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On the Maximal Ideal Space of the Extended Polynomial and Rational Uniform Algebras

S. Moradi[a,*], T. G. Honary[b], D. Alimohammadi[a]

[a]*Department of Mathematics, Faculty of Science, Arak University, Arak, 38156-8-8349, Iran.*
[b]*Faculty of Mathematical Sciences and Computer Engineering, Teacher Training University, 599 Taleghani Avenue, Tehran, 15618, I. R. Iran.*

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Abstract

Let K and X be compact plane sets such that $K \subseteq X$. Let $P(K)$ be the uniform closure of polynomials on K. Let $R(K)$ be the closure of rational functions K with poles off K. Define $P(X,K)$ and $R(X,K)$ to be the uniform algebras of functions in $C(X)$ whose restriction to K belongs to $P(K)$ and $R(K)$, respectively. Let $CZ(X,K)$ be the Banach algebra of functions f in $C(X)$ such that $f|_K = 0$. In this paper, we show that every nonzero complex homomorphism φ on $CZ(X,K)$ is an evaluation homomorphism e_z for some z in $X\backslash K$. By considering this fact, we characterize the maximal ideal space of the uniform algebra $P(X,K)$. Moreover, we show that the uniform algebra $R(X,K)$ is natural.

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1. Introduction

Let A be a commutative Banach algebra. A linear functional $\varphi : A \longrightarrow C$ is called a complex homomorphism on A if $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in A$. If φ is a complex homomorphism on A and $\varphi(f) \neq 0$ for some $f \in A$, then φ is called a nonzero complex homomorphism or a multiplicative linear functional on A. Every complex homomorphism on A is continuous. It is known that if A is with unit 1, then A has at least a nonzero complex homomorphism and $\varphi(1) = 1$ for each nonzero complex homomorphism φ on A. We denote by M_A the set of all nonzero complex homomorphisms

*Corresponding author
Email addresses: s-moradi@araku.ac.ir (S. Moradi), honary@saba.tmu.ac.ir (T. G. Honary), d-alimohammadi@araku.ac.ir (D. Alimohammadi)

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on A. It is known that M_A is a compact (locally compact) Hausdorff space with the Gelfand topology, if A is with (without) unit [5]. M_A with the Gelfand topology is called the maximal ideal space of A.

Let Ω be a locally compact Hausdorff space. We denote by $C(\Omega)(C^b(\Omega))$ the algebra of continuous (bounded continuous) complex-valued functions on Ω. For each $f \in C^b(\Omega)$, we define

$$\|f\|_\Omega = \sup\{|f(w)| : w \in \Omega\},$$

and call it the uniform norm of f on Ω. Then $(C^b(\Omega), ||.||_\Omega)$ is a commutative Banach algebra with unit 1. We denote by $C_0(\Omega)$ the set of all functions f in $C(\Omega)$ which vanish at infinity. Then $C_0(X)$ is a closed subalgebra of $C^b(\Omega)$ and $C_0(\Omega) = C^b(\Omega) = C(\Omega)$, when Ω is compact. Note that if $f \in C_0(\Omega)$, then $\|f\|_\Omega = |f(w_0)|$ for some $w_0 \in \Omega$. Also, $C_0(\Omega)$ is without unit if Ω is not compact.

Let Ω be a locally compact Hausdorff space and A be a subalgebra of $C^b(\Omega)$ such that $1 \in A$ or $C_0(\Omega) \subseteq A$. For each $w \in \Omega$, the map $e_w : A \longrightarrow C$, defined by $e_w(f) = f(w)$, is a nonzero complex homomorphism on A, which is called the *evaluation homomorphism* at w on A. Note that, if A separates the points of Ω, then $e_{w_1} \neq e_{w_2}$ whenever $w_1, w_2 \in \Omega$ and $w_1 \neq w_2$.

Let X be a compact Hausdorff space and let A be a subalgebra of $C(X)$ such that A separates the points of X and let $1 \in A$. If there is an algebra norm $\|.\|$ on A such that $\|1\| = 1$ and $(A, \|.\|)$ is a Banach algebra, then A is called a Banach function algebra on X (under the norm $\|.\|$). If the norm of the Banach function algebra A on X is the uniform norm on X, then A is called a *uniform (function) algebra* on X. For example $C(X)$ is a uniform algebra on X.

Let X be a compact Hausdorff space and let A be a *Banach function algebra* on X. It is easy to see that the map $\varepsilon : X \longrightarrow M_A$ defined by $\varepsilon(x) = e_x$, is continuous and one-to-one. We say that A is natural, if ε is surjective. In this case ε is an homeomorphism and we write $M_A \approx X$.

We know that a Banach function algebra A on X is natural when A is self-adjoint and inverse-closed. Therefore, $C(X)$ is natural.

