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Numerical algorithm for pricing of discrete barrier option in a Black-Scholes model

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Abstract

In this article, we propose a numerical algorithm for computing price of discrete single and double barrier option under the *Black–Scholes* model. In virtue of some general transformations, the partial differential equations of option pricing in different monitoring dates are converted into simple diffusion equations. The present method is fast compared to alternative numerical methods presented in previous papers.

Keywords: Discrete barrier option, *Black–Scholes* model, Constant parameters.
2000 MSC: Primary 47H10, 47H09; Secondary 54H25, 55M20.

1. Introduction

Option pricing is one of the most important problems in quantitative finance and many researchers are involved in it. As a description, down–and–out barrier option is that option which deactivated (knock–out) if the price of underlying asset touches the predetermined barrier. In practice and with attention to academic literature, barrier options have been studied under two discrete and continuous monitoring. In the first case, the price of underlying asset has been checked at predetermined monitoring dates. The price of underlying assets is usually modeled as geometric Brownian motion process where the model parameters are constants.

In the present paper, we try to price a down–and–out discrete single and double barrier option on an underlying asset which is modeled as geometric Brownian motion with constant parameters. In

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this regard, a set of transformations are applied to correspond partial differential equations (PDEs) for option price. Afterwards, the obtained PDEs are simply converted to familiar heat equations whose solutions are as multiple integral forms. Finally, a new numerical method is proposed to accurately computation these multiple integrals.

This article is managed as follows. In Section 2, the model structure for pricing discrete down–and–out single and double barrier options is discussed and a recursive method is presented. In Section 3, a numerical algorithm is proposed to evaluate the multiple integral in section 2. In addition, we compare the obtained results in the present paper to the alternative numerical methods in other papers for pricing discrete barrier options like [15] and [17]. At last, obtained conclusions and remarks are offered in Section 4.

2. Discrete barrier option modeling in the Black–Scholes world model

In this section, we focus on pricing discrete down–and–out call option and both down–and–out, up–and–out hedging on a underlying stock which could be expired its worth if a lower or upper barrier touches the continuous path of stock value at predetermined monitoring dates. At first we define some preliminary concepts. With attention to this fact that the summation of in and out call option price (in each case down or up) is equal to the price of a simple European call option [20, 21]. Other kind of barrier options like as down–and–out put option, could be priced using the put call parity given in[12]. Also we suppose that the price of underlying stock, that we denote it with X_t, is a Geometric Brownian Motion process, i.e.

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dW_t, \\ X_0 &= x_0, \end{aligned}$$

where W_t is *Wiener process*, $X_0 = x_0$ is stock price in initial time $t = 0$ and three deterministic constant values D, $\rho_{=}\mu - D$ and σ, are non–dividend–paying equity, *drift* and the time independent instantaneous *volatility* respectively. For more details about SDEs and its application, especially in mathematical finance, refer to [14], [8] and [16].

2.1. Black–Scholes *PDE for single barrier option pricing*

In all over our discussion, we consider $0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T$ the monitoring dates. The price of down–and–out call barrier option with the strike price K and lower barrier L, that is active in all monitoring dates t_n, is denoted by $\mathbb{B}(x, t, n) \equiv \mathbb{B}(x, t, n; L)$. So $\mathbb{B}(x, t, n)$ satisfy in the well–known *Black–Scholes* PDE with relevant initial conditions:

$$\frac{-\partial \mathbb{B}(x,t,n)}{\partial t} + \mu x \frac{\partial \mathbb{B}(x,t,n)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \mathbb{B}(x,t,n)}{\partial^2 x} - \mu \mathbb{B}(x,t,n) = 0, \tag{2.1}$$

$$\mathbb{B}(x, t_0, 0) = (x - K)\mathbf{1}_{(x \geq max(K,L))}; \qquad n = 0, \tag{2.2}$$

$$\mathbb{B}(x, t_n, n) = \mathbb{B}(x, t_n, n-1)\mathbf{1}_{(x \geq L)}; \qquad n = 1, 2, \ldots, N-1, \tag{2.3}$$

where $\mathbb{B}(x, t_n, n-1)$ is defined as $\mathbb{B}(x, t_n, n-1) := lim_{t \to t_n^-} \mathbb{B}(x, t, n-1)$ and $\mathbf{1}_{(x \geq L)}$ is characteristic function. Keeping away from making other symbols, we attempt to infer a way to reach the suitable option pricing for discrete barrier in monitoring dates.

