}$$

for all $x_1, \ldots, x_n \in A$. Thus

$$D_1L(x_1, \ldots, x_n) = 0$$

for all $x_1, \ldots, x_n \in A$. By Lemma 3.1 of [15], the mapping $L : A \to A$ is Cauchy additive. By the same reasoning as in the proof of Theorem 2.1 of [14], the mapping $L : A \to A$ is linear. Also letting $\mu = 1$ and $x_1 = \cdots = x_n = x$ in (2.11), we get

$$\|g(qnx) - qng(x)\| \leq n\theta\|x\|^r$$

for all $x \in A$. So

$$\|g(x) - qng(\frac{x}{nq})\| \le \frac{n\theta}{(nq)^r}\|x\|^r$$

for all $x \in A$. We easily prove that by induction that

$$\|(nq)^l g(\frac{x}{(nq)^l}) - (nq)^{l+m} g(\frac{x}{(nq)^{l+m}})\| \tag{2.22}$$

$$\le \sum_{j=l}^{l+m-1} \|(nq)^j g(\frac{x}{(nq)^j}) - (nq)^{j+1} g(\frac{x}{(nq)^{j+1}})\| \tag{2.23}$$

$$\le \frac{n\theta}{(nq)^r}\sum_{j=l}^{l+m-1}\frac{(nq)^j}{(nq)^{rj}}\|x\|^r \tag{2.24}$$

for all nonnegative integers m and l with $x \in A$. It follows from (2.22) that the sequence $\{(nq)^m g(\frac{x}{(nq)^m})\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{(nq)^m g(\frac{x}{(nq)^m})\}$ converges. So one can define the mapping $\sigma : A \to A$ by

$$\sigma(x) := \lim_{m\to\infty}(nq)^m g(\frac{x}{(nq)^m})$$

for all $x \in A$. we easily prove that by (2.11) that $\sigma(\mu x + \mu y) = \mu\sigma(x) + \mu\sigma(y)$ and by letting $l = 0$ and taking the limit $m \to \infty$ in (2.9), we get

$$\|g(x) - \sigma(x)\| \le \frac{n\theta}{(nq)^r}\sum_{j=0}^{m-1}\frac{(nq)^j}{(nq)^{rj}}\|x\|^r$$

for all $x \in A$. So (2.15) holds for all $x \in A$. similarly, there exist linear mapping τ and ξ on A such that (2.16) and (2.17) hold for all $x \in A$. It follows from (2.14) that

$$\begin{aligned}
&\|L([x,y,z]) - [L(x)yz]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}\| \\
&= \lim_{m\to\infty}(nq)^{3m}\left\|f\left(\frac{[x,y,z]}{(nq)^{3m}}\right) - [f\left(\frac{x}{(nq)^m}\right)\frac{y}{(nq)^m}\frac{z}{(nq)^m}]_{(g,h,k)}\right. \\
&\quad \left. + [f\left(\frac{z}{(nq)^m}\right)\frac{y}{(nq)^m}\frac{x}{(nq)^m}]_{(g,h,k)}\right\| \\
&\le \lim_{m\to\infty}\frac{(nq)^{3m}\theta}{(nq)^{mr}}(\|x\|^r + \|y\|^r + \|z\|^r) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$L([x,y,z]) = [L(x)yz]_{(\sigma,\tau,\xi)} - [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x, y, z \in A$. Now, let $L' : A \to A$ be another Euler–Lagrange type additive mapping satisfying (2.18). Then we have

$$\begin{aligned}
\|L(x) - L'(x)\| &= (nq)^m\left\|L(\frac{x}{(nq)^m}) - L'(\frac{x}{(nq)^m})\right\| \\
&\le (nq)^m\left\|L(\frac{x}{(nq)^m}) - f(\frac{x}{(nq)^m})\right\| + \left\|L'(\frac{x}{(nq)^m}) - f(\frac{x}{(nq)^m})\right\| \\
&\le \frac{2(nq)^m\theta}{((nq)^r - nq)(nq)^{mr}}\|x\|^r,
\end{aligned}$$

which tends to zero as $m \to \infty$ for all $x \in A$. So we can conclude that $L(x) = L'(x)$ for all $x \in A$. This prove the uniqueness of L. Thus the mapping $L : A \to A$ is a unique C^*–Lie ternary (σ, τ, ξ)–derivation satisfying (2.18) and similarly, we can prove that σ, τ and ξ are unique on A and the proof of the theorem is complete. □

Theorem 2.4. *Assume that $0 < r < 1$ if $nq > 1$ and that $r > 3$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying (2.10)–(2.14). Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*–Lie ternary (σ, τ, ξ)–derivation $L : A \to A$ satisfying*