Definition 1.1. *Let X be a compact Hausdorff space and let K be a nonempty compact subset of X. We denote by $CZ(X, K)$ the set of complex-valued functions f on X such that $f|_K = 0$. Then $CZ(X, K)$ is a uniformly closed subalgebra of $C(X)$ which is without unit.*

Let K be a compact plane set and let $P_0(K)$ and $R_0(K)$ be the algebras of all polynomials and rational functions in z on K with poles off K, respectively. The uniform closure of $P_0(K)$ and $R_0(K)$ are denoted by $P(K)$ and $R(K)$, respectively, which are uniform algebras on K. $P(K)$ and $R(K)$ are called polynomial and rational uniform algebra on K, respectively. It is known that $K = \{z \in C : |R(z)| \leq \|R\|_K \ for\ all\ R \in R_0(K)\}$ and $R(K)$ is natural. The polynomial convex hull of K is

$$\hat{K} = \{z \in C : |p(z)| < \|p\|_K \ for\ all\ polynomials\ p\ in\ z\}.$$

In fact, $\hat{K}$ is the union of K and the bounded components of $C\backslash K$. The set K is called polynomially convex if $\hat{K} = K$. It is known that $P(\hat{K}) = R(\hat{K})$ and $P(K)$ is isometricall isomorphic to $P(\hat{K})$. Also, $M_{P((K)}$ is homeomorphic to $\hat{K}$. In fact, if $\varphi \in M_{P(K)}$, then there exists a unique $z \in \hat{K}$ such that $\varphi(f) = \lim_{n\to\infty} p_n(z)$, where $f \in P(K)$ is the uniform limit of the sequence of polynomials $\{p_n\}_{n=1}^{\infty}$. For more details, see [1].

Definition 1.2. *Let K and X be nonempty compact plane set such that $K \subseteq X$. We define the algebras $P_0(X, K)$, $R_0(X, K)$, $P(X, K)$ and $R(X, K)$ as the following:*

$$P_0(X, K) := \{f \in C(X) \ : \ f|_K \in P_0(K)\},$$

$$R_0(X, K) := \{f \in C(X) \ : \ f|_K \in R_0(K)\},$$

$$P(X, K) := \{f \in C(X) \ : \ f|_K \in P(K)\},$$

$$R(X, K) := \{f \in C(X) \ : \ f|_K \in R(K)\}.$$

Clearly, $P(X, K)$ and $R(X, K)$ are the uniform closure of $P_0(X, K)$ and $R_0(X, K)$, respectively. $P(X, K)$ and $R(X, K)$ are called extended polynomial and rational uniform algebras on X (with respect to K), respectively. We take $A(X, K) = \{f \in C(X) \ : \ f|_K \in A(K)\}$, where $A(K) = \{f \in C(K) : f\}$ is analytic on interior of $K\}$. It is easy to show that $P(X, K)$, $R(X, K)$ and $A(X, K)$ are uniform algebras on X. We know that $A(X, K)$ is natural [1]. *$A(X, K)$ is called extended analytic uniform algebra on X (with respect to K). Note that if K is finite then $P_0(X, K) = R_0(X, K) = C(X)$ and so $P(X, K) = R(X, K) = A(X, K) = C(X)$. Hence, we may assume that K is infinite. Moreover, $P_0(X, K) = P_0(X)$, $R_0(X, K) = R_0(X)$, $P(X, K) = P(X)$, $R(X, K) = R(X)$ and $A(X, K) = A$ if $X \backslash K$ is finite.*

In 2007, T. G. Honary and S. Moradi determined the maximal ideal space of the certain subalgebras of $A(X, K)$ [2]. Next, they studied the uniform approximation by polynomials, rationals and analytic functions in these uniform algebras and also extended Vitushkin's theorem and Hartogs-Rozental theorem [3].

We intend to characterize of nonzero complex homomorphisms on $P(X, K)$ and prove that $R(X, K)$ is natural.

In Section 2, we prove that for every nonzero complex homomorphism φ on $C_0(\Omega)$, there exists a unique $w \in \Omega$ such that $\varphi = e_w$, where Ω is a locally compact Hausdorff space.

In Section 3, we assume that X is a compact Hausdorff space and K is a nonempty compact subset of X and show that $(CZ(X, K), \|.\|_X)$ is isometrically isomorphic to $(C_0(X \backslash K), \|.\|_{X \backslash K})$ and characterize nonzero complex homomorphisms on $CZ(X, K)$.

In Sections 4 and Sections 5, we assume that K and X are compact plane sets such that $K \subseteq X$ and characterize nonzero complex homomorphisms on $P(X, K)$ and prove that $R(X, K)$ is natural.

2. Nonzero Complex Homomorphisms on $C_0(\Omega)$

Let Ω be a locally compact Hausdorff space with topology τ. Then for each $w \in \Omega$, the evaluation map $e_w : C_0(\Omega) \longrightarrow C$ is a nonzero complex homomorphism on $C_0(\Omega)$. If Ω is compact, then $C_0(\Omega) = C(\Omega)$ and so every nonzero complex homomorphism on $C_0(\Omega)$ is an evaluation homomorphism. Now, we assume that Ω is not compact. Set $\Omega_\infty := \Omega \cup \{\infty\}$, such that $\infty \notin \Omega$. Define the topology τ_∞ on Ω_∞ by

$$\tau_\infty := \tau \cup \{\Omega_\infty \backslash S \ : \ S \subseteq \Omega \text{ and } S \text{ is a compact set in} (\Omega, \tau)\}.$$

So Ω_∞ is a compact Hausdorff space with the topology τ_∞ and $\tau = \{\Omega \cap W \ : \ W \in \tau_\infty\}$. The topological space Ω_∞ (with topology τ_∞) is called one point compactification of Ω (with topology τ).

Throughout this section we assume that Ω is a locally compact Hausdorff space which is not compact and $\Omega_\infty := \Omega \cup \{\infty\}$ is the one point compactification of Ω.

For characterizing of nonzero complex homomorphism on $C_0(\Omega)$, we need the following lemma.

