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(JCLR) property and fixed point in non-Archimedean fuzzy metric spaces

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Abstract

The aim of the present paper is to introduce the concept of joint common limit range property ((JCLR) property) for single–valued and set–valued maps in non–Archimedean fuzzy metric spaces. We also list some examples to show the difference between (CLR) property and (JCLR) property. Further, we establish common fixed point theorems using implicit relation with integral contractive condition. Several examples to illustrate the significance of our results are given.

Keywords: fixed point; (JCLR) property; non–Archimedean fuzzy metric space; hybrid map.

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1. Introduction

Kramosil and Michalek [7] introduced the notion of fuzzy metric spaces (FMS) as a generalization of probabilistic metric spaces by using continuous t–norms. George and Veeramani [6] modified the Kramosil and Michalek [7] notion and obtained a Hausdorff topology on these FMS. Recently, some results in non-Archimedean FMS appeared [1, 3, 13].

On the other hand, Sintunavarat and Kumam [14] generalized (E.A) property into the notion of (CLR) property. Also, Chauhan et al. [5] introduced the concept of (JCLR) property in FMS. For more results on (CLR) property in FMS, see [2, 4, 8, 11] and others.

In this paper, we introduce the notion of (JCLR) property for single–valued and set–valued maps in non–Archimedean FMS. Further, we obtain coincidence and fixed points using implicit relation.

Now, we list some basic definitions.

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Definition 1.1. (Schweizer and Sklar [12]) A binary operation $\star : [0,1]^2 \to [0,1]$ is said to be continuous t–norm, if

(i) $\star$ is commutative and associative;
(ii) $\star$ is continuous;
(iii) $a \star 1 = a$ for all $a \in [0,1]$;
(iv) $a \star b \leq c \star d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Two classical examples of t–norms are $a \star b = ab$ and $a \star b = \min\{a, b\}$.

Definition 1.2. (George and Veeramani [6]) Let X be a nonempty set, $\star$ be a continuous t–norm and M be a fuzzy set on $X \times X \times (0,\infty)$. If the following conditions are satisfied for all $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ iff $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, z, t + s) \geq M(x, y, t) \star M(y, z, s)$;
(v) $M(x, y, .) : (0,\infty) \to (0,1]$ is continuous,

then $(X, M, \star)$ is said to be a FMS. One may replace the triangle inequality (iv) with $M(x, z, \max\{t, s\}) \geq M(x, y, t) \star M(y, z, s)$, in this case the triplet $(X, M, \star)$ is said to be a *Non–Archimedean FMS.*

Definition 1.3. (George and Veeramani [6]) (i) A sequence $\{x_n\}$ in a FMS $(X, M, \star)$ is said to be convergent to $x \in X$, if $\lim\limits_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(ii) A subset $A \subset X$ is called open, if for each x in A, there exist $t > 0$ and $0 < r < 1$ such that $\{y \in X : M(x, y, t) > 1 - r\} \subset A$. A subset B of X is called closed if its complement is open.
(iii) A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

Definition 1.4. (Rodríguez–López and Romaguera [10]) Let $CP(X)$ be the set of all nonempty compact subsets of a FMS $(X, M, \star)$. Then for every $A, B \in CP(X)$ and $t > 0$,

$$M(A, B, t) = \min\{\inf_{a\in A} M(a, B, t), \inf_{b\in B} M(A, b, t)\},$$

also $(M, \star)$ is a fuzzy metric on $CP(X)$.

Definition 1.5. (Ahmed and Nafadi [2]) Let $CL(X)$ be the set of all nonempty closed subsets of a FMS $(X, M, \star)$. Two mappings $f : X \to X$ and $F : X \to CL(X)$ are said to be satisfy the (CLR$_{\mathrm{f}}$) property if there exists a sequence $\{x_n\}$ in X such that for some $u, v \in X$,

$$\lim_{n\to\infty} fx_n = u = fv \in A = \lim_{n\to\infty} Fx_n.$$

Definition 1.6. (Beg et al. [4]) Let $(X, M, \star)$ be a non–Archimedean FMS, $F, G : X \to CL(X)$ and $f, g : X \to X$. Then (f, F) and (g, G) are said to have the (JCLR) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X and $A, B \in CL(X)$ such that for some $u, v, w \in X$ and $u \in A \cap B$:

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u = fv = gw, \lim_{n\to\infty} Fx_n = A, \lim_{n\to\infty} Gy_n = B.$$

Remark 1.7. (i) It is clear that (JCLR) property generalizes (CLR) property but the converse is not true in general.