(i)
$$\|g(x) - \sigma(x)\| \le \frac{n\theta}{nq - (nq)^r}\|x\|^r, \tag{2.25}$$

(ii)
$$\|h(x) - \tau(x)\| \le \frac{n\theta}{nq - (nq)^r}\|x\|^r, \tag{2.26}$$

(iii)
$$\|k(x) - \xi(x)\| \le \frac{n\theta}{nq - (nq)^r}\|x\|^r, \tag{2.27}$$

such that
$$\|f(x) - L(x)\| \le \frac{\theta}{nq - (nq)^r}\|x\|^r. \tag{2.28}$$

Proof . Letting $\mu = 1$ and $x_1 = \cdots = x_n = x$ in (2.10), we get

$$\|nf(nqx) - n^2qf(x)\| \le n\theta\|x\|^r$$

for all $x \in A$. So
$$\|f(x) - \frac{1}{nq}f(nqx)\| \le \frac{\theta}{nq}\|x\|^r$$

for all $x \in A$. So

$$\|\frac{1}{(nq)^l}f((nq)^l x) - \frac{1}{(nq)^{l+m}}f((nq)^{l+m}x)\| \tag{2.29}$$
$$\le \sum_{j=l}^{l+m-1}\|\frac{1}{(nq)^j}f((nq)^j x) - \frac{1}{(nq)^{j+1}}f((nq)^{j+1}x)\| \tag{2.30}$$
$$\le \frac{\theta}{nq}\sum_{j=l}^{l+m-1}\frac{(nq)^{rj}}{(nq)^j}\|x\|^r \tag{2.31}$$

for all nonnegative integers m and l with $x \in A$. It follows from (2.29) that the sequence $\{\frac{1}{(nq)^m}f((nq)^m x)\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{\frac{1}{(nq)^m}f((nq)^m x)\}$ converges. So one can define the mapping $L : A \to A$ by

$$L(x) := \lim_{m\to\infty}\frac{1}{(nq)^m}f((nq)^m x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.29), we get

$$\|f(x) - L(x)\| \leq \frac{\theta}{nq} \sum_{j=0}^{m-1} \frac{(nq)^{rj}}{(nq)^j} \|x\|^r$$

for all $x \in A$. So (2.28) holds for all $x \in A$. The rest of the proof is similar to the proof of Theorem 2.3. □

Theorem 2.5. *Assume that $r > 1$ if $nq > 1$ and that $0 < r < 1$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.11)–(2.13) *and*

$$\|D_\mu f(x_1, \ldots, x_n)\| \leq \theta \prod_{j=1}^{n} \|x_j\|^r, \tag{2.32}$$

such that

$$\|f([x, y, z]) - [f(x)yz]_{(g,h,k)} - [f(z)yx]_{(g,h,k)}\| \leq \theta \|x\|^r \|y\|^r \|z\|^r \tag{2.33}$$

for all $x, y, z \in A$. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*–Lie ternary (σ, τ, ξ)–derivation $L : A \to A$ satisfying (2.15)-(2.17) *such that*

$$\|f(x) - L(x)\| \leq \frac{\theta}{n((nq)^{nr} - nq)} \|x\|^{nr}.$$

Proof . Let the mapping $L : A \to A$ be defined by

$$L(x) := \lim_{m\to\infty} (nq)^m f(\frac{x}{(nq)^m}),$$

for all $x \in A$. It follows from (2.33) that

$$\begin{aligned}
&\|L([x, y, z]) - [L(x)yz]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}\| \\
&= \lim_{m\to\infty} (nq)^{3m} \left\| f\left(\frac{[x, y, z]}{(nq)^{3m}}\right) - \left[f\left(\frac{x}{(nq)^m}\right) \frac{y}{(nq)^m} \frac{z}{(nq)^m}\right]_{(g,h,k)} \right. \\
&\quad \left. + \left[f\left(\frac{z}{(nq)^m}\right) \frac{y}{(nq)^m} \frac{x}{(nq)^m}\right]_{(g,h,k)} \right\| \\
&\leq \lim_{m\to\infty} \frac{(nq)^{3m}\theta}{(nq)^{3mr}} (\|x\|^r.\|y\|^r.\|z\|^r) = 0
\end{aligned}$$

for all $x \in A$. So

$$L([x, y, z]) = [L(x)yz]_{(\sigma,\tau,\xi)} - [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x, y, z \in A$ and the proof of the theorem is complete. □

Theorem 2.6. *Assume that $r > 1$ if $nq < 1$ and that $0 < r < 1$ if $nq > 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.11)–(2.13), (2.32) *and* (2.33). *Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*–ternary (σ, τ, ξ)–derivation $D : A \to A$ satisfying* (2.25)–(2.27) *such that*

$$\|f(x) - L(x)\| \leq \frac{\theta}{n(nq - (nq)^{nr})} \|x\|^{nr}. \tag{2.34}$$

Proof . Letting $\mu = 1$ and $x_1 = \cdots = x_n = x$ in (2.32), we get

$$\|nf(nqx) - n^2qf(x)\| \le \theta\|x\|^{nr}$$

for all $x \in A$. So

$$\|f(x) - \frac{1}{nq}f(nqx)\| \le \frac{\theta}{n^2q}\|x\|^{nr}$$

for all $x \in A$. So

$$\|\frac{1}{(nq)^l}f((nq)^l x) - \frac{1}{(nq)^{l+m}}f((nq)^{l+m}x)\| \tag{2.35}$$

