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On fixed points of fundamentally nonexpansive mappings in Banach spaces

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Abstract

We first obtain some properties of a fundamentally nonexpansive self-mapping on a nonempty subset of a Banach space and next show that if the Banach space is having the Opial condition, then the fixed points set of such a mapping with the convex range is nonempty. In particular, we establish that if the Banach space is uniformly convex, and the range of such a mapping is bounded, closed and convex, then its the fixed points set is nonempty, closed and convex.

Keywords: fixed point; fundamentally nonexpansive mappings; nonexpansive mappings; Opial's condition; uniformly convex Banach spaces.
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1. Introduction and Preliminaries

Let K be a nonempty subset of a Banach space $(X, \|\cdot\|)$, and let $T : K \to K$ be a nonexpansive mapping or a generalized nonexpansive mapping. Under the various conditions on K, X and T, existence a fixed point for T has been investigated, and some fixed point theorems and convergence theorems for T have been presented by many authors, for example, see [2, 3, 4, 5, 7, 6].

In this paper, we give many fixed point theorems and convergence theorems for fundamentally nonexpansive self-mappings in Banach spaces and show that under certain conditions, the fixed points set of such self-mappings is nonempty, closed and convex.

We now review the needed preliminaries. Let K be a nonempty subset of a Banach space $(X, \|\cdot\|)$. Throughout this note, we will use $\mathbb{N}$ to denote the set of all positive integers, $\mathbb{R}$ to denote the set of all real numbers, $B(x, r)$ to denote the open ball with center x and radius r, $cl(K)$ to denote the closure of K and $x_n \rightharpoonup x$ to denote the weak convergence of the sequence $\{x_n\}$ in X to $x \in X$, respectively.

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Let T be a self-mapping of K. The mapping T is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$. We denote by $F(T)$ the fixed points set of T, i.e., $F(T) = \{x \in K : Tx = x\}$. T is called quasi-nonexpansive if its fixed points set is nonempty and $\|Tx - u\| \leqslant \|x - u\|$ for all $x \in K$ and $u \in F(T)$. The mapping T is said to satisfy condition (C) [6] if $\frac{1}{2}\|x - Tx\| \leqslant \|x - y\|$ implies $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$. One can verify that condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

You can find the following definition in [3].

Definition 1.1. Let K be a nonempty subset of a normed space $(X, \|\cdot\|)$. A mapping $T : K \to X$ is said to be fundamentally nonexpansive if it satisfies

$$\|T^2x - Ty\| \leqslant \|Tx - y\| \quad \text{for each } x, y \in K.$$

It is clear that fundamental nonexpansiveness is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

Example 1.2. Define a mapping T on $[0, 4]$ as follows:

$$Tx = \begin{cases} 1 & \text{if x} \neq 4 \\ 2.5 & \text{if x} = 4 \end{cases}$$

for all $x \in [0, 4]$. Then T is fundamentally nonexpansive, but it dose not satisfy condition (C). Therefore, T is not nonexpansive.

Let X be a Banach space. X is said to satisfy the Opial condition [5] if whenever a sequence $\{x_n\}$ in X converges weakly to $x \in X$, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \in X$ with $x \neq y$. It is said to be strictly convex if $\left\|\frac{x+y}{2}\right\| < 1$ for each $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$ (see [1]). X is said to be uniformly convex [1] if for any $\epsilon \in (0, 2]$ there exists some $\delta = \delta(\varepsilon) > 0$, whenever $x, y \in X$, $\|x\| \leqslant 1, \|y\| \leqslant 1$ and $\|x - y\| \geqslant \epsilon$, then $\left\|\frac{x+y}{2}\right\| \leqslant 1 - \delta$. It is evident that uniform convexity implies strict convexity.