(ii) If $f = g$, $\{x_n\} = \{y_n\}$ and $F = G$ in Definition 1.6, then (JCLR) property reduces to (CLR) property.

Example 1.8. Let $X = [0, 1]$. Define the maps f, g, F, G on X as $fx = \frac{x}{2}$, $gx = \frac{x}{3}$,

$$Fx = \begin{cases} \left[0, \frac{x}{2}\right), & \text{if} \quad 0 \leq x < \frac{1}{3}, \\ \left[\frac{x}{2}, 1\right], & \text{if} \quad \frac{1}{3} \leq x \leq 1 \end{cases}$$

and

$$Gx = \begin{cases} \left[0, \frac{x}{3}\right), & \text{if} \quad 0 \leq x < \frac{1}{2}, \\ \left[\frac{x}{3}, 1\right], & \text{if} \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

for all $x, y \in X$. Define two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\{x_n\} = \{\frac{1}{3} + \frac{1}{n}\}, \{y_n\} = \{\frac{1}{2} + \frac{1}{2n}\}, n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} fx_n = \frac{1}{6}$ and $\lim\limits_{n\to\infty} Fx_n = [\frac{1}{6}, 1]$, then $\lim\limits_{n\to\infty} fx_n = \frac{1}{6} \in [\frac{1}{6}, 1] = \lim\limits_{n\to\infty} Fx_n$. Similarly, $\lim\limits_{n\to\infty} gy_n = \frac{1}{6}$ and $\lim\limits_{n\to\infty} Gy_n = [\frac{1}{6}, 1]$. Now, $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gy_n = \frac{1}{6} \in [\frac{1}{6}, 1] = \lim\limits_{n\to\infty} Fx_n = \lim\limits_{n\to\infty} Gy_n$, that is, the hybrid pairs (f, F) and (g, G) satisfy the property (JCLR). If $f = g$, $F = G$ and $\{x_n\} = \{y_n\}$, then (f, F) satisfies the property (CLR).

Example 1.9. In Example 1.8, if $gx = \frac{x}{5}$, $\{y_n\} = \{\frac{1}{2} + \frac{1}{2n}\}$ and

$$Gx = \begin{cases} \left[0, \frac{x}{5}\right), & \text{if} \quad 0 \leq x < \frac{1}{2}, \\ \left[\frac{x}{5}, 1\right], & \text{if} \quad \frac{1}{2} \leq x \leq 1, \end{cases}$$

then $\lim\limits_{n\to\infty} gy_n = \frac{1}{10}$ and $\lim\limits_{n\to\infty} \{Gy_n\}_{\frac{1}{4}} = [\frac{1}{10}, 1]$, i.e., both of (f, F) and (g, G) satisfy the property (CLR) but does not satisfy the property (JCLR).

Definition 1.10. (Pathak and Rodríguez–López [9]) Let (X, d) be a metric spaces. A map $f : X \to X$ is said to be occasionally coincidentally idempotent w.r.t. a mapping $F : X \to CL(X)$ if $ffx = fx$ for some $x \in C(f, F)$, where $C(f, F)$ is the set of coincidence point of f and F.

Remark 1.11. (Pathak and Rodríguez–López [9]) Coincidentally idempotent pairs of mappings are occasionally coincidentally idempotent, but the converse is not necessarily true.

Definition 1.12. (Beg et al. [4]) Let $(X, M, \star)$ be a non–Archimedean FMS. A map $f : X \to X$ is said to be F–weakly commuting at $x \in X$ if $ffx \in Ffx$.