Afterwards, we solve this PDE with a new method which is suitable for this kind of equations and compare it with other implemented methods applied in [9]. After applying the following transforms in each separate time interval:

$$\mathbb{B}(x,t,n) = \mathcal{B}(Z,t,n), \qquad Z = ln\left(\frac{x}{L}\right), \qquad k = ln\left(\frac{K}{L}\right), \tag{2.4}$$

and rearranging (2.1), based on well–known converter $\mathcal{B}(Z,t,n)$, a new PDE is concluded:

$$-\frac{\partial \mathcal{B}}{\partial t} + m\frac{\partial \mathcal{B}}{\partial Z} + \frac{\sigma^2}{2}\frac{\partial^2 \mathcal{B}}{\partial Z^2} - \mu\mathcal{B} = 0, \tag{2.5}$$

that $m = \mu - \sigma^2/2$, and according to the last conversion the initial condition (2.2) and (2.3) converts to following condition:

$$\mathcal{B}(Z,t_0,0) = L(e^Z - e^k)\mathbf{1}_{(Z\geq\delta)}, \qquad \delta = \max\{k,0\} \tag{2.6}$$

$$\mathcal{B}(Z,t_n,n) = \mathcal{B}(Z,t_n,n-1)\mathbf{1}_{(Z\geq 0)}, \qquad n = 1,2,\ldots,N-1. \tag{2.7}$$

By following transform

$$\mathcal{B}(Z,t,n) = e^{\alpha Z+\beta t}g(Z,t,n), \qquad n = 0,1,2,\ldots,N-1, \tag{2.8}$$

where αandβ are defined as

$$\alpha = -\frac{m}{\sigma^2}, \qquad \beta = \alpha m + \alpha^2\frac{\sigma^2}{2} - \mu, \tag{2.9}$$

we reach the Heat equation

$$-\frac{\partial g}{\partial t} + C^2\frac{\partial^2 g}{\partial Z^2} = 0, \qquad C^2 = \frac{\sigma^2}{2}, \qquad n = 0,1,2,\ldots,N-1. \tag{2.10}$$

In addition, the initial conditions (2.6) and (2.7) convert to following

$$g(Z,t_0,0) \;=\; Le^{-\alpha_0 Z}(e^Z - e^k)\mathbf{1}_{(Z\geq\delta)}, \qquad \delta = \max\{k,0\}, \tag{2.11}$$

$$g(Z,t_n,n) \;=\; g(Z,t_n,n-1)\mathbf{1}_{(Z\geq 0)}, \quad 1 \leq n \leq N-1 \tag{2.12}$$

which has unique analytical solution in each time interval $[t_n, t_{n+1}]$ (see[19]):

$$g(Z,t,n) \;=\; L\int_0^\infty S_n(Z-\xi,t-t_n)e^{-\alpha\xi}(e^\xi - e^k)\mathbf{1}_{(\xi\geq\delta)}d\xi \;,\; n = 0, \tag{2.13}$$

$$g(Z,t,n) \;=\; \int_0^\infty S_n(Z-\xi,t-t_n)g(\xi,t_n,n-1)\mathbf{1}_{(\xi\geq 0)}d\xi \;,\; n = 1,2,\ldots,N-1. \tag{2.14}$$

In above equality kernel $S(Z,t)$, is the normal distribution function $\mathcal{N}\left(0,\sqrt{4C^2t}\right)$

$$S_n(Z,t) = \frac{1}{\sqrt{4\pi C^2 t}}\exp\left(\frac{-Z^2}{4C^2t}\right), \qquad n = 0,1,2,\ldots,N-1. \tag{2.15}$$

According to the concluded results, the price of the discrete barrier option at monitoring dates t_n, can be calculated by following theorem.

