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The Effects of the Moving Load and the Attached Mass-Spring-Damper System Interactions on the Dynamic Responses of the Composite Plates: An Analytical Approach

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ABSTRACT

In the current study, the effects of interactions of the moving loads and the attached mass-spring-damper systems of the composite plates on the resulting dynamic responses are investigated comprehensively, for the first time, using the classical plate theory. The solution of the coupled governing system of equations is accomplished through tracing the spatial variations using a Navier-type solution and the time variations by means of a Laplace transform. Therefore, the results are exact. The effects of various material, stiffness, and kinematic parameters of the system on the responses are investigated comprehensively and the results are illustrated graphically. Apart from the novelties presented in the modeling and solution stages, some practical conclusions have been drawn such as the fact that the amplitude of vibration increases for both the free and forced vibrations of the plate and the suspended mass, when the magnitude of suspended mass increases.

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1. Introduction

The investigation of the forced vibration behavior of the structures carrying mass-spring systems is an important issue in many engineering applications, including the automotive, aerospace, marine, and civil structures. The control and measurement instruments attached to a control board and the navigation instrumentations, as well as the passengers and sprung masses (such as the differential) of a vehicle may be regarded as the attached spring-mass systems [1,2]. Moreover, it has been experimentally verified that a person or a crowd on a bridge or narrow plate-type structure may be regarded as an attached mass-spring or mass-spring-damper system when the interaction between the human and the structure is of concern [3]. Sometimes, the attached mass-spring system may delib-

erately be included into the structure to serve as a dynamic vibration absorber [4,5].

The vibration of continuous systems with mass attachments has been the topic of some investigations recently. Turhan [6] analyzed the fundamental natural frequencies of an Euler-Bernoulli beam with an attached point mass. Kopmaz and Telli [7] investigated the free transverse vibration of a plate carrying a distributed mass through a simple model. Wong [8] analyzed the free bending vibration of a simply supported rectangular plate carrying distributed mass using the Rayleigh-Ritz method. Chiba and Sugimoto [9] studied the vibration characteristics of a cantilever plate with a spring-mass system attached upon it using Rayleigh-Ritz method. Li [10] represented an exact solution for free vibration analysis of rectangular plates with line-concentrated mass and elastic line-support. The vibratory characteristics of rectangular plates attached with contin-

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ously and uniformly distributed spring-mass which might represent free vibration of a human-structure system were studied by Zhou and Tianjian [11]. Khodzhaev and Éshmatov [12,13] analyzed vibration and stability of viscoelastic plates with several attached masses using the numerical methods. Ciancio et al. [14] studied the free vibrations of a cantilever anisotropic plate carrying a concentrated mass approximately, using the energy method. Alibeigloo et al. [15] studied the free vibration of a simply supported laminated composite plate with a distributed patch mass using Hamilton's principle and a double Fourier series solution. Watkins et al. [16] investigated vibration responses of an elastically point-supported plate with attached masses, experimentally. Using a high-order plate theory, Malekzadeh et al. [17] investigated the free vibration characteristics of plates carrying distributed attached masses and the effects of the stiffness of the attached mass on the frequency of the system were included. In a more recent attempt, Amabili and Carra [18] used an experimental set up to investigate large amplitude vibrations of rectangular plates carrying concentrated masses.

The studies on the dynamic response of composite plates to moving loads and masses are of great importance from both the theoretical and practical points of view. The results of such investigations can be employed in various branches of transportation engineering, including designing road beds and bridges for trains, cars, trucks, etc., parking garages, aircraft runways, high-speed precision machining, magnetic disk drives, and so forth. Most of the previous studies on plates subjected to moving loads were accomplished based on the two-dimensional theories such as the Classical Plate Theory (CPT) and the First-order Shear Deformation Theory (FSDT). Agrawal et al. [19] analyzed the dynamic responses of the orthotropic thin plates subjected to moving masses using Green's function based on the CPT. Taheri and Ting [20] analyzed the moving load response of a thin plate using the finite element approach. Zaman et al. [21] presented a finite element algorithm to evaluate the dynamic response of a thick isotropic plate on viscoelastic foundation subjected to moving loads. Faria and Oguamanam [22] analyzed the vibration of Mindlin plates with moving concentrated loads, using the finite element method. Wu [23] investigated dynamic responses of a rectangular plate undergoing a transverse moving line load. Sun [24] studied the dynamic displacements of a thin plate induced by moving harmonic line and point loads. Malekzadeh et al. [25] accomplished three-dimensional dynamic analysis of laminated composite plates subjected to moving loads using the analytical methods.

Several examples of plates with attached mass-spring systems and moving loads may be presented. Moving of the passengers/trolley across the airplane cabin that is carrying the passengers and seats (as mass and springs), moving of a lift truck on deck of a ship that is conveying goods and connexes, passage of the air flow (the induced aerodynamic forces) over the airplane wing with attached engine (that is usually attached through flexible mounts), and passage of a high speed car on a bridge with some other stationary or low velocity commercial vehicles are some typical examples of a moving load on plates with attached mass-spring systems. In the present study, the dynamic responses of a rectangular composite plate with one or two attached mass-spring-damper systems to a moving load have been investigated analytically, for the first time. Since the method is fully analytical (including Laplace transform and Navier solutions) the results are exact. For the simple case of a symmetric cross-ply laminated plate, the present results are compared with the results of the available solutions and a good concordance is noticed. The effects of various parameters (i.e. magnitude of the suspended mass, coefficients of the spring and damper and velocity and magnitude of moving load) on the time history and the frequencies of vibrations of the plate and suspended mass have been investigated comprehensively and illustrated graphically.

2. The Governing Equations of the Problem

According to the classical plate theory, the governing equations of motion of the composite plate are written as follows in the absence of the in-plane forces [26].

$$\begin{aligned} N_{xx,x} + N_{xy,y} &= I_0 \ddot{u}_0 - I_1 \ddot{w}_{0,x} \\ N_{yx,x} + N_{yy,y} &= I_0 \ddot{v}_0 - I_1 \ddot{w}_{0,y} \\ M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + q(x, y, t) &= I_0 \ddot{w}_0 + I_1 (\ddot{u}_{0,x} + \ddot{v}_{0,y}) - I_2 (\ddot{w}_{0,xx} + \ddot{w}_{0,yy}) \end{aligned} \quad (1)$$

where $(I_0, I_1, I_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, \hat{z}, \hat{z}^2) \rho_0 d\hat{z}$, and

$$\begin{aligned} \{N_{yy}\} &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \{v_{0,y}\} - \\ &\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} w_{0,xx} \\ w_{0,yy} \\ 2w_{0,xy} \end{Bmatrix} \end{aligned} \quad (2)$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{21} & B_{22} & B_{26} \\ B_{61} & B_{62} & B_{66} \end{bmatrix} \begin{Bmatrix} u_{0,x} \\ v_{0,y} \\ u_{0,y} + v_{0,x} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{21} & D_{22} & D_{26} \\ D_{61} & D_{62} & D_{66} \end{bmatrix} \begin{Bmatrix} w_{0,xx} \\ w_{0,yy} \\ 2w_{0,xy} \end{Bmatrix} \quad (3)$$

where

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}^{(k)} (1, \hat{z}, \hat{z}^2) d\hat{z} \quad (4)$$

where \hat{z} is measured for the mid-plane of the plate.