Let Φ be the family of continuous mappings $\phi : (0, 1]^4 \to [0, 1]$, which is non–decreasing in the first coordinate and satisfying the following condition for each $u, v \in (0, 1]$: if $\phi(u, v, v, u) \geq 0$ or $\phi(u, v, u, v) \geq 0$, then $u \geq v$.

Example 1.13. (i) $\phi(t_1, t_2, t_3, t_4) = t_1 - \min\{t_2, \frac{t_3+t_4}{2}\}$; (ii) $\phi(t_1, t_2, t_3, t_4) = t_1 - t_2 - \frac{t_3+t_4}{2}$.

Let Ψ be the family of continuous mappings $\psi : (0,1]^3 \to [0,1]$, which is non–decreasing in the first coordinate and satisfying the following condition for each $u, v \in (0,1]$: if $\psi(u,v,v) \geq 0$ or $\psi(u,v,u) \geq 0$, then $u \geq v$.

Example 1.14. (i) $\psi(t_1,t_2,t_3) = t_1 - \frac{t_2+t_3}{2}$; (ii) $\psi(t_1,t_2,t_3) = t_1 - \frac{3(t_2+t_3)}{4}$; (iii) $\psi(t_1,t_2,t_3) = t_1 - \frac{n(t_2+t_3)}{n+1}$, $n \in \mathbb{N}$.

Let Θ be the family of continuous mappings $\theta : (0,1]^2 \to [0,1]$, which is non–decreasing in the first coordinate and satisfying the following condition for each $u, v \in (0,1]$: if $\theta(u,v) \geq 0$ or $\theta(u,v) \geq 0$, then $u \geq v$.

Example 1.15. (i) $\theta(t_1,t_2) = t_1 - t_2$; (ii) $\theta(t_1,t_2) = {t_1}^2 - {t_2}^2$; (iii) $\theta(t_1,t_2) = {t_1}^n - {t_2}^n$; (iv) $\theta(t_1,t_2) = t_1 - nt_2$; (v) $\theta(t_1,t_2) = t_1 - \frac{t_1+t_2}{2}$; (vi) $\theta(t_1,t_2) = t_1 - \frac{3(t_1+t_2)}{4}$; (vii) $\theta(t_1,t_2) = t_1 - \frac{n(t_1+t_2)}{n+1}$.

2. Main results

The first result is the following.

Theorem 2.1. *Let $(X, M, \star)$ be a non–Archimedean FMS, $f, g : X \to X$ and $F, G : X \to 2^X$ such that for each $x \in X$, $Fx, Gx \in CL(X)$, suppose that there exist $\phi \in \Phi$ such that*

$$\int_0^{\phi(Q)} \alpha(s)ds - L\int_0^{\max(Q)} \beta(e)de \geq 0, \tag{2.1}$$

where

$$Q = (M(Fx,Gy,t), M(fx,gy,t), M(fx,Fx,t), M(gy,Gy,t)),$$

where $0 \leq L < 1$ and $\alpha, \beta : [0,\infty) \to [0,\infty)$ are summable non negative Lebesgue integrable functions such that for each $\epsilon \in (0,1]$, $\int_0^\epsilon \alpha(s)ds > 0$ and $\int_0^\epsilon \beta(e)de > 0$. If (f,F) and (g,G) satisfy (JCLR) *property, weakly commuting and occasionally coincidentally idempotent, then f, g, F, G have a common fixed point.*

Proof . By (JCLR) property, we have $\lim_{n\to\infty} Fx_n = A$, $\lim_{n\to\infty} Gy_n = B$, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A \cap B$, where $\{x_n\}, \{y_n\} \in X$, $A, B \in CL(X)$ and $u = fv = gw$, for some $u, v, w \in X$. Now, v is a coincidence point of f and F, that is $fv \in Fv$. To prove this, suppose not, i.e., $M(Fv, fv, t) \neq 1$, since