$$\le \sum_{j=l}^{l+m-1} \|\frac{1}{(nq)^j}f((nq)^j x) - \frac{1}{(nq)^{j+1}}f((nq)^{j+1}x)\| \tag{2.36}$$

$$\le \frac{\theta}{n^2q}\sum_{j=l}^{l+m-1} \frac{(nq)^{nrj}}{(nq)^j}\|x\|^{nr} \tag{2.37}$$

for all nonnegative integers m and l with $x \in A$. It follows from (2.35) that the sequence $\{\frac{1}{(nq)^m}f((nq)^m x)\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{\frac{1}{(nq)^m}f((nq)^m x)\}$ converges. So one can define the mapping $L : A \to A$ by

$$L(x) := \lim_{m\to\infty} \frac{1}{(nq)^m}f((nq)^m x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.35), we get (2.34). Uniqueness L is similar to the proof of Theorem 3.1. Also there exist unique linear mappings σ, τ and ξ on A similar to the proof of Theorem 2.3. It follows from (2.33) that

$$\begin{aligned}
&\|L([x, y, z]) - [L(x)yz]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}\| \\
&= \lim_{m\to\infty} \frac{1}{(nq)^{3m}}\|f((nq)^{3m}[x, y, z]) - [f((nq)^m x)(nq)^m y(nq)^m z]_{(g,h,k)} \\
&\quad + [f((nq)^m z)(nq)^m y(nq)^m x]_{(g,h,k)}\| \\
&\le \lim_{m\to\infty} \frac{(nq)^{3mr}\theta}{(nq)^{3m}}(\|x\|^r.\|y\|^r.\|z\|^r) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$L([x, y, z]) = [L(x)yz]_{(\sigma,\tau,\xi)} - [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x, y, z \in A$ and the proof of the theorem is complete. □

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan. 2 (1950) 64–66.

[2] F. Bagarello and G. Morchio, *Dynamics of mean–field spin models from basic results in abstract differential equations*, J. Stat. Phys. 66 (1992) 849–866.

[3] N. Bazunova, A. Borowiec and R. Kerner, *Universal differential calculus on ternary algebras*, Lett. Matt. Phys. 67 (2004) 195–206.

[4] A. Cayley, *On the* 34 *concomitants of the ternary cubic*, Am. J. Math. 4 (1881) 1–15.

[5] S. Czerwik (ed), *Stability of Functional Equations of Ulam–Hayers–Rassias Type*, Hadronic Press, (2003).

[6] M. Eshaghi and S. Abbaszadeh, *Approximate generalized derivations close to derivations in Lie C*-algebras*, J. Appl. Anal., 21 (2015) 37-43.
[7] P. Gavrtua, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 184 (1994) 431–436.
[8] M.E. Gordji and S. Abbaszadeh, *Theory of Approximate Functional Equations: In Banach Algebras, Inner Product Spaces and Amenable Groups*, Academic Press, 2016.
[9] R. Haag and D. Kastler, *An algebraic approach to quantum field theory*, J. Math. Phys. 5 (1964) 848–861.
[10] S. M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic press, Palm Harbor, Florida (2001).
[11] M. Kapranov, I.M. Gelfand and A. Zelevinskii, *Discrimininants, Resultants and Multidimensional Determinants*, Birkhauser, Berlin (1994).
[12] R. Kerner, *Ternary algebraic structures and their applications in physics*, Univ. P. M. Curie preprint, Paris (2000).
[13] R. Kerner, *The cubic chessboard*, Geometry and physics, Class. Quantum Grav. 14, A 203 (1997).
[14] C. Park, *Generalized Hyers–Ulam Stability of C^*–Ternary Algebtra Homomorphisms*, Math. Anal. 16 (2009) 67–79.
[15] C. Park, *Homomorphisms between Poisson JC^*–algebras*, bull. Braz. Math. Soc. 36 (2005) 79–97.
[16] C. Park, *Hyers–Ulam–Rassias stability of a generalized Euler–Lagrange type additive mapping and isomorphisms between C^*–algebras*, Bull. Belgian Math. Soc. Simon Stevin 13 (2006) 619–631.
[17] Th.M. Rassias, *On the stability of the linear mapping in banach spaces*, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[18] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. 62 (2000) 23–130.
[19] Th.M. Rassias (ed), *Functional Equations and Inequalities*, Kluwer Academic Publishers, Dordrecht, Boston, London (2000).
[20] S.M. Ulam, *Problems in Modern Mathematics*, Chapter VI, science ed. Wiley, New York 1940.
[21] H. Zettl, *A characterization of ternary rings of operators*, Adv. Math. 48 (1983) 117–143.