Assuming that: (i) the in-plane displacement components u and v , have no direct effects on the suspended mass-spring system, i.e., the horizontal displacements of the mass are restrained and negligible, (ii) the deflections of the plate are small, and (iii) the equivalent spring exhibits a linear behavior, the mass-spring system can be modeled as a system of one degree of freedom whose motion can be described by the absolute vertical position of the suspended mass $z(t)$ shown in Fig. 1. Therefore for the plate shown in Fig. 1, using Newton's second law of motion and the Dirac delta function (δ), the distributed transverse load is expressed as follows:

$$q(x, y, t) = M(g - \ddot{z})\delta(x - x_0)\delta(y - y_0) + f(x, y, t) \quad (5)$$

where $f(x, y, t)$ can be described based on Fig. 1, as follows:

$$f(x, y, t) = P_0\delta(x - V_x t)\delta(y - b_0) \quad (6)$$

On the other hand, the governing Equation of the suspended mass can be written as:

$$M\ddot{z} + C\dot{z} + Kz = Kw_0(x_0, y_0, t) + C\dot{w}_0(x_0, y_0, t) \quad (7)$$

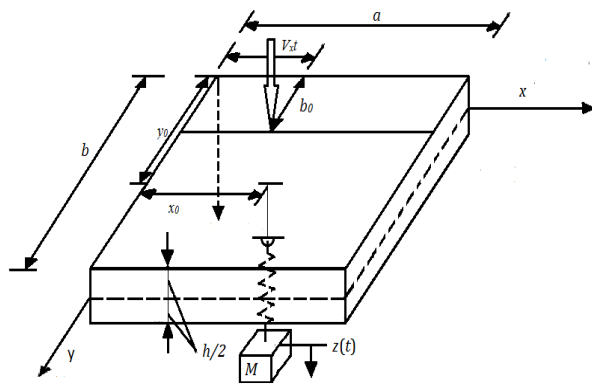


Figure 1. A composite plate with an attached mass-spring system under moving load

Substituting Eq. (6) with Eq. (5) and subsequently, substituting Eqs. (2)-(5) with Eq. (1) lead to the governing equations of the plate in terms of the displacement components as the following:

$$A_{11} \frac{\partial^2 u_0}{\partial x^2} + 2A_{16} \frac{\partial^2 u_0}{\partial x \partial y} + A_{66} \frac{\partial^2 u_0}{\partial y^2} + A_{16} \frac{\partial^2 v_0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 v_0}{\partial x \partial y} + A_{26} \frac{\partial^2 v_0}{\partial y^2} - \left[B_{11} \frac{\partial^3 w_0}{\partial x^3} + 3B_{16} \frac{\partial^3 w_0}{\partial x^2 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^3 w_0}{\partial x \partial y^2} + B_{26} \frac{\partial^3 w_0}{\partial y^3} \right] = I_0 \ddot{u}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial x} \quad (8)$$

$$A_{16} \frac{\partial^2 u_0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u_0}{\partial x \partial y} + A_{26} \frac{\partial^2 u_0}{\partial y^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} + 2A_{26} \frac{\partial^2 v_0}{\partial x \partial y} + A_{22} \frac{\partial^2 v_0}{\partial y^2} - \left[B_{16} \frac{\partial^3 w_0}{\partial x^3} + 3B_{26} \frac{\partial^3 w_0}{\partial x \partial y^2} + (B_{12} + 2B_{66}) \frac{\partial^3 w_0}{\partial x^2 \partial y} + B_{22} \frac{\partial^3 w_0}{\partial y^3} \right] = I_0 \ddot{v}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial y} \quad (9)$$

$$B_{11} \frac{\partial^3 u_0}{\partial x^3} + 3B_{16} \frac{\partial^3 u_0}{\partial x^2 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^3 u_0}{\partial x \partial y^2} + B_{26} \frac{\partial^3 u_0}{\partial y^3} + B_{16} \frac{\partial^3 v_0}{\partial x^3} + 3B_{26} \frac{\partial^3 v_0}{\partial x \partial y^2} + (B_{12} + 2B_{66}) \frac{\partial^3 v_0}{\partial x^2 \partial y} + B_{22} \frac{\partial^3 v_0}{\partial y^3} - \left[D_{11} \frac{\partial^4 w_0}{\partial x^4} + 4D_{16} \frac{\partial^4 w_0}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w_0}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w_0}{\partial y^4} \right] + q(x, y, t) = I_1 \left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y} \right) + I_0 \ddot{w}_0 - I_2 \left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2} \right) \quad (10)$$

3. The Analytical Solution of the Governing Equations

3.1. Composite Plate with One-attached Mass-spring System

The simply-supported boundary conditions for the classical linear plate theory (CLPT) are as the following:

$$u_0(x, 0, t) = 0, \quad u_0(x, b, t) = 0, \quad v_0(0, y, t) = 0, \quad v_0(a, y, t) = 0 \quad (11)$$

$$\begin{aligned}
w_0(x, 0, t) = 0, \quad w_0(x, b, t) = 0, \quad w_0(0, y, t) = 0, \\
w_0(a, y, t) = 0, \quad w_{0,x}(x, 0, t) = 0, \quad w_{0,x}(x, b, t) = 0, \\
w_{0,y}(0, y, t) = 0, \quad w_{0,y}(a, y, t) = 0, \\
N_{xx}(0, y, t) = 0, \quad N_{xx}(a, y, t) = 0, \\
N_{yy}(x, 0, t) = 0, \quad N_{yy}(x, b, t) = 0, \\
M_{xx}(0, y, t) = 0, \quad M_{xx}(a, y, t) = 0, \\
M_{yy}(x, 0, t) = 0, \quad M_{yy}(x, b, t) = 0
\end{aligned} \tag{12}$$

The boundary conditions appeared in Eq. (11) can be satisfied through the following Navier-type solutions

$$\begin{aligned}
u_0(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{mn}(t) \cos(\alpha x) \sin(\beta y) \\
v_0(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{mn}(t) \sin(\alpha x) \cos(\beta y) \\
w_0(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn}(t) \sin(\alpha x) \sin(\beta y)
\end{aligned} \tag{13}$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$. $q(x, y, t)$ can also be expressed in the following double Fourier series form:

$$q(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn}(t) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{14}$$

where

$$\begin{aligned}
Q_{mn}(t) &= \frac{4}{ab} \int_0^a \int_0^b q(x, y, t) \sin\left(\frac{m\pi x}{a}\right) \\
&\quad \cdot \sin\left(\frac{n\pi y}{b}\right) dx dy
\end{aligned} \tag{15}$$

Substituting $q(x, y, t)$ from Eq. (5) with Eq. (15) leads to the following:

$$\begin{aligned}
Q_{mn}(t) &= \frac{4}{ab} \left[M(g - \ddot{z}) \sin(\alpha x_0) \sin(\beta y_0) \right. \\
&\quad \left. + P_0 \sin(\alpha V_x t) \sin(\beta b_0) \right]
\end{aligned} \tag{16}$$

Substituting Eq. (13) with Eqs. (2) and (3), and substituting the resulting equations with Eq. (12), reveal that the Navier solution (13) exists only if $A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = A_{45} = 0, I_1 = 0$

It follows that plates with a single generally orthotropic layer, symmetrically laminated plates with multiple especially orthotropic layers, and anti-symmetric cross-ply laminated plates, which satisfy Eq. (16), admit the Navier solutions for simply-supported boundary conditions. Substituting the Navier solution of Eq. (13) with governing Eqs. (8)-(10) leads to the following equation in terms of unknown coefficients of the Navier solution $U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn}$ and the vertical position of the suspended mass $z(t)$.

$$\begin{aligned}
\begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} \\ \hat{c}_{12} & \hat{c}_{22} & \hat{c}_{23} \\ \hat{c}_{13} & \hat{c}_{23} & \hat{c}_{33} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} + \begin{bmatrix} \hat{m}_{11} & 0 & 0 \\ 0 & \hat{m}_{22} & 0 \\ 0 & 0 & \hat{m}_{33} \end{bmatrix} \\
\begin{Bmatrix} \ddot{U}_{mn} \\ \ddot{V}_{mn} \\ \ddot{W}_{mn} \end{Bmatrix} = \begin{cases} 0 \\ 0 \\ \frac{4}{ab} \left[M(g - \ddot{z}) \sin(\alpha x_0) \sin(\beta y_0) \right. \right. \\ \left. \left. + P_0 \sin(\alpha V_x t) \sin(\beta b_0) \right] \right\}
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
\hat{c}_{11} &= (A_{11}\alpha^2 + A_{66}\beta^2) \\
\hat{c}_{12} &= (A_{12} + A_{66})\alpha\beta \\
\hat{c}_{13} &= -(B_{11}\alpha^3 + \tilde{B}_{12}\alpha\beta^2) \\
\hat{c}_{22} &= (A_{66}\alpha^2 + A_{22}\beta^2) \\
\hat{c}_{23} &= -(\tilde{B}_{12}\alpha^2\beta + B_{22}\beta^3) \\
\hat{c}_{33} &= (D_{11}\alpha^4 + 2\tilde{D}_{12}\alpha^2\beta^2 + D_{22}\beta^4) \\
\hat{m}_{11} &= \hat{m}_{22} = I_0 \\
\hat{m}_{33} &= (I_0 + I_2(\alpha^2 + \beta^2))
\end{aligned} \tag{18}$$

