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Functionally closed sets and functionally convex sets in real Banach spaces

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Abstract

Let X be a real normed space, then $C(\subseteq X)$ is functionally convex (briefly, F-convex), if $T(C) \subseteq \mathbb{R}$ is convex for all bounded linear transformations $T \in B(X, R)$; and $K(\subseteq X)$ is functionally closed (briefly, F-closed), if $T(K) \subseteq \mathbb{R}$ is closed for all bounded linear transformations $T \in B(X, R)$. We improve the Krein-Milman theorem on finite dimensional spaces. We partially prove the Chebyshev 60 years old open problem. Finally, we introduce the notion of functionally convex functions. The function f on X is functionally convex (briefly, F-convex) if epi f is a F-convex subset of $X \times \mathbb{R}$. We show that every function $f : (a, b) \longrightarrow \mathbb{R}$ which has no vertical asymptote is F-convex.

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1. Introduction

In 1965, L.P. Vlasov defined an approximately convex subset M of a linear normed space X, by denoting the multivalued mapping which puts into correspondence with each point $x \in X$, the set Tx of all points $y \in M$ which satisfy the condition $d(x, y) = d(x, M)$. Then the set M is called approximately convex if, for $x \in X$ the set Tx is nonempty and convex. He proved that, in Banach spaces which are uniformly smooth in each direction, each approximately compact and approximately convex set is convex [12]. Another generalization of convexity defined by Green and Gustin [9]. They called a set $S \subseteq \mathbb{R}^n$ nearly convex, if there is $\alpha \in (0, 1)$ such that $\alpha x + (1 - \alpha)y \in S$ for all $x, y \in S$.

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Every convex set is nearly convex, while $\mathbb{Q}$, the rational numbers is nearly convex (with $\alpha = \frac{1}{2}$), which is not convex.

In this work, by defining two notions F-convexity and F-closedness of subsets of Banach spaces, we improve some basic theorems in functional analysis. The Krein-Milman theorem has been generalized on finite dimensional spaces. Hence, we show that the set of extreme points of every bounded, F-convex and F-closed subset of a finite dimensional space is nonempty. Additionally, we partially prove the famous Chebyshev open problem (which asks whether or not every Chebyshev set in a Hilbert space is convex?). Hence, we show that, if A is a Chebyshev subset of a Hilbert space and the metric projection P_A is continuous, then A is F-convex. Finally, we introduce the notion of F-convex functions and improve some results in convexity.

2. Main results

Throughout this paper we assume that X is a real vector space.

Definition 2.1. In a normed space X, we say that $K(\subseteq X)$ is m-functionally convex (briefly, m-F-convex) (for $m \in \mathbb{N}$) if for every bounded linear transformation $T \in B(X, R^m)$, the set $T(K)$ is convex. A 1-F-convex set is called F-convex. A subset K of X is called permanently F-convex if K is m- F-convex, for all $m \in \mathbb{N}$.

Proposition 2.2. *If T is a bounded linear mapping from a normed space X into a normed space Y, and K is F-convex in X, then $T(K)$ is F-convex in Y.*

Proof . For $g \in Y^*$, we have $g \circ T \in X^*$. So by assumption, $g(T(K))$ is convex. □

Proposition 2.3. *Let A,B be two F-convex subsets of a normed space X and λ be a real number, then*

$$A + B = \{a + b : a \in A, b \in B\}, \quad \lambda A = \{\lambda.a : a \in A\}$$

are F-convex. Moreover, $\overline{A}$, the closure of A is F-convex.

Proof . It follows from Proposition 2.2. □

As an example of a big class of F-convex sets, we have next theorem.

Theorem 2.4. Every arcwise connected subset of a normed space X is F-convex.

Proof . Let K be an arcwise connected subset of X and $f \in X^*$. For $f(x), f(y) \in f(K)$ and every $\lambda(0 \leq \lambda \leq 1)$, there is a continuous function $g : [0, 1] \longrightarrow K$ which, $g(0) = x$ and $g(1) = y$. Since $f \circ g$ is continuous, then the intermediate value theorem implies that $\lambda f(x) + (1 - \lambda)f(y) = f(g(t_0))$, for some $t_0 \in [0, 1]$. This completes the proof. □

Definition 2.5. Let X be a normed space and let $A \subseteq X$. A is functionally closed (briefly, F-closed), if $f(A)$ is closed for all $f \in X^*$.