Theorem 2.1. The price of down–and–out discrete barrier call option with stock price x, strike price K, and barrier level L, at monitoring dates t_{n+1}, are evaluated as follow

$$\mathbb{B}(x,t_{n+1},n) = g\left(ln(\frac{x}{L}),t_{n+1},n\right)\exp\{\alpha ln(\frac{x}{L}) + \beta t_{n+1}\}, \qquad n = 0,1,2,\ldots,N-1, \tag{2.16}$$

where the constants α and β are defined in (2.9) and $g(.,t_{n+1},n)$ is evaluated recursively in (2.13) and (2.14).

2.2. Black–Scholes *PDE for double barrier option pricing*

In this subsection, the price of down–and–out and up–and–out call double barrier option with the Strike price K, the constant lower and upper barrier L_1 and L_2, is denoted by $\mathbb{DB}(x,t,n) \equiv \mathbb{DB}(x,t,n,L_1,L_2)$. The double barrier option price $\mathbb{DB}(x,t,n)$, under the *Black–Scholes* world framework satisfy in the well–known *Black–Scholes* PDE

$$-\frac{\partial \mathbb{DB}(x,t,n)}{\partial t} + \mu x \frac{\partial \mathbb{DB}(x,t,n)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \mathbb{DB}(x,t,n)}{\partial^2 x} - \mu \mathbb{DB}(x,t,n) = 0, \tag{2.17}$$

with these initial conditions

$$\mathbb{DB}(x,t_0,0) = (x-K)\mathbf{1}_{(L_2 \geq x \geq max(K,L_1))}; \qquad n = 0, \tag{2.18}$$

$$\mathbb{DB}(x,t_n,n) = \mathbb{DB}(x,t_n,n-1)\mathbf{1}_{(L_2 \geq x \geq L_1)}; \qquad n = 1,2,\ldots,N-1, \tag{2.19}$$

where $\mathbb{DB}(x,t_n,n-1)$ is defined as $\mathbb{DB}(x,t_n,n-1) := lim_{t \to t_n^-} \mathbb{DB}(x,t,n-1)$. By applying following transform

$$\bar{\mathbb{DB}}(\bar{x},t,n) = \mathcal{DB}(Z,t,n), \qquad Z = ln(\frac{\bar{x}}{L_1}), \qquad k = ln(\frac{K}{L_1}) \tag{2.20}$$

and rewriting PDE (2.17) and initial conditions (2.18), based on $\mathcal{DB}(Z,t,n)$, we have

$$-\frac{\partial \mathcal{DB}}{\partial t} + m \frac{\partial \mathcal{DB}}{\partial Z} + \frac{\sigma^2}{2}\frac{\partial^2 \mathcal{DB}}{\partial Z^2} - \mu \mathcal{DB} = 0, \tag{2.21}$$

$$\mathcal{DB}(Z,t_0,0) = L_1(e^Z - e^k)\mathbf{1}_{(ln(\frac{L_2}{L_1}) \geq Z \geq \delta)}, \qquad \delta = \max\{k, 0\} \tag{2.22}$$

$$\mathcal{DB}(Z,t_n,n) = \mathcal{DB}(Z,t_n,n-1)\mathbf{1}_{(ln(\frac{L_2}{L_1}) \geq Z \geq 0)}, \qquad n = 1,2,\ldots,N-1, \tag{2.23}$$

where $m = \mu - \sigma^2/2$. Another conversion as follow is done in each time interval

$$\mathcal{DB}(Z,t,n) = e^{\alpha Z + \beta t} g(Z,t,n), \qquad n = 0,1,2,\ldots,N-1, \tag{2.24}$$