Lemma 2.1. *(i) If $g \in C_0(\Omega)$ and $g_\infty : \Omega_\infty \longrightarrow C$ is defined by*

$$g_\infty(w) = \begin{cases} g(w) & w \in \Omega \\ 0 & w = \infty, \end{cases}$$

then $g_\infty \in C(\Omega_\infty)$ and $\|g_\infty\|_{\Omega_\infty} = \|g\|_\Omega$.

(ii) If $f \in C(\Omega_\infty)$ and $g : \Omega \longrightarrow C$ is defined by

$$g(w) = f(w) - f(\infty),$$

then $g \in C_0(\Omega)$, $\|g\|_\Omega \le \|f\|_{\Omega_\infty} + |f(\infty)|$ and $f = g_\infty$.

(iii) If $\Psi : C_0(\Omega) \longrightarrow C(\Omega_\infty)$ is defined by $\Psi(g) = g_\infty$, then Ψ is an isometrical homomorphism from $(C_0(\Omega), \|.\|_\Omega)$ into $(C(\Omega_\infty), \|.\|_{\Omega_\infty})$ and $\Psi(C_0(\Omega)) = M_\infty$, where M_∞ is the maximal ideal $\{f \in C(\Omega_\infty) \ : \ f(\infty) = 0\}$ in $C(\Omega_\infty)$.

Proof .

(i) To prove the continuity of g_∞ at ∞, take $\varepsilon > 0$. Since $g \in C_0(\Omega)$, there exists a compact subset S of Ω such that

$$g(\Omega \backslash S) \subseteq \{z \in C \ : \ |z| < \varepsilon\}. \qquad (1)$$

Clearly, $\Omega_\infty \backslash S$ is an open set in Ω_∞ and $\infty \in \Omega_\infty \backslash S$. Since $g_\infty(\infty) = 0$ and $g_\infty = g$ on Ω, we have

$$g_\infty(\Omega_\infty \backslash S) \subseteq \{z \in C \ : \ |z| < \varepsilon\},$$

by (1). Therefore, g_∞ is continuous at ∞. Let $w_0 \in \Omega$ and take $\varepsilon > 0$. Since g is continuous at w_0, there exists an open set U in Ω with $w_0 \in \Omega$ such that

$$g(U) \subseteq \{z \in C \ : \ |z - g(w_0)| < \varepsilon\}. \qquad (2)$$

Clearly, U is an open set in Ω_∞. Since $w_0 \in \Omega$ and $g_\infty = g$ on Ω, we have

$$g_\infty(U) \subseteq \{z \in C \ : \ |z - g_\infty(w_0)| < \varepsilon\},$$

by (2). It follows that g_∞ is continuous at w_0. Therefore, $g_\infty \in C(\Omega_\infty)$.

Now, we show that $\|g_\infty\|_{\Omega_\infty} = \|g\|_\Omega$. Since $\|g\|_\Omega = |g(w_1)|$ for some $w_1 \in \Omega$, $g_\infty(\infty) = 0$ and $g_\infty = g$ on Ω, we have

$$\|g\|_{\Omega_\infty} = |g(w_1)| = \|g\|_\Omega.$$

(ii) Since the topological space Ω is a subspace of the topological space Ω_∞, $g = f - f_\infty$ on Ω and $f \in C(\Omega_\infty)$, we have $g \in C(\Omega)$. Take $\varepsilon > 0$. Continuity of f at ∞ implies that there exists a compact set S in Ω such that

$$f(\Omega_\infty \backslash S) \subseteq \{z \in C \ : \ |z - f(\infty)| < \varepsilon\}. \qquad (3)$$

If $w \in \Omega \backslash S$, then $w \in \Omega_\infty \backslash S$ and so

$$|g(w)| = |f(w) - f(\infty)| < \varepsilon,$$

by (3). Therefore, $g \in C_0(\Omega)$. Also, $g_\infty = f$ and

$$\|g\|_\Omega \le \|f\|_\Omega + |f(\infty)|,$$

by the definition of g.

(iii) Clearly, Ψ is a homomorphism. Also, Ψ is an isometry by (i). If $g \in C_0(\Omega)$, then $g_\infty \in C(\Omega_\infty)$, $g_\infty(\infty) = 0$ by (i) and so $\Psi(g) = g_\infty \in M_\infty$.

Conversely, if $f \in M_\infty$, then $f \in C(\Omega_\infty)$ and $f(\infty) = 0$. Define $g = f|_\Omega$. It follows that $g \in C_0(\Omega)$, by (ii). Since $f(\infty) = 0$ and $f|_\Omega = g$, we have $f = g_\infty = \Psi(g)$. Hence, (iii) holds. □

Definition 2.2. *Let $g \in C_0(\Omega)$. The map $g_\infty : \Omega_\infty \longrightarrow C$ defined by*

$$g_\infty(w) = \begin{cases} g(w) & w \in \Omega \\ 0 & w = \infty, \end{cases}$$

is called the standard extension of g on Ω_∞.