Equations (17) and (7) are the governing equations of the system shown in Fig. 1, which must be solved simultaneously, using the following initial conditions:

$$\begin{aligned}
U_{mn}(0) = 0, \quad \dot{U}_{mn}(0) = 0, \quad V_{mn}(0) = 0, \\
\dot{V}_{mn}(0) = 0, \quad W_{mn}(0) = 0, \quad \dot{W}_{mn}(0) = 0, \\
z(0) = z_0, \quad \dot{z}(0) = 0
\end{aligned} \tag{19}$$

Applying Laplace transform to Eqs. (7) and (17) leads to the following system of equations:

$$\begin{aligned}
\begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} \\ \hat{c}_{12} & \hat{c}_{22} & \hat{c}_{23} \\ \hat{c}_{13} & \hat{c}_{23} & \hat{c}_{33} \end{bmatrix} + s^2 \begin{bmatrix} \hat{m}_{11} & 0 & 0 \\ 0 & \hat{m}_{22} & 0 \\ 0 & 0 & \hat{m}_{33} \end{bmatrix} \\
\begin{Bmatrix} \bar{U}_{mn} \\ \bar{V}_{mn} \\ \bar{W}_{mn} \end{Bmatrix} = \begin{cases} 0 \\ 0 \\ \frac{4}{ab} \left[M \left(\frac{g}{s} - s^2 Z + s z_0 \right) \sin(\alpha x_0) \sin(\beta y_0) \right. \right. \\ \left. \left. + P_0 \frac{\alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \right\}
\end{aligned} \tag{20}$$

$$(Ms^2 + Cs + K)Z = (Cs + K)W_0(x_0, y_0, s) + (Ms + C)z_0 \tag{21}$$

Substituting Z from Eq. (21) with Eq. (20) leads to the following equation:

$$\begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} \\ \hat{c}_{12} & \hat{c}_{22} & \hat{c}_{23} \\ \hat{c}_{13} & \hat{c}_{23} & \hat{c}_{33} \end{pmatrix} + s^2 \begin{pmatrix} \hat{m}_{11} & 0 & 0 \\ 0 & \hat{m}_{22} & 0 \\ 0 & 0 & \hat{m}_{33} \end{pmatrix} \begin{Bmatrix} \bar{U}_{mn} \\ \bar{V}_{mn} \\ \bar{W}_{mn} \end{Bmatrix} = \begin{cases} 0 \\ 0 \\ \frac{4}{ab} \left[H(x_0, y_0, s) \sin(\alpha x_0) \sin(\beta y_0) + P_0 \frac{\alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \end{cases} \tag{22}$$

where

$$H(x_0, y_0, s) = A - BW_0(x_0, y_0, s) \tag{23}$$

$$A = M \left(\frac{g}{s} + \frac{Ksz_0}{Ms^2 + Cs + K} \right)$$

$$B = Ms^2 \frac{Cs + K}{Ms^2 + Cs + K}$$

Now suppose that

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{12} & \varphi_{22} & \varphi_{23} \\ \varphi_{13} & \varphi_{23} & \varphi_{33} \end{pmatrix} = \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} \\ \hat{c}_{12} & \hat{c}_{22} & \hat{c}_{23} \\ \hat{c}_{13} & \hat{c}_{23} & \hat{c}_{33} \end{pmatrix} + s^2 \begin{pmatrix} \hat{m}_{11} & 0 & 0 \\ 0 & \hat{m}_{22} & 0 \\ 0 & 0 & \hat{m}_{33} \end{pmatrix} \tag{24}$$

Therefore,

$$\begin{aligned} \bar{U}_{mn} &= \frac{4\varphi_{13}}{ab} \left[H(x_0, y_0, s) \sin(\alpha x_0) \sin(\beta y_0) + \frac{P_0 \alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \\ \bar{V}_{mn} &= \frac{4\varphi_{23}}{ab} \left[H(x_0, y_0, s) \sin(\alpha x_0) \sin(\beta y_0) + \frac{P_0 \alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \\ \bar{W}_{mn} &= \frac{4\varphi_{33}}{ab} \left[H(x_0, y_0, s) \sin(\alpha x_0) \sin(\beta y_0) + \frac{P_0 \alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \end{aligned} \tag{25}$$

Rewriting the third equation of Eq. (25) considering Eq. (23) and using the Laplace transformed version of Eq. (13) lead to what follows:

$$\bar{W}_0(x, y, s) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi_{33} \left[(A - BW_0(x_0, y_0, s)) \sin(\alpha x_0) \sin(\beta y_0) + \frac{P_0 \alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \sin(\alpha x) \sin(\beta y) \tag{26}$$

Replacing (x, y) by (x₀, y₀) in both sides of Eq. (26) results in the following equation for $\bar{W}_0(x_0, y_0, s)$

$$\bar{W}_0(x_0, y_0, s) = \frac{F_1(x_0, y_0)}{1 + F_2(x_0, y_0)} \tag{27}$$

where

$$F_1(x_0, y_0) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi_{33} \left[A \sin(\alpha x_0) \sin(\beta y_0) + \frac{P_0 \alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \sin(\alpha x_0) \sin(\beta y_0) \tag{28}$$

$$F_2(x_0, y_0) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi_{33} B \sin(\alpha x_0) \sin(\beta y_0) \tag{29}$$

Substituting $\bar{W}_0(x_0, y_0, s)$ from Eq. (27) with Eq. (25) and substituting the results into the Laplace transformed form of Eq. (13) lead to the Laplace transformed form of the displacement components ($\bar{U}_0(x, y, s)$, $\bar{V}_0(x, y, s)$ and $\bar{W}_0(x, y, s)$). Eventually, using the inverse Laplace transform, Navier solution (13), i.e., $u_0(x, y, t)$, $v_0(x, y, t)$ and $w_0(x, y, t)$ may be obtained in the time domain. On the other hand, substituting $\bar{W}_0(x_0, y_0, s)$ from Eq. (27) with Eq. (21) results in the following equation for the vertical position Z of the suspended mass in the complex domain of Laplace transform:

$$Z = \frac{(Cs + K)}{(Ms^2 + Cs + K)} \frac{F_1(x_0, y_0)}{1 + F_2(x_0, y_0)} + \frac{(Ms + C)z_0}{(Ms^2 + Cs + K)} \tag{30}$$

The inverse Laplace transform of Eq. (30) is determined by means of the residue theorem and by aid of Maple software. In order to successfully accomplish the process of applying the invers Laplace transform to any rational function by means of the residue theorem [27], the roots of the denominator must be found. In the case of Eq. (30) the most challenging part of denominator is $1 + F_2(x_0, y_0)$, but

the series of $F_2(x_0, y_0)$ converges rapidly by growth of the number of its sentences, so the roots of $1 + F_2(x_0, y_0)$ can be found in any desired degree of accuracy and these roots can be used in implementing of the residue theorem.

3.2. A Composite Plate with Two-attached Mass-spring Systems

In the case of a composite plate with two-attached mass-spring systems, the total external mechanical load can be expressed as follows:

$$q(x, y, t) = M_1(g - \ddot{z}_1)\delta(x - x_1)\delta(y - y_1) + M_2(g - \ddot{z}_2)\delta(x - x_2)\delta(y - y_2) + f(x, y, t) \quad (31)$$

where $f(x, y, t)$ is the moving load defined by Eq. (6), M_1 and M_2 are magnitudes of the suspended masses, z_1 and z_2 are the vertical positions of the masses M_1 and M_2 , respectively and (x_1, y_1) and (x_2, y_2) are positions of the suspended masses, respectively. The governing equations of motion of the suspended masses are as the following:

$$M_1\ddot{z}_1 + C_1\dot{z}_1 + K_1z_1 = K_1w_0(x_0, y_0, t) + C_1\dot{w}_0(x_0, y_0, t) \quad (32)$$

$$M_2\ddot{z}_2 + C_2\dot{z}_2 + K_2z_2 = K_2w_0(x_0, y_0, t) + C_2\dot{w}_0(x_0, y_0, t) \quad (33)$$