which α and β are defined by (2.9). After rewriting PDE (2.21) respect to $g(Z,t,n)$, we obtain the Heat equation:

$$-\frac{\partial g}{\partial t} + C^2 \frac{\partial^2 g}{\partial Z^2} = 0, \qquad C^2 = \frac{\sigma^2}{2}, \qquad n = 0,1,2,\ldots,N-1, \tag{2.25}$$

also the initial conditions (2.22) and (2.23) convert to following

$$g(Z,t_0,0) = L_1 e^{-\alpha_0 Z}(e^Z - e^k)\mathbf{1}_{(ln(\frac{L_2}{L_1}) \geq Z \geq \delta)}, \qquad \delta = \max\{k, 0\}, \tag{2.26}$$

$$g(Z,t_n,n) = g(Z,t_n,n-1)\mathbf{1}_{(ln(\frac{L_2}{L_1}) \geq Z \geq 0)}. \tag{2.27}$$

These are as well–known second order PDEs which have unique analytical solution in each time interval $[t_n, t_{n+1}]$ as follows [19]

$$g(Z,t,n) = L_1 \int_0^\infty S_n(Z-\xi, t-t_n) e^{-\alpha \xi}(e^\xi - e^k)\mathbf{1}_{(ln(\frac{L_2}{L_1}) \geq \xi \geq \delta)} d\xi, \quad n = 0, \tag{2.28}$$

$$g(Z,t,n) = \int_0^\infty S_n(Z-\xi, t-t_n) g(\xi, t_n, n-1)\mathbf{1}_{(ln(\frac{L_2}{L_1}) \geq \xi \geq 0)} d\xi, \; n = 1,2,\ldots,N-1. \tag{2.29}$$

According to the obtained results, the price of the discrete double barrier option at monitoring dates t_n, is given in a theorem.

Theorem 2.2. The price of down–and–out, up–and–out double discrete barrier call option with stock price x, strike price K, and barrier levels L_1 and L_2, at monitoring dates t_{n+1}, are evaluated as follows

$$\mathbb{DB}(x, t_{n+1}, n) = g\left(ln(\frac{x}{L_1}), t_{n+1}, n\right) \exp\{\alpha ln(\frac{x}{L_1}) + \beta t_{n+1}\}, \qquad n = 0, 1, 2, \dots, N-1, \tag{2.30}$$

where the constants α and β are defined in (2.9) and $g(., t_{n+1}, n)$ is evaluated recursively by (2.28) and (2.29).

3. Numerical algorithm and some numerical results

In this section, a fast numerical algorithm for computing price of double and single barrier option with discrete monitoring dates, based on romberg numerical integration method, is presented. Assume that stock price Z_0 is given, we intend to evaluate $g(Z_0, t, N)$ as the price of discrete double barrier option. Recursive formula (2.13) shows that for this purpose, dependent on numerical integration method that is implemented, we must evaluate $g(., t_N, N-1)$ in adequate points belong to $[0, \infty)$ but S function has exponential decay property and its maximum occurs in Z_0, so we could consider integral over finite interval $I_{N-1} = [0, Z_0 + l]$ instead of $[0, \infty)$ where l is chosen as large enough constant. In similar way to compute $g(\xi, t_N, N-1)$ where $\xi \in [0, Z_0 + l]$, we must compute $g(., t_{N-1}, N-2)$ in adequate points of the interval $I_{N-1} = [0, Z_0 + 2l]$. By following this process, finally to evaluate $g(\xi, t_2, 1)$ where $\xi \in [0, Z_0 + (N-1)l]$, we have to evaluate $g(\xi, t_1, 0)$ over $I_0 = [0, Nl]$.

Note that in application we can consider $I_n = [0, \min\{(N-n)l, H\}]$ that H is a practical constant. The algorithm for double barrier option is similar and it is just enough consider all integral interval $[0, \ln(l1/l2)]$. The semi–code of this algorithm is as follows:

Algorithm: Single barrier option pricing with N discrete monitoring dates

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Input:► $m \in \mathbb{N}$ positive integer, $N \in \mathbb{N}$ number of monitoring dates
Output:◄ $X \in \mathbb{R}^+$, option price.