$$\int_0^{\phi(Q)} \alpha(s)ds - L\int_0^{\max(Q)} \beta(e)de \geq 0,$$

where

$$Q = \phi(M(Fv,Gy_n,t), M(fv,gy_n,t), M(fv,Fv,t), M(gy_n,Gy_n,t)) \geq 0.$$

When $n \to \infty$, we get

$$\int_0^{\phi(M(Fv,B,t),1,M(fv,Fv,t),1)} \alpha(s)ds \geq 0.$$

But $\phi(M(Fv,fv,t),1,M(fv,Fv,t),1) \geq \phi(M(Fv,B,t),1,M(fv,Fv,t),1)$, therefore

$$\int_0^{\phi(M(Fv,fv,t),1,M(fv,Fv,t),1)} \alpha(s)ds \geq 0.$$

It further give

$$\phi(M(Fv, fv, t), 1, M(fv, Fv, t), 1) \geq 0.$$

But $\phi(u, v, u, v) \geq 0$ implies $u \geq v$, then $M(Fv, fv, t) \geq 1$, which a contradiction. So that $fv \in Fv$. In similar way, one may use $\phi(u, v, v, u) \geq 0$ and deduce that $gw \in Gw$. By weakly commuting and occasionally coincidentally idempotent, we have $ffv = fv$ and $ffv \in Ffv$. So that $u = fu \in Fu$. Further, $ggw = gw$ and $ggw \in Ggw$ implies $u = gu \in Gu$. Then f, g, F, G have a common fixed point. This concludes the proof. □

Now, we list two examples to illustrate Theorem 2.1.

Example 2.2. Let $(X, M, \star)$ be a non–Archimedean FMS where $X = [0, 1], a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$for all $t > 0$ and $x, y \in X$. Define $\phi : [0, 1]^4 \to [0, 1]$ as $\phi(t_1, t_2, t_3, t_4) = t_1 - t_2$. Also, we define the maps F, G, f, g on X as $Fx = [\frac{3x}{5}, 1], Gx = [x^3, 1]$, $fx = \frac{x}{5}$ and $gx = \frac{x}{2}$ for all $x, y \in X$. Define two sequences $\{x_n\} = \{\frac{1}{n}\}, \{y_n\} = \{\frac{1}{2n}\}, n \in \mathbb{N}$, in X. As $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = 0 \in [0, 1] = \lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gy_n$, then (f, F) and (g, G) are satisfy (JCLR) property. Further, $ff0 = f0 \in [0, 1] = Ff0$ and $gg0 = g0 \in [0, 1] = Gg0$, i.e., (f, F) and (g, G) are weakly commuting and occasionally coincidentally idempotent besides

$$\int_0^{\phi(M(Fx_n,Gy_n,t),M(fx_n,gy_n,t),M(fx_n,Fx_n,t),M(gy_n,Gy_n,t))} \alpha(s)ds = 0,$$

which gives

$$\phi(M(Fx_n, Gy_n, t), M(fx_n, gy_n, t), M(fx_n, Fx_n, t), M(gy_n, Gy_n, t)) = 0.$$

Thus, all the conditions of Theorem 2.1 are satisfied and 0 is a common fixed point for the maps f, g, F, G.

Example 2.3. Let $(X, M, \star)$ as in Example 2.2. Define the maps f, g, F, G on X as $fx = gx = \frac{1}{6}$,

$$Fx = \begin{cases} \left[0, \frac{1}{6}\right), & \text{if} \quad 0 \leq x < \frac{1}{3}, \\ \left[\frac{1}{6}, 1\right], & \text{if} \quad \frac{1}{3} \leq x \leq 1 \end{cases}$$

and

$$Gx = \begin{cases} \left[0, \frac{1}{6}\right), & \text{if} \quad 0 \leq x < \frac{1}{2}, \\ \left[\frac{1}{6}, 1\right], & \text{if} \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

for all $x, y \in X$. Define two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\{x_n\} = \{\frac{1}{3} + \frac{1}{n}\}, \{y_n\} = \{\frac{1}{2} + \frac{1}{2n}\}, n \in \mathbb{N}$. Also, let $\phi(t_1, t_2, t_3, t_4) = {t_1}^2 - {t_2}^2$, then all conditions of Theorem 2.1 are satisfied and $\frac{1}{6}$ be common fixed point of (f, F) and (g, G).