Note that every compact set is F-closed. Also, every closed subset of real numbers $\mathbb{R}$ is F-closed. In $X = \mathbb{R}^2$, the set $A = \{(x, y) : x, y \geq 0\}$ is (non-compact) F-closed whereas, the set $A = \mathbb{Z} \times \mathbb{Z}$ is closed but it is not not F-closed (by taking $f(x, y) = x + \sqrt{2}y$, the set $f(A)$ is not closed in $\mathbb{R}$). By taking $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$ a nonconvex F-closed and F-convex set is obtained. Also, the set $B = \{(x, y) : x \in [0, \frac{\pi}{2}), y \geq \tan(x)\}$ is a closed convex set which is not F-closed. On the other hand, $A = \{(x, y) : 1 < x^2 + y^2 \leq 4\}$ is a non-closed and F-closed set. The two last examples show that weakly closed and F-closed sets are different.

Theorem 2.6. ([7]) If K_1 and K_2 are disjoint closed convex subsets of a locally convex linear topological space X, and if K_1 is compact, then there exist constants c and $\epsilon > 0$, and a continuous linear functional f on X, such that

$$f(K_2) \leq c - \epsilon < c \leq f(K_1).$$

Lemma 2.7. *If A is a subset of a Banach space X, then*

$$\bigcap_{f \in X^*} f^{-1}(f(A)) \subseteq \overline{co}(A).$$

Proof . If there exists x in $\bigcap_{f \in X^*} f^{-1}(f(A)) - \overline{co}(A)$, then for all $f \in X^*$, $f(x) \in f(A)$ and x is outside of $\overline{co}(A)$. By Theorem 2.6, there exist constants c and $\epsilon > 0$, and a continuous linear functional f on X, such that

$$f(\overline{co}(A)) \leq c - \epsilon < c \leq f(x).$$

On the other hand, $f(x) \in f(A) \subseteq f(\overline{co}(A))$. This is a contradiction and the proof is completed. □

Corollary 2.8. Let A be an F-closed subset of a Banach space X. Then A is F-convex if and only if

$$\overline{co}(A) = \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Corollary 2.9. *A compact subset A in a Banach space X is convex if and only if A is F-convex and X^* separates A and every element of $X - A$.*

Proof . If A is a compact convex subset of X, then by Theorem 2.6, the assertion holds. Conversely, assume that A is a compact F-convex subset of X. Hence, $\overline{co}(A) = \bigcap_{f \in X^*} f^{-1}(f(A))$. On the other hand, there is $f \in X^*$ such that for every $x \in X - A$, we have $f(A) < f(x)$. This implies that x is outside of $f^{-1}(f(A))$. Thus $f^{-1}(f(A)) = A$ and $\overline{co}(A) = A$. □

It follows from the Krein-Milman theorem that if K is a nonempty compact convex subset of a locally convex space X, then the set of extreme points of K is nonempty [6].

In what follows, we would like to replace "boundedness and F-closedness" instead of "compactness" in Krein-Milman theorem. Indeed, we show that the set of extreme points of every bounded, F-closed and F-convex subset of a finite dimensional space is nonempty.

Theorem 2.10. *Let X be a real Banach space with dimensional $n \in \mathbb{N}$. If A is a bounded, F-closed and F-convex subset of X, then*

$$\overline{co}(A) = \overline{co}(Ext(A)) = \bigcap_{f \in X^*} f^{-1}(f(A)).$$

Proof . First we prove the assertion for the case $n = 2$. Obviously, we have

$$\overline{co}(Ext(A)) \subseteq \overline{co}(A) = \bigcap_{f \in X^*} f^{-1}(f(A)). \tag{2.1}$$

On the other hand, if there exists $c \in \bigcap_{f \in X^*} f^{-1}(f(A)) - (\overline{co}(Ext(A)) \triangleq B)$, then $f(c) \in f(A)$ for all $f \in X^*$ and c does not belong to B. By Hahn–Banach separation theorem, there exists $g \in X^*$

such that $g(c) > \sup_B g$. By taking $\sup_A g = \alpha = g(a_0)$ and $H = g^{-1}(\alpha)$, we claim that there exists x in $Ext(A) \cap H$. If a_0 do not lie in $Ext(A)$, then there are $a_1, a_2 \in A$ such that $a_0 = \frac{a_1+a_2}{2}$. Hence, $2\alpha = 2g(a_0) = g(a_1) + g(a_2)$. This implies that $g(a_1) = g(a_2) = \alpha$ and so, $a_1, a_2 \in H$. We can define totally order relation " $\leq$ " on H. By this, let $a_1 < a_0 < a_2$. Since, $A \cap H \subset g^{-1}(g(A)) \subset \mathbb{R}^2$ and $g^{-1}(g(A))$ is closed and bounded so $\sup A \cap H = a^*$ exists. There is a sequence $\{h_n\} \subset A \cap H$ such that tends to a^*. This implies that $\lim g(h_n) = g(a^*) = \alpha$ and so a^* is in $A \cap H$. If a^* do not lie in $Ext(A)$, then there are $b, c \in A$ so that $a^* = \frac{b+c}{2}$. By the above manner, one may show that $b, c \in A \cap H$. Assume that $b < a^* < c$, but this is a contradiction. Therefore, there exists $x \in Ext(A) \cap H$. In this case, we have

$$g(x) = \alpha = \sup_A g \geq g(c) > \sup_B g. \tag{2.2}$$

Therefore, x does not lie in B, a contradiction.