Theorem 2.3. *If φ is a nonzero complex homomorphism on $C_0(\Omega)$, then there exists a unique $w \in \Omega$ such that $\varphi = e_w$ on $C_0(\Omega)$.*

Proof . Define the map $\psi : C(\Omega_\infty) \longrightarrow C$ by

$$\psi(f) = \varphi(f|_\Omega).$$

It is easy to see that ψ is a complex homomorphism on $C_0(\Omega)$. Since φ is a nonzero complex homomorphism on $C(\Omega_\infty)$, there exists $h \in C_0(\Omega)$ such that $\psi(h) \neq 0$. Let h_∞ be the standard extension of h on Ω_∞. Then $h_\infty \in C(\Omega_\infty)$ and $h = h_\infty|_\Omega$, by part (i) of Lemma 2.1. Therefore,

$$\psi(h_\infty) = \varphi(h) \neq 0.$$

It follows that ψ is a nonzero complex homomorphism on $C(\Omega_\infty)$. Therefore, there exists $w \in \Omega_\infty$ such that $\psi(f) = f(w)$ for all $f \in C(\Omega_\infty)$. We claim that $w \neq \infty$. If $w = \infty$, then $\psi(f) = f(\infty)$ for all $f \in C(\Omega_\infty)$. Now, let $g \in C_0(\Omega)$ and let g_∞ be the standard extension of g on Ω_∞. By part (i) of Lemma 2.1, $g_\infty \in C(\Omega_\infty)$. Therefore, $\psi(g) = \varphi(g_\infty) = g_\infty(\infty) = 0$. So $\psi \equiv 0$ on $C_0(\Omega)$. Hence, our claim is justified.

Let $g \in C_0(\Omega)$ and let g_∞ be the standard extension of g on Ω_∞. Now, we have

$$\varphi(g) = \psi(g_\infty) = g_\infty(w) = g(w) = e_w(g).$$

Therefore $\varphi = e_w$ on $C_0(\Omega)$. □

3. Nonzero Complex Homomorphisms on $CZ(X, K)$

Throughout of this section we assume that X is a compact Hausdorff space and K is a nonempty compact subset of X. Since $X\backslash K$ is an open set in X, we conclude that $X\backslash K$ with the subspace topology is a locally compact Hausdorff space.

Lemma 3.1. *(i) $CZ(X, K)$ separates the points of $X\backslash K$.*

(ii) If $g \in C_0(X\backslash K)$ and $g_0 : X \longrightarrow C$ is defined by

$$g_0(x) = \begin{cases} g(x) & x \in X\backslash K \\ 0 & x \in K, \end{cases}$$

then $g_0 \in CZ(X, K)$.

(iii) If $f \in CZ(X, K)$ and $g = f|_{X\backslash K}$, then $g \in C_0(X\backslash K)$.

Proof . (i) Let $x_1, x_2 \in X \backslash K$ with $x_1 \neq x_2$. There exists $f \in C(X)$ such that $f(x) = 0$ for all $x \in K \cup \{x_1\}$. and $f(x_2) = 1$, by the Urysohn's lemma. Therefore, $f \in CZ(X, K)$ and $f(x_1) \neq f(x_2)$. So (i) holds.
(ii) Since $g_0|_K = 0$, it is enough to show that $g_0 \in C(X)$.
We first assume that $x_0 \in X$ with $g_0(x) \neq 0$. Thus, $x_0 \in X \backslash K$. Take $\varepsilon > 0$. Since $g : X \backslash K \longrightarrow C$ is continuous in x_0, there exists an open set W_0 in X with $x_0 \in W_0$ such that $|g(x) - g(x_0)| < \varepsilon$, for all $x \in W_0$. If $U_0 = W_0 \cap (X \backslash K)$, then U_0 is an open set in X with $x_0 \in U_0$ and

$$|g_0(x) - g_0(x_0)| = |g(x) - g(x_0)| < \varepsilon,$$

for all $x \in U$. Therefore, g_0 is continuous at x_0.
We now assume $x_0 \in X$ with $g_0(x_0) = 0$. Take $\varepsilon > 0$. If $S = \{x \in X \ : \ |g_0(x)| \geq \varepsilon\}$, then S is a compact set in X, $S \subseteq X \backslash K$ and $x_0 \notin S$. Therefore, there exists an open set V_0 in X with $x_0 \in V_0$ such that $V_0 \cap S = \emptyset$. It follows that

$$|g_0(x) - g_0(x_0)| = |g(x_0)| < \varepsilon,$$

for all $x \in V_0$. Therefore, g_0 is continuous at x_0. Consequently, $g_0 \in C(X)$ and (ii) holds.
(iii) Clearly, $g \in C(X \backslash K)$. Take $\varepsilon > 0$ and set

$$S = \{x \in X \ : \ |f(x)| \geq \varepsilon\}.$$

Since $f \in C(X)$ and $f|_K = 0$, S is a compact set in X and $S \subseteq X \backslash K$. So S is a compact set in $X \backslash K$ and $|g(x)| < \varepsilon$ for all $x \in (X \backslash K) \backslash S$. Therefore, $g \in C_0(X \backslash K)$ and (iii) holds. □

Theorem 3.2. *The map $\Phi : CZ(X, K) \longrightarrow C_0(X \backslash K)$ defined by $\Phi(f) = f|_{X \backslash K}$, is an isometrical isomorphism from $(CZ(X, K), ||.||_X)$ onto $(C_0(X \backslash K), ||.||_{X \backslash K})$.*

Proof . Part (ii) of Lemma 3.1 implies that Φ is well-defined. Clearly, Φ is an homomorphism. Let $f \in CZ(X, K)$. Then, $f \in C(X)$ and $f|_K = 0$. Therefore,

$$\|\Phi(f)\|_{X \backslash K} = \|f|_{X \backslash K}\|_{X \backslash K} = \|f\|_{X \backslash K} = \|f\|_X.$$

Thus Φ is an isometry.