According to Eqs. (15) and (31), the following $Q_{mn}(t)$ expression may be used for a plate with two-attached mass-spring systems:

$$Q_{mn}(t) = \frac{4}{ab} \left[M_1(g - \ddot{z}_1)\sin(\alpha x_1)\sin(\beta y_1) + M_2(g - \ddot{z}_2)\sin(\alpha x_2)\sin(\beta y_2) + P_0 \sin(\alpha V_x t)\sin(\beta b_0) \right] \quad (34)$$

Thus, the Laplace transformed version of the relevant governing equation takes the following form.

$$\begin{Bmatrix} \bar{U}_{mn} \\ \bar{V}_{mn} \\ \bar{W}_{mn} \end{Bmatrix} = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{12} & \varphi_{22} & \varphi_{23} \\ \varphi_{13} & \varphi_{23} & \varphi_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \bar{Q}_{mn}(s) \end{Bmatrix} \quad (35)$$

$$(M_1s^2 + C_1s + K_1)Z_1 = (C_1s + K_1)W_0(x_1, y_1, s) + (M_1s + C_1)z_{01} \quad (36)$$

$$(M_2s^2 + C_2s + K_2)Z_2 = (C_2s + K_2)W_0(x_2, y_2, s) + (M_2s + C_2)z_{02} \quad (37)$$

where

$$\bar{Q}_{mn}(s) = \frac{4}{ab} \left[M_1 \left(\frac{g}{s} - s^2 Z_1 + sz_{01} \right) \sin(\alpha x_1) \sin(\beta y_1) + M_2 \left(\frac{g}{s} - s^2 Z_2 + sz_{02} \right) \sin(\alpha x_2) \sin(\beta y_2) + P_0 \frac{\alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \quad (38)$$

Substituting Z_1 and Z_2 from Eqs. (36) and (37) with Eq. (38) results in:

$$\bar{Q}_{mn}(s) = \frac{4}{ab} \left[H_1(x_1, y_1, s) \sin(\alpha x_1) \sin(\beta y_1) + H_2(x_2, y_2, s) \sin(\alpha x_2) \sin(\beta y_2) + P_0 \frac{\alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \quad (39)$$

where

$$\begin{aligned} H_1(x_1, y_1, s) &= A_1 - B_1 W_0(x_1, y_1, s) \\ A_1 &= M_1 \left(\frac{g}{s} + \frac{K_1 s z_{01}}{M_1 s^2 + C_1 s + K_1} \right) \\ B_1 &= M_1 s^2 \frac{C_1 s + K_1}{M_1 s^2 + C_1 s + K_1} \\ H_2(x_2, y_2, s) &= A_2 - B_2 W_0(x_2, y_2, s) \\ A_2 &= M_2 \left(\frac{g}{s} + \frac{K_2 s z_{02}}{M_2 s^2 + C_2 s + K_2} \right) \\ B_2 &= M_2 s^2 \frac{C_2 s + K_2}{M_2 s^2 + C_2 s + K_2} \end{aligned} \quad (40)$$

Eq. (35) and the Laplace transformed version of Eq. (13) lead to the following:

$$\bar{U}_0(x, y, s) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi_{13} \left[H_1(x_1, y_1, s) \sin(\alpha x_1) \sin(\beta y_1) + H_2(x_2, y_2, s) \sin(\alpha x_2) \sin(\beta y_2) + P_0 \frac{\alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \cos(\alpha x) \sin(\beta y) \quad (41)$$

$$\bar{V}_0(x, y, s) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi_{23} \left[H_1(x_1, y_1, s) \sin(\alpha x_1) \sin(\beta y_1) + H_2(x_2, y_2, s) \sin(\alpha x_2) \sin(\beta y_2) + P_0 \frac{\alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \sin(\alpha x) \cos(\beta y) \quad (42)$$

$$\bar{W}_0(x, y, s) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi_{33} \left[H_1(x_1, y_1, s) \sin(\alpha x_1) \sin(\beta y_1) + H_2(x_2, y_2, s) \sin(\alpha x_2) \sin(\beta y_2) + P_0 \frac{\alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \right] \sin(\alpha x) \sin(\beta y) \quad (43)$$

Replacing (x, y) by (x_1, y_1) and (x_2, y_2) in both sides of Eq. (43) results in the following equations respectively:

$$\begin{aligned} [1 + F_2(x_1, y_1, s)] W_0(x_1, y_1, s) &= F_1(x_1, y_1, s) \\ &+ G_1(x_1, y_1, s) - W_0(x_2, y_2, s) G_2(x_1, y_1, s) \end{aligned} \quad (44)$$

$$\begin{aligned} [1 + G_2(x_2, y_2, s)] W_0(x_2, y_2, s) &= F_1(x_2, y_2, s) \\ &+ G_1(x_2, y_2, s) - W_0(x_1, y_1, s) F_2(x_2, y_2, s) \end{aligned} \quad (45)$$

where

$$\begin{aligned} F_1(x, y, s) &= \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_1 \varphi_{33} \sin\left(\frac{m\pi x_1}{a}\right) \\ &\cdot \sin\left(\frac{n\pi y_1}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) + \frac{4}{ab} \end{aligned} \quad (46)$$

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\varphi_{33} P_0 \alpha V_x}{s^2 + (\alpha V_x)^2} \sin(\beta b_0) \\ \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \end{aligned}$$

$$\begin{aligned} F_2(x, y, s) &= \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_1 \varphi_{33} \sin\left(\frac{m\pi x_1}{a}\right) \\ &\cdot \sin\left(\frac{n\pi y_1}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \end{aligned} \quad (46)$$

$$\begin{aligned} G_1(x, y, s) &= \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_2 \varphi_{33} \sin\left(\frac{m\pi x_2}{a}\right) \\ &\cdot \sin\left(\frac{n\pi y_2}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \end{aligned} \quad (46)$$

$$\begin{aligned} G_2(x, y, s) &= \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_2 \varphi_{33} \sin\left(\frac{m\pi x_2}{a}\right) \\ &\cdot \sin\left(\frac{n\pi y_2}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \end{aligned} \quad (46)$$

So that

$$\begin{aligned} W_0(x_1, y_1, s) &= [F_1(x_1, y_1, s)(1 + G_2(x_2, y_2, s)) \\ &- F_1(x_2, y_2, s) G_2(x_1, y_1, s)] / [(1 + F_2(x_1, y_1, s)) \end{aligned} \quad (47)$$

$$(1 + G_2(x_2, y_2, s)) - F_2(x_2, y_2, s) G_2(x_1, y_1, s)]$$

$$\begin{aligned} W_0(x_2, y_2, s) &= [G_1(x_2, y_2, s)(1 + F_2(x_1, y_1, s)) \\ &- G_1(x_1, y_1, s) F_2(x_2, y_2, s)] / [(1 + G_2(x_2, y_2, s)) \end{aligned} \quad (48)$$

$$(1 + F_2(x_1, y_1, s)) - G_2(x_1, y_1, s) F_2(x_2, y_2, s)]$$

Substituting $W_0(x_1, y_1, s)$ and $W_0(x_2, y_2, s)$ from Eqs. (47) and (48) with Eqs. (41)-(43) leads to the Laplace transformed version of the displacement components. Using the inverse Laplace transform, the displacement components may be obtained in the time domain. The most important conceptual consequence of this analysis is that the superposition principle is not allowed to be used in case of a plate with several mass-spring-damper systems,

because the systems have mutual dynamical interactions on their dynamic responses as may be readily seen in Eqs. (47) and (48), but the presented method can be extended to the case of a composite plate with several mass-spring-damper attachments. However, we analyzed the case of a composite plate with only two-attached mass-spring-damper systems, for the sake of simplicity and brevity.

4. Results and Discussions

4.1. Verification of the Results

Since the present problem has not been investigated before, a special example of a plate with a moving load but without attached mass already treated by Malekzadeh et al. [25] is reexamined, for verification purposes. The considered plate is a symmetrically laminated [0/90/90/0] composite plate with the following specifications:

$a = 1 \text{ m}$, $b = 1 \text{ m}$, $E_1 = 144.8 \text{ GPa}$, $E_2 = 9.65 \text{ GPa}$
 $G_{12} = 4.136 \text{ GPa}$, $\nu_{12} = 0.25$, $\rho = 1389.297 \text{ Kg/m}^3$
 Thickness of each layer is 1 mm.