Suppose that the assertion holds for real Banach spaces with dimension less than n. By the same reason as the case $n = 2$, it is sufficient to show that the set $Ext(A \cap H)$ is nonempty. Note that the set $\overline{A} \cap H$ is a closed and bounded subset of H, On the other hand, the supporting manifold H is isomorphic to a finite dimensional space with dimension less than n hence the set $C = \overline{A} \cap H$ is an F-closed subset of H. Therefore by the assumption $E = Ext(C)$ is nonempty. Let $e \in A$ and $e \notin Ext(A)$, we claim that $e \notin E$. Suppose that e is not an extreme point of A, then there are $a_1, a_2 \in A$ such that $e = \frac{a_1+a_2}{2}$. If $e \notin H$ then $e \notin \overline{A} \cap H$, hence $e \notin E$. If $e \in H$ then, $2\alpha = 2g(e) = g(a_1) + g(a_2)$. This implies that $g(a_1) = g(a_2) = \alpha$ and so, $a_1, a_2 \in A \cap H \subseteq C$, hence $E \subset Ext(A)$ and so, $Ext(A)$ is non-empty. □

We can not prove the above theorem for infinite dimensional spaces. Hence, it may be happened in every Banach space.

Remark 2.11. The set $A = \{(0, x) : \frac{1}{2} \leq x \leq 1\} \cup \{(x, y) : 1 < x^2 + y^2 \leq 4\}$ is a bounded F-closed set which is not compact. Note that $Ext(A) = \{(\frac{1}{2}, 0\} \cup \{(x, y) : x^2 + y^2 = 4\}$ and $\overline{co}(A) = \overline{co}(Ext(A))$.

Let X be a normed linear space and K be a nonempty subset of X. Note that the set-valued mapping $P_K : X \longrightarrow 2^K$ is defined by

$$P_K(x) = \{y \in K : \|x - y\| = d(x, K) = \inf_{k \in K} \|x - k\|\},$$

is called the metric projection or best approximation operator. K is called proximinal (semi-Chebyshev) if $P_K(x)$ contains at least (at most) one element for every $x \in X$. K is said to be Chebyshev if it is both proximinal and semi-Chebyshev, i.e., $P_K(x)$ is singleton for every $x \in X$. By the nearest point theorem, every nonempty closed convex set in a Hilbert space is Chebyshev. However, a famous unsolved problem is whether or not every Chebyshev set in a Hilbert space is convex.

If $A \subseteq X$ and $\overline{x} \in P_A(x)$, it is always true that $\overline{x} \in P_A(\lambda x + (1 - \lambda)\overline{x})$, for $\lambda \in (1, \infty)$. That is, $\overline{x}$ is a solar point in A for x, if $\overline{x} \in P_A(y)$, for every y in the half-line $R = \{\lambda x + (1 - \lambda)\overline{x} : \lambda \geq 0\}$. A set A is said to be a *Sun* in X, if for each $x \in X - A$, the set $P_A(x)$ contains a solar point for x (the half-line R is then a *ray* of the sun which passes through x).

Proposition 2.12. (Suns, [2]) Let A be a closed set in a Hilbert space. Then the following assertions are equivalent:
(i) A is convex;
(ii) A is a Sun;
(iii) the metric projection P_A is nonexpansive.

A more flexible notion than that of a Sun is that of an approximately convex set, [2]. We call $A \subseteq X$ *approximately convex* if, for any closed norm ball $D \subseteq X$ disjoint from A, there exists a closed ball $D' \supseteq D$ disjoint from A with arbitrary large radius. Every Sun is approximately convex.

For a closed set A in a Banach space X some sufficient conditions for existence of proximal points are proved. For instance:

1) X is reflexive and the norm is (sequentially) Kadec-Klee, (see [3, 5, 10]);

2) X has the Radon-Nikodym property [8] and A is bounded (see [3]);

3) X has norm closed and boundedly relatively weakly compact, (see [4]).

Now, we are ready to prove the following theorem which can be consider as a partially proof for Chebyshev open problem.