Now, we show that Φ is surjective. Let $g \in C_0(X \backslash K)$. Define the complex-valued function g_0 on X by

$$g_0(x) = \begin{cases} g(x) & x \in X \backslash K \\ 0 & x \in K. \end{cases}$$

By part (i) of Lemma 3.1, $g_0 \in CZ(X \backslash K)$. Definition of Φ implies that $\Phi(g_0) = g_0|_K = g$. Thus Φ is surjective. □

Theorem 3.3. *Let $A = CZ(X, K)$.*

(i) If $x \in X \backslash K$ and $e_x : A \longrightarrow C$ is defined by $e_x(f) = f(x)$, then $e_x \in M_A$.

(ii) If $x_1, x_2 \in X \backslash K$ with $x_1 \neq x_2$, then $e_{x_1} \neq e_{x_2}$.

(iii) If $\psi \in M_A$, there exists a unique $x \in X$ such that $\psi = e_x$.

Proof . (i) Clearly, e_x is a complex homomorphism on A. Since $x \in X\backslash K$, there exists $f_1 \in C(X)$ such that $f_1|_K = 0$ and $f_1(x) = 1$, by the Uryshon's lemma. Hence, $f_1 \in CZ(X, K)$ and $e_x(f_1) = 1 \neq 0$. Therefore, $e_x \in M_A$.
(ii) Since A separates the points of $X\backslash K$, there is $f \in A$ such that $f(x_1) \neq f(x_2)$ and so $e_{x_1}(f) \neq e_{x_2}(f)$. Thus $e_{x_1} \neq e_{x_2}$.
(iii) Define the map $\Phi : A \longrightarrow C_0(X\backslash K)$ by $\Phi(f) = f|_{X\backslash K}$. Then Φ is an isometrical isomorphism from $(A, \|.\|_X)$ onto $(C_0(X\backslash K), \|.\|_X)$, by Theorem 3.2. Therefore, $\psi o \Phi^{-1}$ is a complex homomorphism on $C_0(X\backslash K)$. Since $\psi \in M_A$, there exists $f_0 \in A\backslash\{0\}$ such that $\psi(f_0) \neq 0$. If $g_0 = \Phi(f_0)$, then $g_0 \in C_0(X\backslash K)$, $g_0 \not\equiv 0$ and $(\psi o \Phi^{-1})(g_0) = \psi(f_0) \neq 0$. Therefore, $\psi o \Phi^{-1} \in M_{C_0(X\backslash K)}$. It follows that there exists $x \in X\backslash K$ such that $(\psi o \Phi^{-1})(g) = g(x)$ for all $g \in C_0(X\backslash K)$, by Theorem 2.3. Let $f \in A$. Set $g = f|_{X\backslash K}$. Therefore, $g \in C_0(X\backslash K)$ by part (iii) of Lemma 3.1 and $\Phi(f) = g$ by Theorem 3.2. Also, we have

$$\psi(f) = \psi(\Phi^{-1}(g)) = (\psi o \Phi^{-1})(g)$$

$$= g(x) = f(x) = e_x(f).$$

Hence $\psi = e_x$. $\square$

4. Nonzero Complex Homomorphisms on $P(X, K)$

Let X and K be compact plane sets such that $K \subseteq X$. We intend to characterize of the nonzero complex homomorphisms on uniform algebra $P(X, K)$. If K is finite, then $P(X, K) = C(K)$ and so $P(X, K)$ is natural. If $K = X$, then $P(X, K) = P(X)$ and so $M_{P(X,K)} \approx \hat{K}$. We now study the cases in which K is infinite and $X\backslash K$ is nonempty.

Lemma 4.1. *Let X be a compact plane set. If K is an infinite compact subset of X, then*

$$P_0(X, K) = P_0(X) \oplus CZ(X, K).$$

Proof . The case $K = X$ is trivial. We assume that $X\backslash K \neq \emptyset$. If p is a polynomial in z and $g \in CZ(X, K)$, then $p|_X + g \in P(X, K)$. Let $f \in P_0(X, K)$. Then, $f \in C(X)$ and there exists a polynomial p in z such that $f|_K = p|_K$. Define the complex-valued function g on X by $g(z) = f(z) - p(z)$. Clearly, $g \in CZ(X, K)$ and $f = p|_X + g$. Thus

$$P_0(X, K) = P_0(X) + CZ(X, K).$$

If $f \in P_0(X) \cap CZ(X, K)$, then $f \in C(X)$ and there exists a polynomial p in z with $f = p|_X$ such that $p|_K = 0$. It follows that $p \equiv 0$, since K is an infinite subset of C. Therefore, $f = 0$ and this completes the proof. $\square$

Lemma 4.2. *Let X be a compact plane set and let K be an infinite compact subset of X such that $X\backslash K$ is nonempty.*

(i) If $f \in P(X, K)$, if $\{p_n\}_{n=1}^{\infty}$ is a sequence of polynomials in z and if $\{g_n\}_{n=1}^{\infty}$ is a sequence in $CZ(X, K)$ with $\lim_{n\to\infty} \|p_n + g_n - f\|_X = 0$, then there exists $g \in C(\hat{K})$ such that $g|_K = f|_K$ and $\lim_{n\to\infty} \|p_n - g\|_{\hat{K}} = 0$.

(ii) If $f \in P(X, K)$, if $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ are sequences of polynomials in z, if $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ are sequences in $CZ(X, K)$ with $\lim_{n\to\infty} \|p_n + g_n - f\|_X = 0$ and if $\lim_{n\to\infty} \|q_n + h_n - f\|_X = 0$, then the sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ are uniformly convergence on $\hat{K}$ and $\lim_{n\to\infty} p_n(\lambda) = \lim_{n\to\infty} q_n(\lambda)$ for all $\lambda \in \hat{K}$.