$step \leftarrow 0$
$numnode_1 \leftarrow 2^m.Ceil(length(I_0)) + 1$
$h \leftarrow length(I_0)/numnode_1$
for $i = 0 : numnode_1$ **do**
  $\xi_i \leftarrow i.h$
end
for $i = 0 : numnode_1$ **do**
  Compute $g(\xi_i, t_1, 0)$ *by gaussian quadrature rule.*
end
for $step = 1 : N-2$ **do**
  $numnode_{step} \leftarrow 2^m.Ceil(length(I_{step})) + 1$
  $h \leftarrow length(I_{step})/numnode_{step}$
  for $i = 0 : numnode_{step}$ **do**
    $\xi_i \leftarrow i.h$

    end
    for $i = 0 : numnode_{step}$ **do**
        Compute $g(\xi_i, t_{step}, step-1)$ *by Romberg method based on Simpson's rule using nodal points* $g(\xi_j, t_{step-1}, step-2),\ 0 \leq j \leq numnode_{step-2}$.
    end
end
$\mathbf{X} \leftarrow g(z_0, t_N, N-1)$ *by Romberg method based on Simpson's rule using nodal points* $g(\xi_j, t_{N-1}, N-2),\ 0 \leq j \leq numnode_{N-2}$.

Example 3.1. Consider the problem of pricing down–and–out discrete barrier call option on stock for different levels of L, maturity time T, and monitoring dates. The employed parameters in this example are stock price = 100, strike = 100, $\mu = 0.1$, $\sigma = 0.3$, and $T = 0.2$ [9]. In Table 1, the pricing of a single barrier down–and–out call option for lower level L and different monitoring dates N has been presented. Also, the other methods which have been brought in this sample are the recursive integration method (RI) in [1] with 2000 used points; the continuous monitoring formula(CC) with the barrier level shifting which has been demonstrated in [7]; Trinomial tree method (TT) indicated in [6]; Monte Carlo (MC) in [3]. The Wiener–Hopf method (WH) is an analytical solution of discrete barrier option pricing [9].

Table 1: Discrete barrier option pricing of Example 3.1: $\mu = 0.1, \sigma^2 = 0.09$

N	L	Presented Method	AS	RI	CC	TT	Monte Carlo
5	89	6.2807	6.2807	6.2763	6.284	6.281	6.28092
5	95	5.6711	5.6711	5.6667	5.646	5.671	5.67124
5	97	5.1672	5.1672	5.1628	5.028	5.167	5.16739
25	89	6.2097	6.2099	6.2003	6.210	6.210	6.21059
25	95	5.0812	5.0814	5.0719	5.084	5.081	5.08203
25	97	4.1159	4.1158	4.1064	4.113	4.115	4.11621

Example 3.2. Consider an especial case of discrete double barrier option with constant drift and volatility which has been mentioned in [11]. Consider the problem of pricing down–and–out and up–and–out discrete double barrier call option on stock for different levels of L and U, maturity time $T = 1$, and monitoring dates. The parameters are Stock price=2, $\mu(t) = 0.05$, and $\sigma^2 = 0.5$. Obtained results are demonstrated in Table 2.

Table 2: Double Discrete Barrier option contract pricing of Example 3.2: $\mu = 0.05, \sigma^2 = 0.5$

K	L1	L2	Presented Method	Pricing by [11]	Monte Carlo
1.75	1	3	0.076181	0.076172	0.0772
2	1.5	3	0.018124	0.017856	0.0191

4. Conclusions and remarks

In this article, pricing of double and single discrete double barrier option under the *Black–Scholes* model with constant parameters, is investigated. The partial differential equations of option pricing in different monitoring dates are converted into simple diffusion equations and a fast numerical algorithm is presented. The accuracy of the numerical results shows the reliability and validity of this algorithm.

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