Theorem 2.13. *let X be a Banach space if $A \subseteq X$ is a Chebyshev set and the metric projection P_A is continuous, then A is F-convex.*

Proof . If A is not F-convex, then there exists a linear functional $f \in X^*$ such that $f(A)$ is not convex. Then there are $a_1, a_2 \in A$ and $\lambda \in (0,1)$ such that $f(\lambda a_1 + (1-\lambda)a_2)$ is not in $f(A)$. Therefore, by taking $x = \lambda a_1 + (1-\lambda)a_2$ and $K \doteq \mathrm{Ker}(f)$, $x - a$ is outside of K for all $a \in A$. Since the quotient space $\frac{X}{K}$ is isomorphic to $\mathbb{R}$, then there exists $x_0 \in X - K$ such that

$$\frac{X}{K} \simeq K^{\perp} = \{\alpha x_0 : \alpha \in \mathbb{R}\}.$$

This implies that there is $\lambda_a \in \mathbb{R}$ for all $a \in A$ such that $x - a = \lambda_a x_0$. By assumption A is a Chebyshev set, then there exists a unique $a_0 \in A$ such that

$$\|x - a_0\| = d(x, A) = \inf_{a \in A} \|x - a\| = \inf_{a \in A} |\lambda_a| \|x_0\|.$$

On the other hand, $\|x - a_0\| = |\lambda_0| \|x_0\|$ for some $\lambda_0 \in \mathbb{R} - \{0\}$ and then $0 < |\lambda_0| \leq |\lambda_a|$ for all $a \in A$. Thus for every α which $|\alpha| < |\lambda_0|$, we have

$$\forall a \in A; x - a \neq \alpha x_0.$$

.

Also, we have

$$\|x - a\| \geq \|x - a_0\| = |\lambda_0| \|x_0\| \doteq r$$

for all $a \in A$, then $\overline{B_r(x)} \cap A = \{a_0\}$ and $B_r(x) \cap A = \varnothing$. This is contrary to continuouity of P_A. □

Remark 2.14. The converse of the above mentioned theorem is not true. For example, if $A = \mathbb{R} \times \mathbb{R} - \{(x, y) : |x| \leq 1, y \geq 0\}$ is a closed F-convex subset of $X = \mathbb{R} \times \mathbb{R}$ which is not a Chebyshev set. Hence, every point on the nonnegative part of the y-axis has two nearest point in A.

Theorem 2.15. *Every Chebyshev and approximately convex set in a Hilbert space is F-convex.*

Proof . In the process of the proof of Theorem 2.13, we prove that if A is not F-convex then there are $a_1, a_2 \in A$ and an element $x \in X - A$, between them and $r > 0$ such that $B_r(x) \cap A = \varnothing$. But $B_{2d(x,a_1)}(x) \cap A \neq \varnothing$, then A is not approximately convex. □

Definition 2.16. ([7]) Let X be a real vector space and let f be a mapping from X into $\mathbb{R}$. The epigraph of f is the subset of $X \times \mathbb{R}$ defined by

$$epif := \{(x,r) \in X \times \mathbb{R} : f(x) \leq r\}.$$

The function f on X is convex if and only if epi f is a convex subset of $X \times \mathbb{R}$. In what follows, we define the notion of functionally convex (F-convex) functions.

Definition 2.17. The function f on X is F-convex if epi f is a F-convex subset of $X \times \mathbb{R}$.

Theorem 2.18. *If $f : (a,b) \longrightarrow \mathbb{R}$ is continuous then f is F-convex.*

Proof . Let (x_1, r_1) and (x_2, r_2) be two members of epi f. By joining point (x_1, r_1) to $(x_1, f(x_1))$ and (x_2, r_2) to $(x_2, f(x_2))$, we find a path which joins two members of epi f. So, epi f is a path-connected subset of $X \times \mathbb{R}$ and by Theorem 2.4, epi f is a F-convex subset. □

Theorem 2.19. *Every bounded function $f : (a,b) \longrightarrow \mathbb{R}$ is F–convex.*

Proof . There is $M \geq 0$ so that for all $x \in (a,b)$, $|f(x)| \leq M$. If (x_1, r) and (x_2, s) are elements of $epi(f)$ then the path

$$C = \{(x_1,t) : r \leq t \leq M\} + \{(t,M) : x_1 \leq t \leq x_2\} + \{(x_2,t) : s \leq t \leq M\}$$

joines this two points of $epi(f)$. This means that the epigragh of the function is path-connected. So, it is F–convex. □

One may verify that the Dirichlet function is F-convex. If the function $f : I \longrightarrow \mathbb{R}$ is not F-convex then there exists $x_0 \in I$ such that $f(x_o) = \infty$. Since, in this case there is a linear functional ϕ and elements (x_1, r_1) and (x_2, r_2) in epi f and $\lambda \in (0,1)$ such that $\phi(\lambda x_1 + (1-\lambda)x_2, \lambda r_1 + (1-\lambda r_2)) \doteq \phi(x_0, r_0)$ do not belong to the image of epi f under the linear functional ϕ. This implies that $f(x_0) > r$ for all $r \geq r_0$.

By applying Proposition 2.3, if $f, g : X \longrightarrow \mathbb{R}$ are two F-convex functions and $\alpha \in \mathbb{R}$, then $f + g$ and αf also are F-convex.

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