Let K be a nonempty subset of a Banach space X, and let $\{x_n\}$ be a bounded sequence in X. Define the function $r_a(., \{x_n\}) : X \to [0, \infty)$ by

$$r_a(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\| \quad \text{for each } x \in X.$$

The function $r_a(., \{x_n\})$ is convex and weakly lower semicontinuous (see [1, page 128]). The infimum of $r_a(., \{x_n\})$ over K is said to be the asymptotic radius of $\{x_n\}$ with respect to K and denoted by $r_a(K, \{x_n\})$. A point $u \in K$ is said to be asymptotic center of the sequence $\{x_n\}$ with respect to K if $r_a(u, \{x_n\}) = r_a(K, \{x_n\})$. The set of all asymptotic center of the sequence $\{x_n\}$ with respect to K is denoted by $Z_a(K, \{x_n\})$ (see [1] for more details).

Definition 1.3. ([1]) Let K be a nonempty subset of a Banach space X, and let T be a self-mapping of K. A sequence $\{x_n\}$ in K is called an approximate fixed point sequence for T if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

The following lemma can be found in [2]

Lemma 1.4. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that

$$x_{n+1} = \lambda y_n + (1 - \lambda)x_n \quad \text{and} \quad \|y_n - y_{n+1}\| \leqslant \|x_n - x_{n+1}\|$$

for all $n \in \mathbb{N}$, where $\lambda \in (0, 1)$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

2. Main results

The following lemmas are useful to prove our main results.

Lemma 2.1. *Let K be a nonempty subset of a normed space X, and let $T : K \to X$ be a fundamentally nonexpansive mapping. Then*

$$\|x - Ty\| \leqslant 3\|x - Tx\| + \|x - y\|$$

holds for each $x, y \in K$.

Proof . For each $x, y \in K$, we have

$$\begin{aligned} \|x - Ty\| &\leqslant \|x - Tx\| + \|Tx - T^2x\| + \|T^2x - Ty\| \\ &\leqslant 2\|x - Tx\| + \|Tx - y\| \\ &\leqslant 3\|x - Tx\| + \|x - y\|. \end{aligned}$$

Therefore, we obtain the demanded result. □

The following lemma is easy to verify.

Lemma 2.2. Let K be a nonempty subset of a normed space X, and let $T : K \to X$ be a mapping. Then the following hold:

(i) If T is nonexpansive, then it is fundamentally nonexpansive.

(ii) If T is fundamentally nonexpansive, and $F(T)$ is nonempty, then T is quasi-nonexpansive.

Lemma 2.3. *Let T be a fundamentally nonexpansive self-mapping on a nonempty subset K of a Banach space X, and let $T(K)$ be bounded and convex. Define a sequence $\{Tx_n\}$ in $T(K)$ by $x_1 \in K$ and*

$$Tx_{n+1} = \lambda T^2x_n + (1-\lambda)Tx_n \quad \text{for all } n \in \mathbb{N},$$

where $\lambda \in (0, 1)$. Then $\{Tx_n\}$ is an approximate fixed point sequence for T.

Proof . Since T is fundamentally nonexpansive, we have

$$\|T^2x_{n+1} - T^2x_n\| \leqslant \|Tx_{n+1} - Tx_n\| \text{ for all } n \in \mathbb{N}.$$

Lemma 1.4 implies that $\lim_{n\to\infty} \|Tx_n - T^2x_n\| = 0$. This completes the proof of the lemma. □

Proposition 2.4. Let $T : K \to K$ be a fundamentally nonexpansive mapping, where K is a nonempty subset of a Banach space X. Then $F(T)$ is closed. Moreover, if X is strictly convex, and K or $T(K)$ is convex, then $F(T)$ is also convex.