Remark 2.4. (i) In Theorem 2.1, we can replace the equation (2.1) by one of the following:

(a) suppose that there exist $\psi \in \Psi$ such that

$$\int_0^{\psi(Q)} \alpha(s)ds - L \int_0^{\max(Q)} \beta(e)de \geq 0,$$

where

$$Q = (M(Fx, Gy, t), M(fx, gy, t), M(fx, Fx, t));$$

(b) suppose that there exist $\theta \in \Theta$ such that

$$\int_0^{\theta(Q)} \alpha(s)ds - L\int_0^{\max(Q)} \beta(e)de \geq 0,$$

where

$$Q = (M(Fx, Gy, t), M(fx, gy, t)).$$

(ii) One may put one of $\beta(e) = 1$ or $\alpha(s)$ or both in Theorem 2.1, in this case, condition (2.1) take the following versions receptively:

$$\int_0^{\phi(Q)} \alpha(s)ds - L\max(Q) \geq 0,$$

$$\phi(Q) - L\int_0^{\max(Q)} \beta(e)de \geq 0,$$

$$\phi(Q) - L\max(Q) \geq 0,$$

where

$$Q = (M(Fx, Gy, t), M(fx, gy, t), M(fx, Fx, t), M(gy, Gy, t)).$$

Corollary 2.5. Let $(X, M, \star)$ be a non–Archimedean FMS, $f : X \to X$ and $F : X \to 2^X$ such that Fx is a closed subset of X there exist $\phi \in \Phi$ such that

$$\phi(M(Fx, Fy, t), M(fx, fy, t), M(fx, Fx, t), M(fy, Fy, t)) \geq 0.$$

If (f, F) are satisfy (CLR) property, weakly commuting and occasionally coincidentally idempotent, then f, g, F, G have a common fixed point.

Corollary 2.6. Let $(X, M, \star)$ be a non–Archimedean FMS, $f, g, F, G : X \to X$ such that there exist $\phi \in \Phi$ such that

$$\phi(M(Fx, Gy, t), M(fx, gy, t), M(fx, Fx, t), M(gy, Gy, t)) \geq 0.$$

If (f, F) and (g, G) are satisfy (JCLR) property, weakly commuting and occasionally coincidentally idempotent, then f, g, F, G have a common fixed point.

Corollary 2.7. Let $(X, M, *)$ be a non–Archimedean FMS, $f, F : X \to X$ such that there exist $\phi \in \Phi$ such that

$$\phi(M(Fx, Fy, t), M(fx, fy, t), M(fx, Fx, t), M(fy, Fy, t)) \geq 0.$$

If (f, F) are satisfy (CLR) property, weakly commuting and occasionally coincidentally idempotent, then f, g, F, G have a common fixed point.

Theorem 2.8. *Let $(X, M, \star)$ be a non–Archimedean FMS, $f, g : X \to X$ and $F_n : X \to 2^X$ such that for each $x \in X$ and $n \in \mathbb{N}$, $Fx \in CL(X)$. Suppose that there exist $\phi \in \Phi$ such that*

$$\int_0^{\phi(Q)} \alpha(s)ds - L \int_0^{\max(Q)} \beta(e)de \geq 0,$$

where

$$Q = (M(F_k x, F_l y, t), M(fx, gy, t), M(fx, F_k x, t), M(gy, F_l y, t)),$$

$0 \leq L < 1$, $k = 2n+1$, $l = 2n+2$ and $\alpha, \beta : [0, \infty) \to [0, \infty)$ are summable non negative lebesgue integrable functions such that for each $\epsilon \in (0, 1]$, $\int_0^\epsilon \alpha(s)ds > 0$ and $\int_0^\epsilon \beta(e)de > 0$. If (f, F_k) and (g, F_l) are satisfy (JCLR) *property, weakly commuting and occasionally coincidentally idempotent, then f, g, F_k, F_l have a common fixed point.*

Proof . Put $F = F_k$ and $G = F_l$ in Theorem 2.1. This concludes the proof. □

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