The present results are compared in Fig. 2 with results of the formulation proposed by Malekzadeh et al. [25], for the following parameters of the moving load:

$V_x = 1 \text{ m/s}$, $P_0 = 100 \text{ N}$, $b_0 = 0.5 \text{ m}$

Fig. 2 reveals that the results are coincident.

4.2. The Effect of M , C and K on the Forced Vibration

In Fig. 3 the effects of the magnitude of the suspended mass on the vibration of the plate treated in the previous section are depicted. As may be deduced from Fig. 3, the amplitude of the vibration for both the free (after $t = 1 \text{ s}$) and forced (before $t = 1 \text{ s}$) vibrations of the plate increases when the magnitude of the suspended mass is increased. This happens mainly because the exerted force by the attached system on the plate increases as the weight of the suspended mass Mg increases. The amplitude of the free vibration increases with the suspended mass, almost linearly. The effects of the magnitude of suspended mass on the vibration frequencies of the middle point of the plate are not quite clear in this figure. In section 4-3, the effect of the suspended mass on the frequencies of the vibrations is explained in more details.

If a damper is included in parallel with the spring, the amplitude of vibration of the middle point of the plate decays as time elapses (see Fig. 4). As may be deduced from Fig. 4 when coefficient of the damper increases, the decay in the amplitude of vibration happens with a higher rate, as it has been already expected.

It is known that the vibration response of a system may be considered as a superposition of the

effects of the various mode shapes (mode superposition). For a continuous system, infinite natural frequencies and consequently, infinite vibration modes may be defined. Fig. 4 shows that the effects of the higher vibration modes have been superimposed on the main response. Moreover, as Fig. 4 implies, the amplitudes of the higher vibration modes are considerably smaller, due to their smaller reversal times. Therefore, the effects of these higher modes are secondary ones. One of the superiorities of the present Laplace transform approach is that it retains all the frequency contents, as Fig. 4 confirms.

The effect of the stiffness of the spring (K) is shown in Fig. 5. It is obvious that increasing the spring stiffness reduces the amplitude of vibration

slightly. This fact stems from the evidence that the average stiffness of the system of the plate with attached mass-spring system increases when the stiffness of the spring increases. The effect of the spring stiffness on the vibration frequency is investigated in section 4.3 in more details.

4.3. The Effects of M and K on the Frequencies of the Vibration

In this section the effects of the parameters of the mass-spring system (M and K) on the frequencies of the vibrations of the system are investigated. For this purpose imaginary parts of roots of characteristic equation of the system are analyzed.

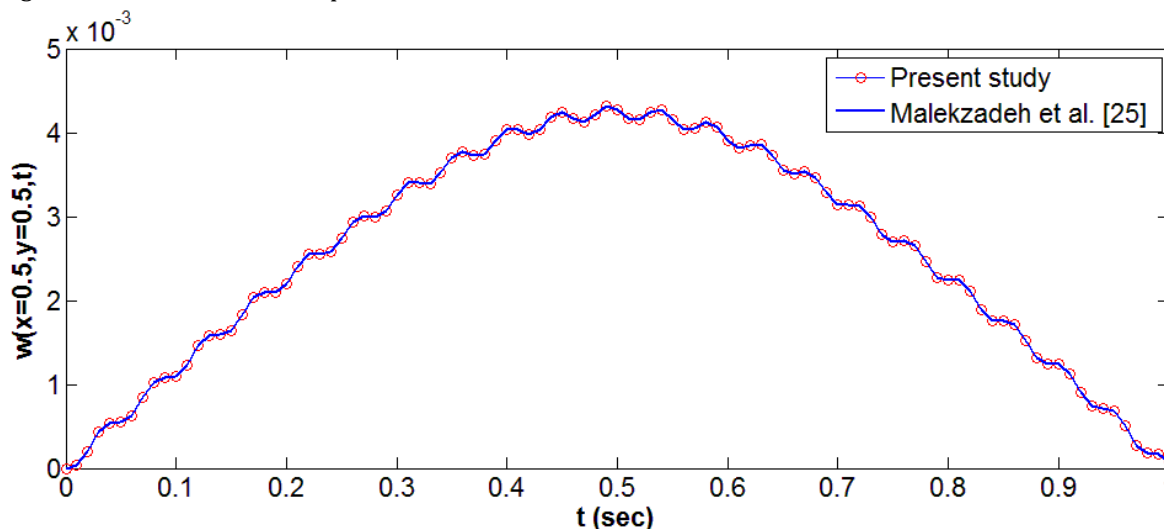


Figure 2. A comparison between the present results and the results of Malekzadeh et al. [25] for a laminated plate [0/90/90/0] with moving load but without the suspended mass ($a = 1\text{ m}, b = 1\text{ m}, V_x = 1\text{ m/s}, P_0 = 100\text{ N}, b_0 = 0.5\text{ m}$)

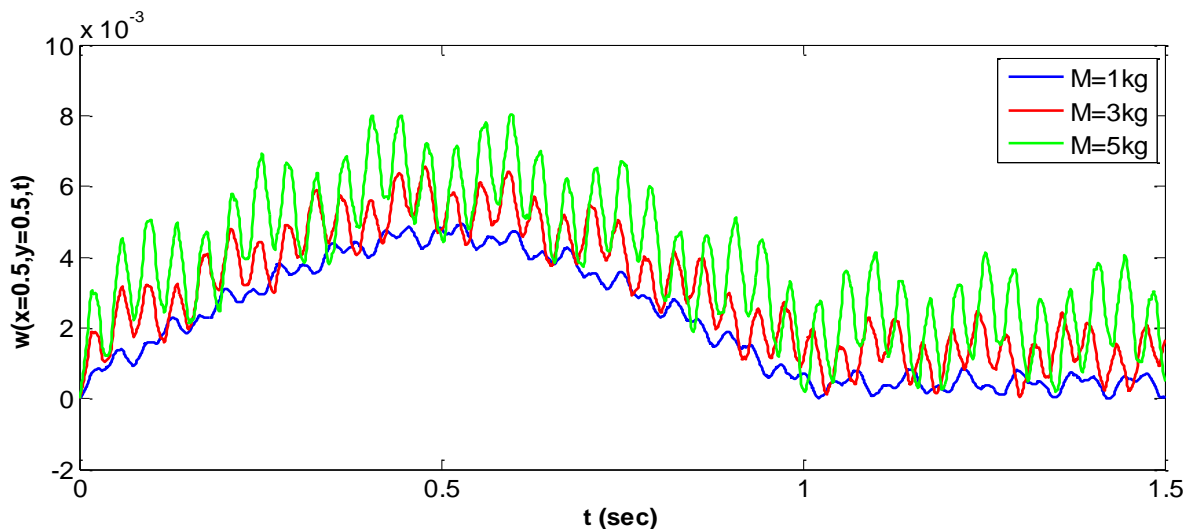


Figure 3. The effects of the suspended mass on the time history of the transverse vibration of the middle point of the laminated [0/90/90/0] plate with attached mass-spring system ($a = b = 1\text{ m}, V_x = 1\text{ m/s}, P_0 = 100\text{ N}, b_0 = 0.5\text{ m}, x_0 = y_0 = 0.5\text{ m}, C = 0, K = 10000\text{ N/m}$)