Proof . Suppose, for contradiction, that there is an element x of $cl(F(T))$ such that $x \notin F(T)$. Set $r = \frac{\|x-Tx\|}{3}$. Since $x \in cl(F(T))$, $B(x, r) \cap F(T)$ is nonempty. Let $u \in B(x, r) \cap F(T)$. Then $\|x - u\| < r$ and $Tu = u$ hold. By Lemma 2.2 (ii), we have

$$\begin{aligned} \|x - Tx\| &\leqslant \|x - u\| + \|Tx - u\| \\ &\leqslant 2\|x - u\| \\ &< 2r \\ &= \tfrac{2\|x-Tx\|}{3}, \end{aligned}$$

which is a contradiction. Thus F(T) is closed. Assume that X is strictly convex, we show that $F(T)$ is convex. Let $\lambda \in (0,1)$ and $x, y \in F(T)$ with $x \neq y$. Put $u = \lambda x + (1-\lambda)y$. By Lemma 2.2 (ii), we have

$$\begin{aligned} \|x-y\| &\leqslant \|Tu-x\| + \|Tu-y\| \\ &\leqslant \|u-x\| + \|u-y\| \\ &= \|x-y\|. \end{aligned}$$

Thus,

$$\|x-y\| = \|Tu-x\| + \|Tu-y\| = \|u-x\| + \|u-y\|. \tag{2.1}$$

As X is strictly convex, Proposition 2.1.7 of [1] implies that there exists $t > 0$ such that $Tu - y = t(x - Tu)$. So $Tu = \frac{t}{t+1}x + \frac{1}{1+t}y$. From (2.1), we conclude that

$$\|Tu-x\| = \|u-x\| \text{ and } \|Tu-y\| = \|u-y\|,$$

which implies $\lambda = \frac{t}{t+1}$. Hence $Tu = \lambda x + (1-\lambda)y = u$, that is, $u \in F(T)$. Therefore, we obtain the desired result. □

Theorem 2.5. *Let K be a nonempty compact subset of a Banach space X. Assume that $T : K \to K$ is fundamentally nonexpansive, and $T(K)$ is convex. Then the sequence $\{Tx_n\}$ in $T(K)$ defined by $x_1 \in K$ and*

$$Tx_{n+1} = \lambda T^2 x_n + (1-\lambda)Tx_n \quad \textit{for all} \ \ n \in \mathbb{N}, \tag{2.2}$$

where $\lambda \in (0,1)$, converges strongly to a fixed point of T.

Proof . Since K is compact, there is a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\lim\limits_{k\to\infty} Tx_{n_k} = u$ for some $u \in K$. From (2.2), we obtain $\lim\limits_{k\to\infty} T^2 x_{n_k} = u$. Since T is fundamentally nonexpansive, we have

$$\|T^2 x_{n_k} - Tu\| \leqslant \|Tx_{n_k} - u\| \quad \text{for all } k \in \mathbb{N}.$$

This implies that $\lim\limits_{k\to\infty} T^2 x_{n_k} = Tu$, hence u is a fixed point of T. On the other hand, we have

$$\begin{aligned} \|Tx_{n+1} - u\| &= \|\lambda T^2 x_n + (1-\lambda)Tx_n - Tu\| \\ &\leqslant \lambda \|T^2 x_n - Tu\| + (1-\lambda)\|Tx_n - u\| \\ &\leqslant \|Tx_n - u\| \end{aligned}$$

for all $n \in \mathbb{N}$. It follows that the sequence $\{\|Tx_n - u\|\}$ is bounded and decreasing, hence it is convergent. As $\lim\limits_{k\to\infty} Tx_{n_k} = u$, we conclude that $\lim\limits_{n\to\infty} Tx_n = u$. □

Theorem 2.6. *Let K be a nonempty weakly compact subset of a Banach space X with the Opial condition. Suppose $T : K \to K$ is a fundamentally nonexpansive mapping, and $T(K)$ is convex. Then the sequence $\{Tx_n\}$ in $T(K)$ defined by $x_1 \in K$ and*

$$Tx_{n+1} = \lambda T^2 x_n + (1-\lambda)Tx_n \quad \textit{for all } n \in \mathbb{N},$$

where $\lambda \in (0,1)$, converges weakly to a fixed point of T.

