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Symmetric Rogers-Hölder's inequalities on diamond–α calculus

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Abstract

We present symmetric Rogers–Hölder's inequalities on time scales when $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=0$ and $\frac{r}{p}+\frac{r}{q}$ is not necessarily equal to 1 where p, q and r are nonzero real numbers.

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1. Introduction and Preliminaries

First we need here basic concepts of delta calculus. The results of delta calculus are adapted from [3, 7, 8]. A time scale is an arbitrary nonempty closed subset of the real numbers. It is denoted by $\mathbb{T}$. For $t \in \mathbb{T}$, *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, \infty)$ such that $\mu(t) := \sigma(t) - t$ is called the *graininess*. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, \infty)$ such that $\nu(t) := t - \rho(t)$ is called the *backward graininess*. If $\sigma(t) > t$, we say that t is *right–scattered*, while if $\rho(t) < t$, we say that t is *left–scattered*. Also, if

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$t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right–dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left–dense*. If $\mathbb{T}$ has a left–scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the derivative f^Δ is defined as follows. Let $t \in \mathbb{T}^k$, if there exists $f^\Delta(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$, then f is said to be differentiable at t, and $f^\Delta(t)$ is called the *delta derivative* of f at t. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *right–dense continuous (rd–continuous)* if it is continuous at each right–dense point and there exists a finite left limit at every left-dense point. The set of all rd–continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The next definition is given in [3, 7, 8].

Definition 1.1. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$, then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [3, 6, 7, 8].
If $\mathbb{T}$ has a right–scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}_k = \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is called *nabla differentiable* at $t \in \mathbb{T}_k$, if there exists $f^\nabla(t) \in \mathbb{R}$ with the following property: For any $\epsilon > 0$, there exists a neighborhood U of t, such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|$$

for all $s \in U$. A function $f : \mathbb{T} \to \mathbb{R}$ is *left–dense continuous or ld–continuous* provided it is continuous at left–dense points in $\mathbb{T}$ and its right–sided limits exist (finite) at right–dense points in $\mathbb{T}$. The set of all ld–continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$. The next definition is given in [3, 6, 7, 8].

Definition 1.2. *A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$ provided that $G^\nabla(t) = g(t)$ holds for all $t \in \mathbb{T}_k$, then the nabla integral of g is defined by*

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Now we present short introduction of diamond-α derivative as given in [3, 14].
Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ sense. For $t \in \mathbb{T}_k^k$, where $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$, *diamond–$\alpha$ derivative* $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t) \quad 0 \leq \alpha \leq 1.$$

Thus f is diamond–α differentiable if and only if f is Δ and ∇ differentiable. The diamond–α derivative reduces to the standard Δ–derivative for $\alpha = 1$, or the standard ∇–derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

Theorem 1.3. (Sheng et al. [14]) Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond–α differentiable at $t \in \mathbb{T}$. Then

(i) $f \pm g : \mathbb{T} \to \mathbb{R}$ is diamond–α differentiable at $t \in \mathbb{T}$, with

$$(f \pm g)^{\Diamond_\alpha}(t) = f^{\Diamond_\alpha}(t) \pm g^{\Diamond_\alpha}(t).$$

(ii) $fg : \mathbb{T} \to \mathbb{R}$ is diamond–α differentiable at $t \in \mathbb{T}$, with

$$(fg)^{\Diamond_\alpha}(t) = f^{\Diamond_\alpha}(t)g(t) + \alpha f^\sigma(t)g^\Delta(t) + (1-\alpha)f^\rho(t)g^\nabla(t).$$

(iii) For $g(t)g^\sigma(t)g^\rho(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond–α differentiable at $t \in \mathbb{T}$, with

$$\left(\frac{f}{g}\right)^{\Diamond_\alpha}(t) = \frac{f^{\Diamond_\alpha}(t)g^\sigma(t)g^\rho(t) - \alpha f^\sigma(t)g^\rho(t)g^\Delta(t) - (1-\alpha)f^\rho(t)g^\sigma(t)g^\nabla(t)}{g(t)g^\sigma(t)g^\rho(t)}.$$

Definition 1.4. (Sheng et al. [14]) Let $f : \mathbb{T} \to \mathbb{R}$ be diamond–α differentiable at $t \in \mathbb{T}$. Then:

(i) $(f)^{\Diamond_\alpha\Delta}(t) = \alpha f^{\Delta\Delta}(t) + (1-\alpha)f^{\nabla\Delta}(t)$;

(ii) $(f)^{\Diamond_\alpha\nabla}(t) = \alpha f^{\Delta\nabla}(t) + (1-\alpha)f^{\nabla\nabla}(t)$;

(iii) $(f)^{\Delta\Diamond_\alpha}(t) = \alpha f^{\Delta\Delta}(t) + (1-\alpha)f^{\Delta\nabla}(t) \neq (f)^{\Diamond_\alpha\Delta}(t)$;

(iv) $(f)^{\nabla\Diamond_\alpha}(t) = \alpha f^{\nabla\Delta}(t) + (1-\alpha)f^{\nabla\nabla}(t) \neq (f)^{\Diamond_\alpha\nabla}(t)$;

(v) $(f)^{\Diamond_\alpha\Diamond_\alpha}(t) = \alpha^2 f^{\Delta\Delta}(t) + \alpha(1-\alpha)[f^{\Delta\nabla}(t) + f^{\nabla\Delta}(t)] + (1-\alpha)^2 f^{\nabla\nabla}(t)$
$\neq \alpha^2 f^{\Delta\Delta}(t) + (1-\alpha)^2 f^{\nabla\nabla}(t)$.