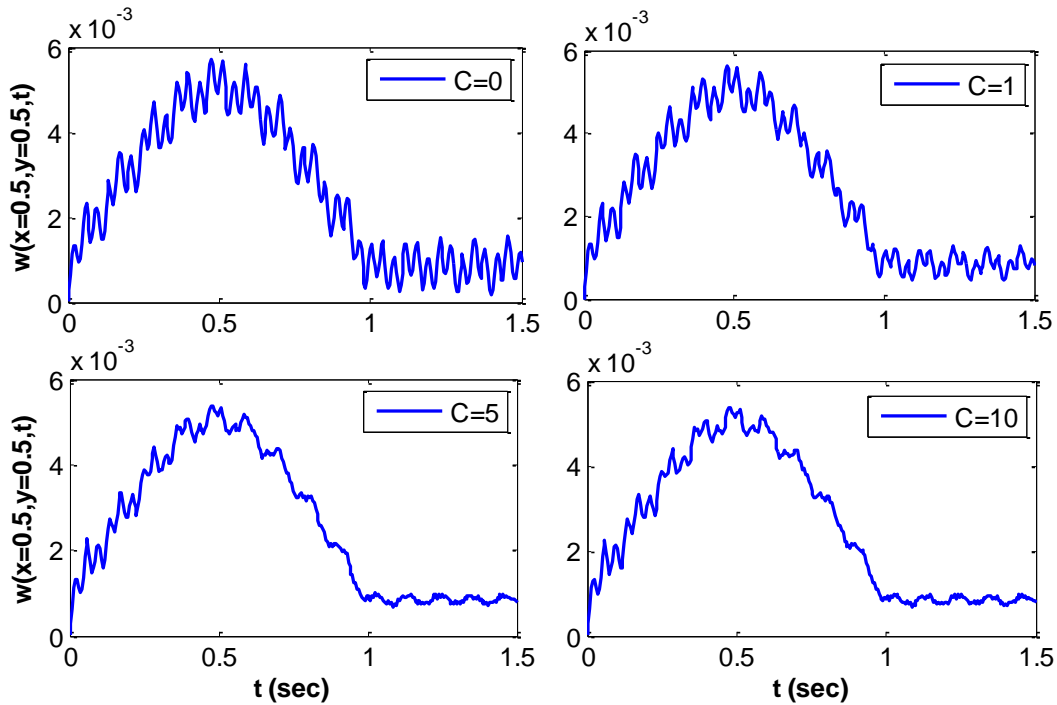


Figure 4. The effects of the damper on time history of the transverse vibration of the middle point of the laminated $[0/90/90/0]$ plate with attached mass-spring-damper system ($a = b = 1 \text{ m}$, $V_x = 1 \text{ m/s}$, $P_0 = 100 \text{ N}$, $b_0 = 0.5 \text{ m}$, $x_0 = y_0 = 0.5 \text{ m}$, $M = 2 \text{ Kg}$, $K = 10000 \text{ N/m}$)

The characteristic equation of the system can be simply deduced from denominator of the right side of Eq. (27) as follows

$$1 + F_2(x_0, y_0, s) = 0 \quad (49)$$

The imaginary components of roots of Eq. (49) are frequencies of vibration of the system. Table 1 reports the effects of the magnitude of the suspended mass on the frequencies of the transverse vibration of the plate of the previous section for a relatively small spring stiffness ($K = 100 \text{ N/m}$) and in Table 2, depicts the effects for a larger stiffness ($K = 10000 \text{ N/m}$). Comparing the results of Table 1 and Table 2 shows that in the larger stiffness, the changes in the magnitude of the suspended mass results in relatively greater changes in the frequencies of the vibration of the plate. This is mainly because the stiffness of the spring is the coupling parameter between the plate and the mass-spring system and when its magnitude experiences an increase, the coupling parameter becomes stronger and the same changes in mass results in a relatively larger change in the frequencies of the vibration. On the other hand, Table 2 shows that attaching a mass-spring system to a plate results in a considerable increase in the first two frequencies of vibration and an increase in the magnitude of the attached mass, results in a relatively smaller decrease in the frequencies. This is because when we attach a mass-spring system to a plate, the total energy of system jumps to a higher level and its increase appears in higher frequencies of vibration and when magnitude

of the suspended mass increases, the average inertia of the system increases and leads to a small reduction in the first two frequencies. On the other hand, comparing results of Tables 1 and 2 shows that in larger spring stiffness, the frequencies of vibration are higher.

4.4. The Effects of P_0 , V_x and b_0

In Fig. 6 the effects of velocity and magnitude of the moving load on the time history of vibration of the composite plate and the suspended mass are illustrated. As it can be seen from Fig. 6, due to the presence of a damper parallel to the spring with damping coefficient of $C = 3$, the amplitude of vibration reduces for all magnitudes of the velocity and the moving load.

Both the velocity and magnitude of the moving load affect the maximum deflection of the middle point of the plate, as it is expected. When the velocity or magnitude of the moving load increases, the maximum deflection increases. The effect of the path line of the moving load on the time history of transverse deflection of middle point of the plate is shown in Fig. (7). It can be deduced from Fig. (7) that the maximum deflection is higher when the path line passes through the center of the plate, mainly because when the load path is closer to the edges of the plate, the supports compensate for the effects of the external load to some extent.

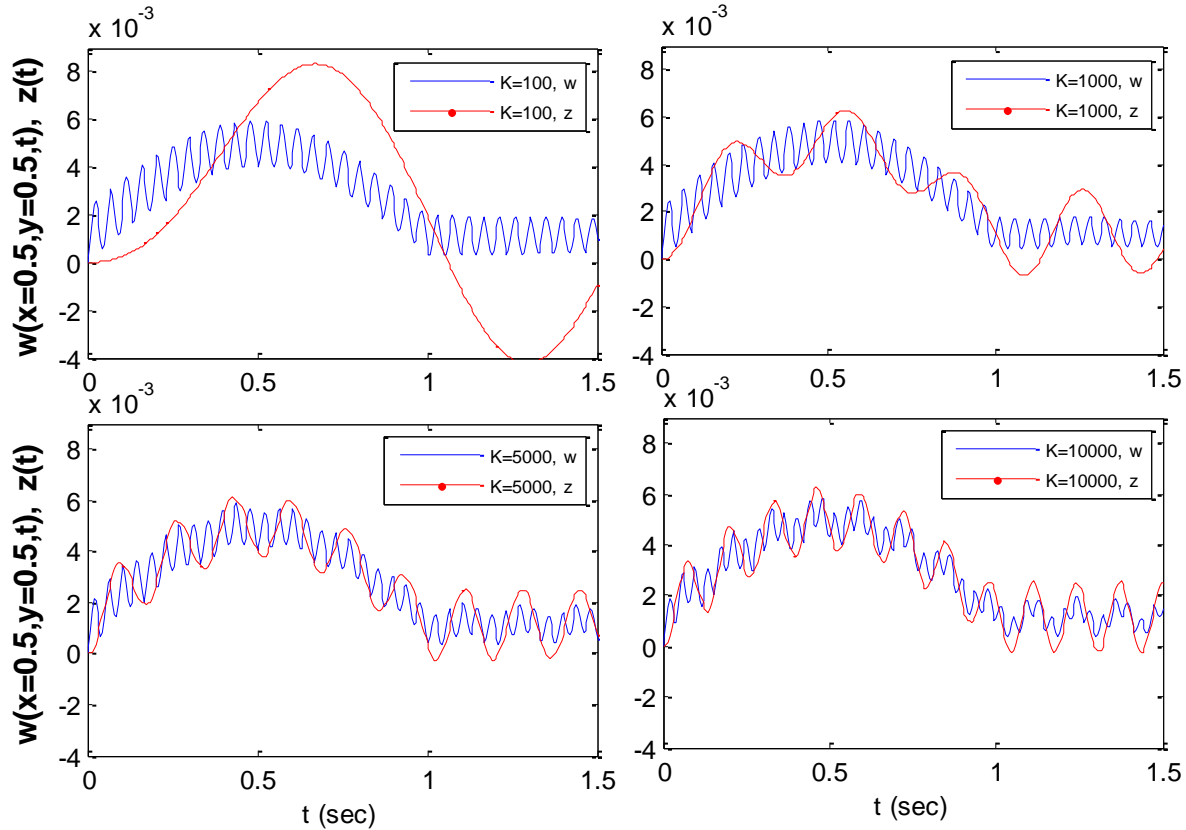


Figure 5. The effect of the spring stiffness on the time histories of the transverse vibration of the middle point of the [0/90/90/0] laminated plate and the suspended mass-spring-damper system ($a = b = 1\text{ m}$, $V_x = 1\text{ m/s}$, $P_0 = 100\text{ N}$, $b_0 = 0.5\text{ m}$, $x_0 = y_0 = 0.5\text{ m}$, $M = 3\text{ Kg}$, $C = 1$)

Table 1. The effect of the suspended mass on the first four frequencies of the transverse vibration of the laminated [0/90/90/0] plate with attached mass-spring system ($a = b = 1\text{ m}$, $x_0 = 0.5\text{ m}$, $y_0 = 0.25\text{ m}$, $C = 0$, $K = 100\text{ N/m}$)

Mass	Frequencies of the mass-spring system		Natural frequencies of the plate			
	ω_M		ω_{11}	ω_{12}	ω_{21}	ω_{22}
	separated	attached				
$M = 0$ (Plate without attached mass-spring)	-	-	137.116	289.881	456.143	548.442
$M = 2$	7.071	7.061	137.247	290.005	456.143	548.444
$M = 5$	4.472	4.465	137.247	290.005	456.143	548.444
$M = 10$	3.162	3.157	137.247	290.005	456.143	548.444