Proof . (i) Since $g_n|_K = 0$ for all $n \in \mathbf{N}$, we have

$$\lim_{n\to\infty} \|p_n - f\|_K = 0. \qquad (1)$$

Therefore, $\{p_n\}_{n=1}^{\infty}$ is uniformly convergence on K. Since $\|p\|_K = \|p\|_{\hat{K}}$ for each polynomial p in z, we concloud that $\{p_n\}_{n=1}^{\infty}$ is uniformly convergence on $\hat{K}$. Thus, there exists $g \in C(\hat{K})$ such that

$$\lim_{n\to\infty} \|p_n - g\|_{\hat{K}} = 0. \qquad (2)$$

Now, we have $f|_K = g|_K$ by (1) and (2).
(ii) There exists $g, h \in C(\hat{K})$ with $f|_K = g|_K = h|_K$ such that

$$\lim_{n\to\infty} \|p_n - g\|_{\hat{K}} = \lim_{n\to\infty} \|q_n - h\|_{\hat{K}} = 0. \qquad (3)$$

Therefore,

$$\lim_{n\to\infty} \|p_n - q_n\|_K = 0. \qquad (4)$$

Since $\|p\|_K = 0$ implies that $\|p\|_{\hat{K}} = 0$ for each polynomial p in z, we concloud that

$$\lim_{n\to\infty} \|p_n - q\|_{\hat{K}} = 0,$$

by (4). Thus, $g = h$ and so $\lim_{n\to\infty} p_n(\lambda) = \lim_{n\to\infty} q_n(\lambda)$ for each $\lambda \in \hat{K}$, by (3). $\square$

Theorem 4.3. *Let X be a compact plane set and let K be an infinite compact subset of X such that $X \backslash K$ is nonempty.*

(i) If $\lambda \in \hat{K}$ and the map $E_\lambda : P(X,K) \longrightarrow C$ defined by

$$E_\lambda(f) = \lim_{n\to\infty} p_n(\lambda),$$

where $\{p_n\}_{n=1}^{\infty}$ is a sequence of polynomials in z, and if $\{g_n\}_{n=1}^{\infty}$ is a sequence in $CZ(X,K)$ such that $\lim_{n\to\infty} \|p_n|_X + g_n - f\|_X = 0$, then $E_\lambda \in M_{P(X,K)}$, $E_\lambda(p|_X) = p(\lambda)$ for each polynomial p in z, $E_\lambda(g) = 0$ for all $g \in CZ(X,K)$, and $E_\lambda(p|_X + g) = p(\lambda)$ for each polynomial p in z and for each $g \in CZ(X,K)$.

(ii) If $\lambda_1, \lambda_2 \in X$ with $\lambda_1 \neq \lambda_2$, then $E_{\lambda_1} \neq E_{\lambda_2}$.

Proof . (i) By Lemma 4.2, E_λ is well-defined. Clearly, E_λ is a complex homomorphism on X and $E_\lambda(1) = 1$. Therefore, $E_\lambda \in M_{P(X,K)}$.

Let p is a polynomial in z. Set $p_n = p$ and $g_n = 0$, for each $n \in N$. Then $\lim_{n\to\infty} \|p_n + g_n - p\|_X = 0$. Therefore,

$$E_\lambda(p|_X) = \lim_{n\to\infty} p_n(\lambda) = p(\lambda).$$

Let $g \in CZ(X,K)$. Set $p_n = 0$ and $g_n = g$ for each $n \in N$. Then $\lim_{n\to\infty} \|p_n + g_n - g\|_X = 0$. Therefore,

$$E_\lambda(g) = \lim_{n\to\infty} p_n(\lambda) = 0.$$

Let p be a polynomial in z and $g \in CZ(X,K)$. Set $p_n = p$ and $g_n = g$ for each $n \in N$. Then $\lim_{n\to\infty} \|p_n + g_n - (p+g)\|_X = 0$. Therefore,

$$E_\lambda(p|_X + g) = \lim_{n\to\infty} p_n(\lambda) = p(\lambda).$$

(ii) The coordinate function $Z : C \longrightarrow C$, defined by $Z(z) = z$, is a polynomial in z. Therefore, $E_\lambda(Z|_X) = Z(\lambda) = \lambda$ for all $\lambda \in \hat{K}$, by (i). Thus,

$$E_{\lambda_1}(Z|_X) = \lambda_1 \neq \lambda_2 = E_{\lambda_2}(Z|_X),$$

and so $E_{\lambda_1} \neq E_{\lambda_2}$. □

Theorem 4.4. *Let X be a compact plane set and let K be an infinite compact subset of X such that $X \backslash K$ is nonempty. Let $\varphi \in M_{P(X,K)}$*

(i) If $\varphi(g) = 0$ for each $g \in CZ(X,K)$, then there exists a unique $\lambda \in \hat{K}$ such that $\varphi = E_\lambda$.

(ii) If $\varphi(g) \neq 0$ for some $g \in CZ(X,K)$, then there exists a unique $\lambda \in X \backslash K$ such that $\varphi = e_\lambda$, where e_λ is the evaluation homomorphism on $P(X,K)$ at λ.