Proof . Since K is weakly compact, there is a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \rightharpoonup u \in K$ as $k \to \infty$. We next show that u is a fixed point of T. Suppose, for contradiction, that $Tu \neq u$. From Lemma 2.1, we obtain

$$\|Tx_{n_k} - Tu\| \leqslant 3\|Tx_{n_k} - T^2x_{n_k}\| + \|Tx_{n_k} - u\| \text{ for all } k \in \mathbb{N}.$$

The above inequality and Lemma 2.3 imply

$$\liminf_{k\to\infty} \|Tx_{n_k} - Tu\| \leqslant \liminf_{k\to\infty} \|Tx_{n_k} - u\|,$$

which contradicts the Opial condition. Therefore, u is a fixed point of T. We now assert that the sequence $\{Tx_n\}$ converges weakly to u. Suppose, for contradiction, that there is a subsequence $\{Tx_{m_j}\}$ of $\{Tx_n\}$ such that $Tx_{m_j} \rightharpoonup v \in K$ as $j \to \infty$ with $u \neq v$. Similarly, one can show that v is a fixed point of T. We know that $\lim_{n\to\infty} \|Tx_n - u\|$ and $\lim_{n\to\infty} \|Tx_n - v\|$ are existent, so the Opial condition implies that

$$\begin{aligned}\lim_{n\to\infty} \|Tx_n - u\| &= \liminf_{k\to\infty} \|Tx_{n_k} - u\| \\ &< \liminf_{k\to\infty} \|Tx_{n_k} - v\| \\ &= \liminf_{j\to\infty} \left\|Tx_{m_j} - v\right\| \\ &< \liminf_{j\to\infty} \left\|Tx_{m_j} - u\right\| \\ &= \lim_{n\to\infty} \|Tx_n - u\|,\end{aligned}$$

which is a contradiction. Therefore, $Tx_n \rightharpoonup u$ as $n \to \infty$. □

Theorem 2.7. *Let K be a nonempty subset of a uniformly convex Banach space X, and let T be a fundamentally nonexpansive self-mapping of K. If $T(K)$ is bounded, closed and convex, then the fixed points set of T is nonempty, closed and convex.*

Proof . Define a sequence $\{Tx_n\}$ in $T(K)$ by $x_1 \in K$ and

$$Tx_{n+1} = \tfrac{1}{2}T^2x_n + \tfrac{1}{2}Tx_n \text{ for all } n \in \mathbb{N}.$$

The theorem assumptions and Theorem 2.2.8 of [1] imply that X is reflexive. Now, consider the convex function $r_a(., \{Tx_n\})$ on X. As $r_a(., \{Tx_n\})$ is weakly lower semicontinuous, Theorem 1.9.20 in [1] along with Theorem 2.5.4 in [1] imply that there exists an element $u \in K$ such that

$$r_a(Tu, \{Tx_n\}) = \inf\{r_a(Tx, \{Tx_n\}) : x \in K\}. \tag{2.3}$$

On the other hand, X is uniformly convex, hence Theorem 3.1.5 of [1] yields $Z_a(T(K), \{Tx_n\})$ is a singleton set. From (2.3), we get $Z_a(K, \{Tx_n\}) = \{Tu\}$. By Lemma 2.1, we have

$$\|Tx_n - T^2u\| \leqslant 3\|Tx_n - T^2x_n\| + \|Tx_n - Tu\| \quad \text{for all } n \in \mathbb{N}.$$

The above inequality and Lemma 2.3 imply $r_a(T^2u, \{Tx_n\}) \leqslant r_a(Tu, \{Tx_n\})$. So $r_a(T^2u, \{Tx_n\}) = r_a(Tu, \{Tx_n\})$, that is, $T^2u \in Z_a(T(K), \{Tx_n\})$. Thus, $T(Tu) = Tu$, i.e., the fixed points set of T is nonempty. Since X is uniformly convex, it is strictly convex. Hence Proposition 2.4 implies that $F(T)$ is closed and convex. Therefore, we obtain the desired result. □

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