Table 2. The effect of the suspended mass on the first four frequencies of the transverse vibration of the laminated [0/90/90/0] plate ($a = b = 1\text{ m}$, $x_0 = 0.5\text{ m}$, $y_0 = 0.25\text{ m}$, $C = 0$, $K = 10000\text{ N/m}$)

Mass	Frequencies of the mass-spring system		Natural frequencies of the plate			
	ω_M		ω_{11}	ω_{12}	ω_{21}	ω_{22}
	separated	attached				
$M = 0$ (Plate without attached mass-spring)	-	-	137.116	289.881	456.143	548.442
$M = 2$	70.710	61.333	151.019	303.411	456.144	548.445
$M = 5$	44.722	39.286	149.349	302.950	456.144	548.445
$M = 10$	31.623	27.884	148.843	302.803	456.144	548.445

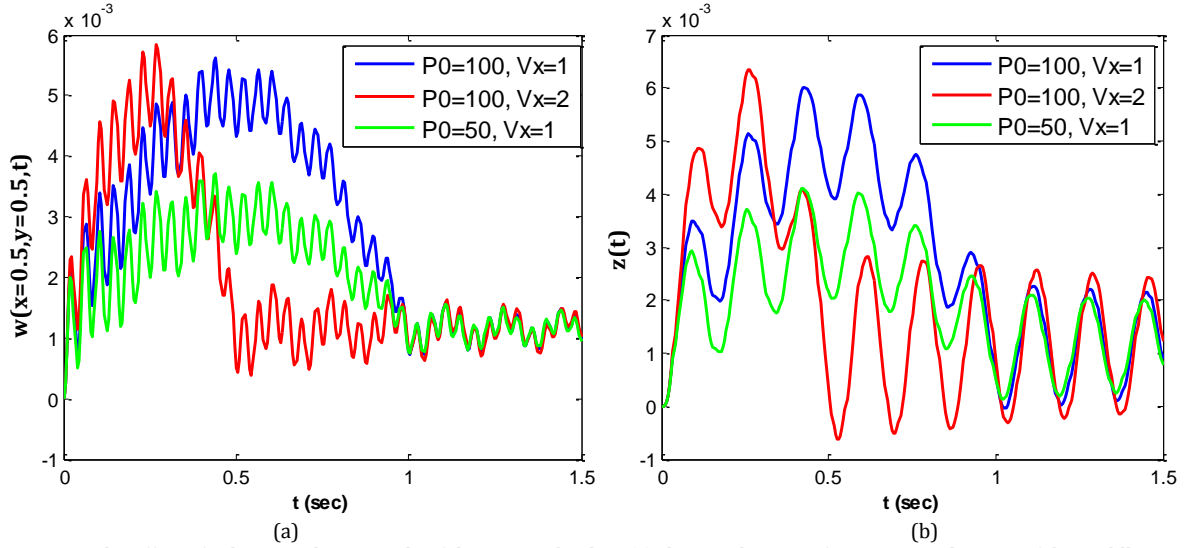


Figure 6. The effect of velocity and magnitude of the moving load on (a) the time history of transverse vibration of the middle point of the laminated [0/90/90/0] plate and (b) the suspended mass ($a = b = 1\text{ m}$, $b_0 = 0.5\text{ m}$, $x_0 = y_0 = 0.5\text{ m}$, $M = 3\text{ Kg}$, $C = 3$, $K = 5000\text{ N/m}$)

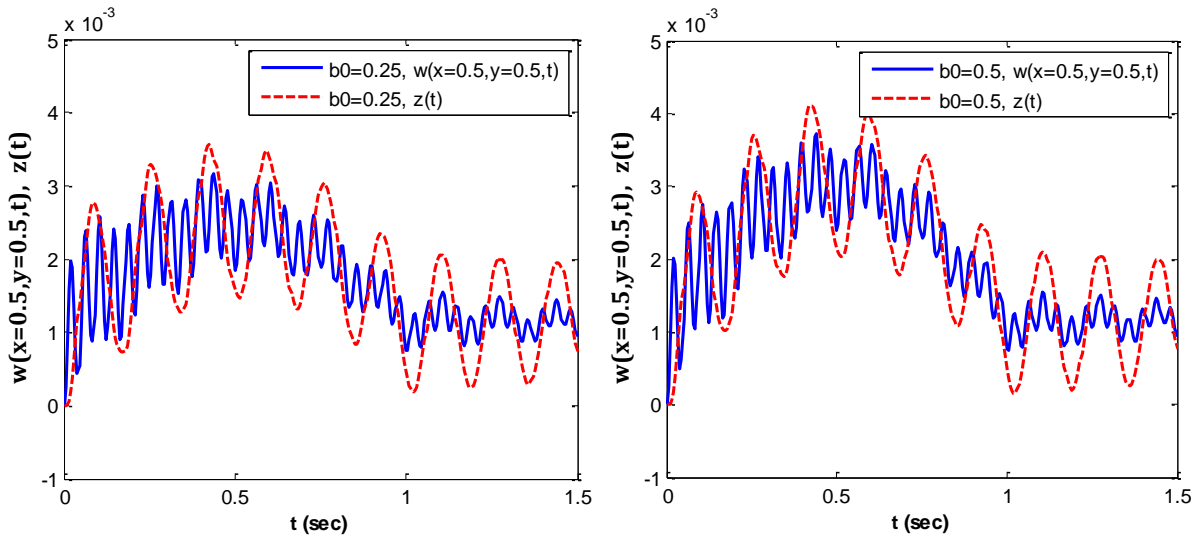


Figure 7. The Effect of the path of the moving load on the time history of transverse vibration of the middle point of the laminated [0/90/90/0] plate and the suspended mass-spring-damper system ($a = b = 1\text{ m}$, $V_x = 1\frac{\text{m}}{\text{s}}$, $P_0 = 50\text{ N}$, $x_0 = y_0 = 0.5\text{ m}$, $M = 3\text{ Kg}$, $C = 3$, $K = 5000\text{ N/m}$)

4.5. The Effects of the Positions of the Attached Mass-Spring Systems on the Frequencies of the Vibrations

The Effect of the positions of the attached mass-spring systems on the frequencies of vibrations of a composite plate with two-attached mass-spring system can be inferred from Table 3. As can be deduced from this table, the general conclusion of section 4.4 regarding distance of the moving load from the plate edge is valid for the frequencies as well, i.e., when the attached systems are closer to the center of plate, the frequencies are higher (in comparison with the first row of Tables 1 and 2 which shows the frequencies of a plate without any attachments).

4.6. The Vibration of the Suspended Mass

In this section, the vibration of the suspended mass, attached to the laminated [0/90/0/90/0/90/0/90] composite plate by means of a spring and a damper, is investigated. The thickness of the plate is $h = 0.016\text{ m}$ (0.002 m for each layer) and the density of the material of the plate is $\rho = 5000\text{ Kg/m}^3$ which imply that the total mass of the plate is 80 Kg. Other properties related to the composite plate are calculated as follows

$$[A] = \begin{bmatrix} 1.876 & 0.0853 & 0 \\ 0.0853 & 1.876 & 0 \\ 0 & 0 & 0.1694 \end{bmatrix} 10^6\text{ N/m}$$

$$[B] = \begin{bmatrix} -0.0959 & 0 & 0 \\ 0 & 0.0959 & 0 \\ 0 & 0 & 0 \end{bmatrix} 10^6 N$$

$$[D] = \begin{bmatrix} 0.1563 & 0.0071 & 0 \\ 0.0071 & 0.1563 & 0 \\ 0 & 0 & 0.0141 \end{bmatrix} 10^6 Nm$$

In Fig. 8, the effect of magnitude of the suspended mass on the time history of its vibration when attached to such a plate is plotted.

It may be deduced from Fig. 8 that an increase in the magnitude of the suspended mass alters its vibration pattern considerably, i.e., increases the amplitude and decreases the frequency of its vibration. The effect of damper's coefficient on the time history of vibration of the middle point of the plate (where the mass-spring-damper system is attached) and the suspended mass is illustrated in Fig. 9.