Proof . (i) Define the map $\eta : P(X) \longrightarrow C$ by $\eta(f) = \varphi(f)$. Clearly, η is a complex homomorphism on $P(X)$. Since $1 \in P(X)$ and $\eta(1) = \varphi(1) = 1$, we have $\eta \in M_{P(X)}$. It follows that there exists $\lambda \in \hat{X}$ such that $\eta(p|_X) = p(\lambda)$ for each polynomial p in z. We claim that $\lambda \in \hat{K}$. If $\lambda \in \hat{X} \backslash \hat{K}$, there exist a polynomial q in z such that $\|q\|_K < |q(\lambda)|$. There exists $f \in C(X)$ such that $f|_K = q|_K$ and $\|f\|_X = \|q\|_K$, by Tietze extension theorem [4;Theorem 20.4]. Clearly, $f \in P_0(X,K)$. By Lemma 3.1, there exists a polynomial q_1 in z and a function $g \in CZ(X,K)$ such that $f = q_1|_X + g$. Since $g|_K = 0$, we have $f|_K = q_1|_K$ and so $q_1|_K = q|_K$. Infiniteness of K implies that $q_1 = q$ on C and so $q_1(\lambda) = q(\lambda)$. Since $\eta(q_1|_X) = q_1(\lambda)$, we have

$$\begin{aligned} |q(\lambda)| &= |q_1(\lambda)| = |\eta(q_1|_X)| = |\varphi(q_1|_X)| \\ &= |\varphi(q_1|_X) + \varphi(g)| = |\varphi(q_1|_X + g)| \\ &= |\varphi(f)| \leq \|f\|_X = \|q\|_K. \end{aligned}$$

This contradiction shows that our claim is justified. Now, we prove that $\varphi = E_\lambda$. Let $f \in P_0(X,K)$. By Lemma 4.1, there exist a polynomial p in z and a function $g \in CZ(X,K)$ such that $f = p|_X + g$. Since $p|_X \in P(X)$ and $\varphi(g) = 0$, we have

$$\begin{aligned} \varphi(f) &= \varphi(p|_X + g) = \varphi(p|_X) + \varphi(g) \\ &= \eta(p|_X) = p(\lambda) = E_\lambda(f). \end{aligned}$$

Now, let $f \in P(X,K)$. There exists a sequence $\{f_n\}_{n=1}^\infty$ in $P_0(X,K)$ such that $f = \lim_{n\to\infty} f_n$ in $(P(X,K), \|.\|_X)$. By continuity of φ and E_λ on $(P(X,K), \|.\|_X)$, we have

$$\varphi(f) = \lim_{n\to\infty} \varphi(f_n) = \lim_{n\to\infty} E_\lambda(f_n) = E_\lambda(f).$$

Thus $\varphi = E_\lambda$.
(ii) Define the map $\psi : CZ(X,K) \longrightarrow C$ by $\psi(g) = \varphi(g)$. Clearly, ψ is a complex homomorphism on $CZ(X,K)$. By hypothesis, $\psi(g_0) \neq 0$. Therefore, $\psi \in M_{CZ(X,K)}$. It follows that there exists $\lambda \in X \backslash K$ such that

$$\varphi(g) = \psi(g) = g(\lambda),$$

for each $g \in CZ(X,K)$, by part (iii) of Theorem 3.3. We now define the map $\eta : P(X) \longrightarrow C$ by $\eta(f) = \varphi(f)$. Clearly, $\eta \in M_{P(X)}$. It follows that there exists $w \in \hat{X}$ such that $\eta(p|_X) = p(w)$ for each polynomial p in z. We show that $\lambda = w$. Since $\lambda \in X \backslash K$, there exists a function g_1 in $CZ(X,K)$

such that $g_1(\lambda) = 1$ by Urysohn's lemma. Let p be a polynomial in z. Then $p|_X g_1 \in CZ(X, K)$. Therefore,

$$\varphi(p|_X g_1) = (p|_X g_1)(\lambda) = p(\lambda)g_1(\lambda) = p(\lambda).$$

On the other hand, $p|_X \in P(X)$ and so we have

$$\begin{aligned}\varphi(p|_X g_1) &= \varphi(p|_X)\varphi(g_1) = \eta(p|_X)\Psi(g_1) \\ &= p(w)g_1(\lambda) = p(w).\end{aligned}$$

Therefore, $p(\lambda) = p(w)$. Consequently, $\lambda = w$. Thus, there exists $\lambda \in X\backslash K$ such that $\varphi(g) = g(\lambda)$ for each $g \in CZ(X, K)$ and $\varphi(p|_X) = p(\lambda)$ for each polynomial p in z.
Now, we show that $\varphi = e_\lambda$. Let $f \in P_0(X, K)$. By Lemma 4.1, there exist a polynomial p in z and a function g in $CZ(X, K)$ such that $f = p|_X + g$. Then

$$\begin{aligned}\varphi(f) &= \varphi(p|_X) + \varphi(g) = p(\lambda) + g(\lambda) \\ &= f(\lambda) = e_\lambda(f).\end{aligned}$$

The density of $P_0(X, K)$ in $(P(X, K), \|.\|_X)$ and continuity of φ and e_λ on $P(X, K)$, imply that $\varphi(f) = e_\lambda(f)$ for each $f \in P(X, K)$. Thus, $\varphi = e_\lambda$. □

5. Maximal Ideal Space of $R(X, K)$

Let X and K be compact plane sets such that $K \subseteq X$. In this section, we show that the uniform algebra $R(X, K)$ is natural. If $K = X$ or K is finite then $R(X, K)$ is natural, since $R(X, K) = R(X)$ or $R(X, K) = C(X)$, respectively. Therefore, $R(X, K)$ is natural.
For proving the naturality of $R(X, K)$ in the case where $X\backslash K$ is nonempty and K is infinite, we need the following lemma.