5. Conclusion

In the present study, the dynamic response and vibration analysis of a composite plate with attached mass-spring-damper system under moving load is accomplished analytically for the first time. Employing the CPT description to explain the behavior of the composite laminated plate, exact results are extracted for the time histories, the frequencies of vibrations of the plate and the suspended mass, and a comprehensive analysis of the effects of various parameters is presented and illustrated graphically. The results are verified by the results available in literature for a special case and an excellent concordance is noticed.

Table 3. The effect of the position of the attached mass-spring systems on the first four frequencies of transverse vibration of the laminated [0/90/90/0] plate with two-attached mass-spring systems ($a = b = 1 m, M_1 = M_2 = 10 Kg, C_1 = C_2 = 0, K_1 = K_2 = 1000 N/m$)

Locations on the attached system	Frequencies of the mass-spring system		Natural frequencies of the plate			
	separated	ω_M attached	ω_{11}	ω_{12}	ω_{21}	ω_{22}
(0.25,0.25), (0.25,0.5)	10	9.684	140.970	291.142	456.143	548.443
(0.25,0.25), (0.25,0.75)	10	9.824	138.403	291.113	456.925	549.771
(0.25,0.25), (0.5,0.5)	10	9.728	140.342	290.506	456.538	549.105
(0.25,0.25), (0.75,0.5)	10	9.802	139.054	290.499	457.323	549.110
(0.25,0.5), (0.75,0.5)	10	9.780	139.709	289.881	457.725	548.444

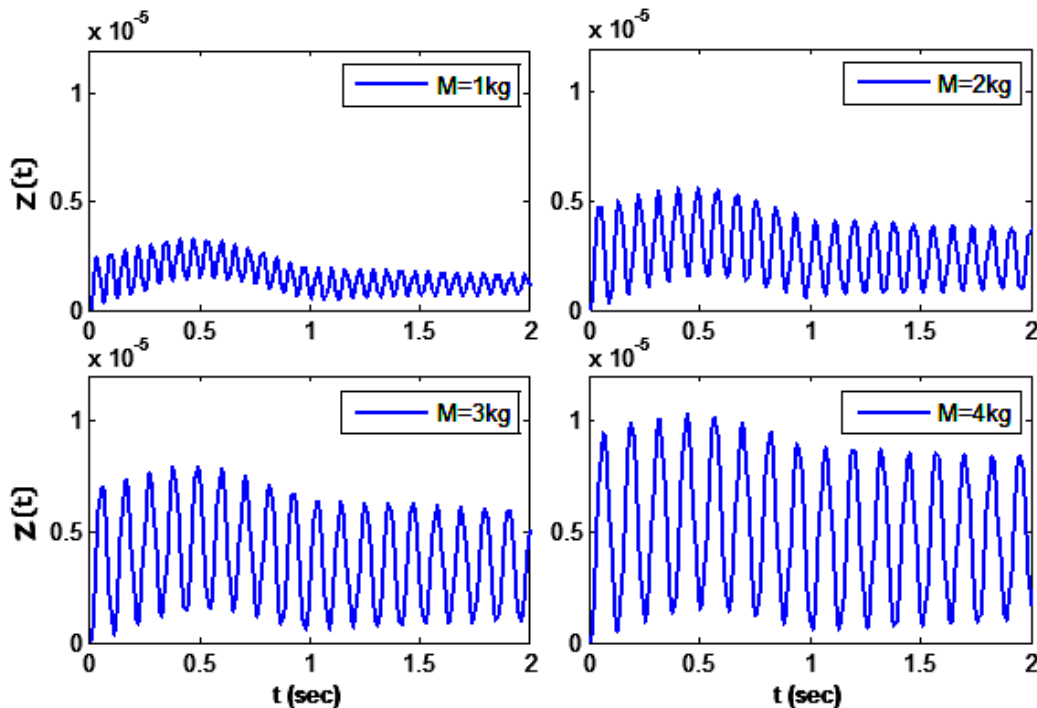


Figure 8. The effect of the suspended mass on the time history vibration of the suspended mass for the laminated [0/90/0/90/0/90/0/90] plate with attached mass-spring-damper system ($a = b = 1 m, b_0 = b/2, x_0 = y_0 = 0.5 m, C = 1, K = 10000 N/m, P_0 = 10 N, V = 1m/s$)

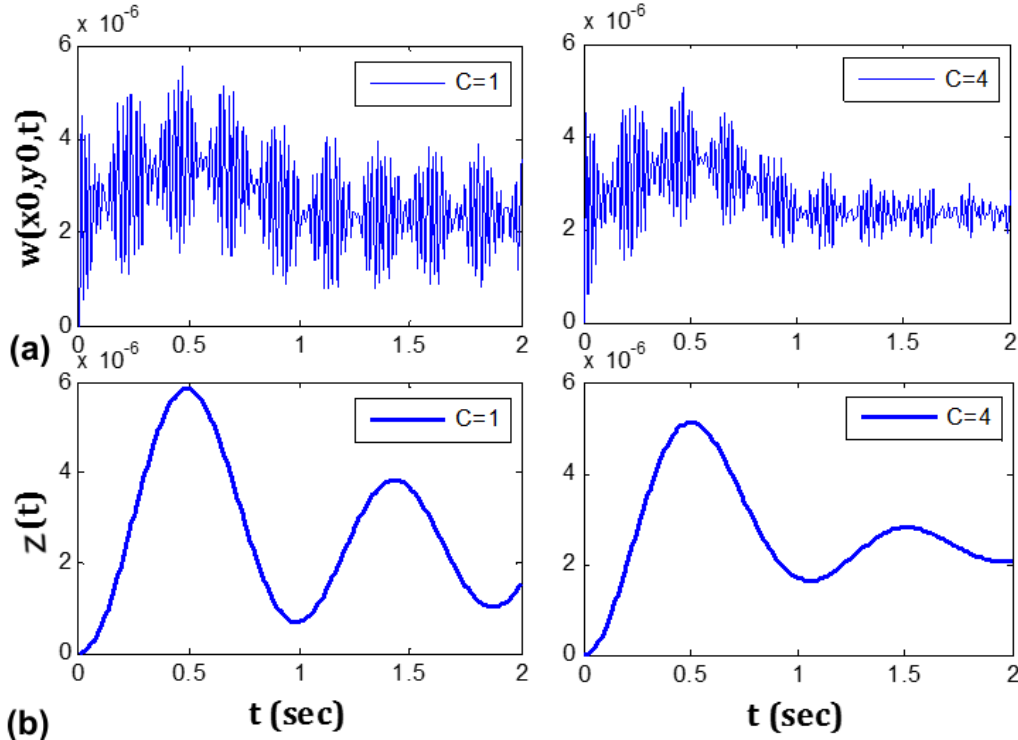


Figure 9. The effect of the damper coefficient on the time histories of vibrations of (a) the attachment point of the laminated $[0/90/0/90/0/90/0/90]$ plate with attached mass-spring-damper system and (b) the suspended mass ($a = b = 1$ m, $b_0 = b/2$, $x_0 = y_0 = 0.5$ m, $M = 2$ Kg, $K = 10000$ N/m, $P_0 = 10$ N, $V = 1$ m/s)

Apart from the novelties presented in the modeling and solution stages, some of the drawn practical conclusions can be summarized as:

- The amplitude of vibration increases for both the free and forced types of vibration of the plate, when the magnitude of suspended mass increases. This fact is also true for the amplitude of vibration of the suspended mass.
- If a damper is added in parallel with the suspended spring, the amplitude of vibration of the plate and the suspended mass diminish with time.
- Increasing the spring's stiffness reduces the amplitude of vibration slightly, and it is mainly due to an increase in the average stiffness of the system of the plate with attached mass-spring system increases.
- For larger spring stiffness, the changes in the magnitude of the suspended mass result in relatively greater changes in the frequencies of vibration of plate with attached mass-spring system. This is mainly because the stiffness coefficient of the spring is the coupling parameter between the plate and the mass-spring system and when its magnitude experiences an increase, the coupling parameter becomes stronger, and increasing the mass results in relatively significant changes in the frequencies of vibration.

- Attaching a mass-spring system to a plate results in a considerable increase in the first two frequencies of vibration and an increase in the magnitude of the attached mass, results in a relatively smaller decrease in the frequencies.
- For larger spring stiffness, the frequencies of vibrations of the plate and suspended mass are higher.

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