Lemma 5.1. *Let X and K be compact plane sets such that $K \subseteq X$, $X\backslash K$ nonempty, and K is infinite. If $\lambda \in X\backslash K$ and q if is a polynomial in z, then there exists $h \in P_0(X, K)$ such that $h|_K = q|_K$ and $h(\lambda) = 1$.*

Proof . By Urysohn's lemma, there exists $h_0 \in C(X)$ such that $h_0|_K = 0$ and $h_0(\lambda) = 1$. Define the function $h : X \longrightarrow C$ by

$$h(z) = q(z) + [1 - q(\lambda)]h_0(z) \qquad (z \in X).$$

Then $h \in P_0(X, K)$, $h|_K = q|_K$ and $h(\lambda) = 1$. □

Theorem 5.2. *Let X and K be compact plane sets such that $K \subseteq X$. If $X\backslash K$ is nonempty and K is infinite, then $R(X, K)$ is natural.*

Proof . Let $\varphi \in M_{R(X,K)}$. Define the map $\psi : P(X, K) \longrightarrow C$ by $\psi(f) = \varphi(f)$ $(f \in P(X, K))$. Clearly, ψ is a complex homomorphism on $P(X, K)$. Since $\psi(1) = \varphi(1) = 1$, so $\psi \in M_{P(X,K)}$.
We first suppose that $\varphi(g) = 0$ for all $g \in CZ(X, K)$. Therefore, there exists $\lambda \in \hat{K}$ such that $\psi(p|_X + g) = p(\lambda)$, for each polynomial p in z and each $g \in CZ(X, K)$, by part (i) of Theorem 4.4. If $f \in R_0(X, K)$, then there exist two polynomials p and q in z such that

$$q(z) \neq 0 \quad , \quad f(z) = \frac{p(z)}{q(z)},$$

for all $z \in K$. Since $q|_X, q|_X f \in P_0(X,K)$, we have

$$\begin{aligned}\varphi(q|_X f) &= \varphi(q|_X)\varphi(f) = \psi(q|_X)\varphi(f)\\ &= q(\lambda)\varphi(f).\end{aligned}$$

On the other hand, $(q|_X f)|_K = p|_K$. Therefore,

$$\varphi(q|_X f) = \psi(q|_X f) = p(\lambda).$$

We claim that $\lambda \in K$. If $\lambda \in \hat{K}\backslash K$, the compactness of K in C implies that there exist two polynomials p_1 and q_1 in z without any common zero on C and with $q_1(z) \neq 0$ for each $z \in K$ such that

$$q_1(\lambda) \neq 0 \quad , \quad \|\frac{p_1}{q_1}\|_K < \frac{|p_1(\lambda)|}{|q_1(\lambda)|}.$$

By Tietze extension theorem, we can extend $\frac{p_1}{q_1}|_K$ to a function $f_1 \in C(X)$ such that

$$\|f_1\|_X = \|\frac{p_1}{q_1}\|_K.$$

Since $q_1|_X f_1 \in C(X)$ and $q_1(z)f_1(z) = p_1(z)$ for each $z \in K$, so $q_1|_X f_1 \in P_0(X,K)$. It follows that

$$q_1(\lambda)\varphi(f_1) = p_1(\lambda),$$

by the above argument. Therefore,

$$\begin{aligned}\frac{|p_1(\lambda)|}{|q_1(\lambda)|} &= |\varphi(f_1)| \leq \|\varphi\|\|f_1\|_X\\ &= \|f_1\|_X = \|\frac{p_1}{q_1}\|_K.\end{aligned}$$

This contradiction shows that our claim is justified. Therefore, $\varphi(f) = f(\lambda) = e_\lambda(f)$ for each $f \in R_0(X,K)$. By the density of $R_0(X,K)$ in $(R(X,K), \|.\|_X)$ and continuity of φ and e_λ on $R(X,K)$, we conclude that $\varphi(f) = e_\lambda(f)$ for each $f \in R(X,K)$.
We now suppose that there exists $g_1 \in CZ(X,K)$ such that $\varphi(g_1) \neq 0$. It follows that there exists $\lambda \in x\backslash K$ such that $\psi(f) = f(\lambda)$ for each $f \in P(X,K)$. If $f \in R_0(X,K)$, then there exist two polynomials p and q in z without any common zeros such that $q(z) \neq 0$ and $f(z) = \frac{p(z)}{q(z)}$ for each $z \in K$. By Lemma 5.1, there exists $h \in P_0(X,K)$ such that $h|_K = q|_K$ and $h(\lambda) = 1$. Clearly, $fh \in P_0(X,K)$. Therefore,

$$\begin{aligned}\varphi(f) &= \varphi(f)h(\lambda) = \varphi(f)\psi(h)\\ &= \varphi(f)\varphi(h) = \varphi(fh)\\ &= \psi(fh) = (fh)(\lambda)\\ &= f(\lambda)h(\lambda) = f(\lambda)\\ &= e_\lambda(f).\end{aligned}$$

Since $R_0(X,K)$ is dense in $(R(X,K).\|.\|_X)$ and the map φ and e_λ are continuous on $R(X,K)$, we conclude that $\varphi(f) = e_\lambda(f)$ for each $f \in R(X,K)$. Consequently, the uniform algebra $R(X,K)$ is natural. $\square$

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