Theorem 1.5. (Sheng et al. [14]) Let $a, t \in \mathbb{T}$, and $h : \mathbb{T} \to \mathbb{R}$. Then, the diamond–$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s)\Diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s \quad 0 \leq \alpha \leq 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 1.6. (Sheng et al. [14]) Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\Diamond_\alpha$–integrable functions on $[a, b]_\mathbb{T}$, then

(i) $\int_a^t [f(s) \pm g(s)]\Diamond_\alpha s = \int_a^t f(s)\ \Diamond_\alpha s \pm \int_a^t g(s)\Diamond_\alpha s$;

(ii) $\int_a^t cf(s)\Diamond_\alpha s = c\int_a^t f(s)\ \Diamond_\alpha s$;

(iii) $\int_a^t f(s)\Diamond_\alpha s = -\int_t^a f(s)\Diamond_\alpha s$;

(iv) $\int_a^t f(s)\Diamond_\alpha s = \int_a^b f(s)\ \Diamond_\alpha s + \int_b^t f(s)\Diamond_\alpha s$;

(v) $\int_a^a f(s)\Diamond_\alpha s = 0$.

Corollary 1.7. (Sheng et al. [14]) Let $t \in \mathbb{T}_k^k$ and $f : \mathbb{T} \to \mathbb{R}$. Then

$$\int_t^{\sigma(t)} f(s)\ \Diamond_\alpha(s) = \mu(t)[\alpha f(t) + (1-\alpha)f^\sigma(t)]$$

and

$$\int_{\rho(t)}^t f(s)\ \Diamond_\alpha(s) = \nu(t)[\alpha f^\rho(t) + (1-\alpha)f(t)].$$

The following result is given in [1, page 147].

Theorem 1.8. If for positive values of sets of f_i, g_i, h_i with condition $f_i g_i h_i = 1$ for $i = 1, \dots, n$, and the nonzero real numbers p, q and r satisfy $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$, then the following inequality holds

$$\left(\sum_{i=1}^{n} f_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} g_i^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} h_i^r\right)^{\frac{1}{r}} \geq 1, \tag{1.1}$$

if all but one of p, q and r are positive. Inequality (1.1) is reversed if all but one of p, q and r are negative for positive values of sets of f_i, g_i and h_i.

The upcoming theorem is given in [4].

Theorem 1.9. *Let p, q, a_k and b_k $(k = 1, 2, \dots, n)$ are positive real numbers, then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^{\frac{1}{p}+\frac{1}{q}} \leq \left(\sum_{k=1}^{n} a_k^p \sum_{k=1}^{n} a_k^{2-p} b_k^2\right)^{\frac{1}{2p}} \left(\sum_{k=1}^{n} b_k^q \sum_{k=1}^{n} a_k^2 b_k^{2-q}\right)^{\frac{1}{2q}}. \tag{1.2}$$

The sign of equality holds in (1.2) *if and only if there exist real constants C_1 and C_2 such that $a_k^{p-1} = C_1 b_k$ and $b_k^{q-1} = C_2 a_k$, $(k = 1, \dots, n)$.*

Inequalities (1.1) and (1.2) can be unified and extended in weighted form on dynamic time scales which was initiated by Stefan Hilger given in [11]. Our aim is to present these applications of Rogers–Hölder's inequalities on diamond–α calculus. Discrete form of Hölder's inequality is given in [12] and found separately by Rogers and Hölder. Integral form of classical Rogers–Hölder's inequality on time scale calculus is given in [2, 3].

Theorem 1.10. (Agarwal et al. [3]) Let $a, b \in \mathbb{T}$ with $a < b$ and let $f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be $\Diamond_\alpha$–integrable functions. If $\frac{1}{p} + \frac{1}{q} = 1$, with $p > 1$, then

$$\int_a^b |f(t)g(t)| \, \Diamond_\alpha t \leq \left(\int_a^b |f(t)|^p \Diamond_\alpha t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \Diamond_\alpha t\right)^{\frac{1}{q}}. \tag{1.3}$$

Theorem 1.11. (Agarwal et al. [3]) Let $a, b \in \mathbb{T}$ with $a < b$ and let $h, f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be $\Diamond_\alpha$–integrable functions. If $\frac{1}{p} + \frac{1}{q} = 1$, with $p > 1$, then

$$\int_a^b |h(t)||f(t)g(t)| \Diamond_\alpha t \leq \left(\int_a^b |h(t)||f(t)|^p \Diamond_\alpha t\right)^{\frac{1}{p}} \left(\int_a^b |h(t)||g(t)|^q \Diamond_\alpha t\right)^{\frac{1}{q}}. \tag{1.4}$$

In the case when $p = q = 2$, then Rogers–Hölder's inequality reduces to the following diamond–α Cauchy–Schwarz's inequality on time scales as

$$\int_a^b |h(t)||f(t)g(t)| \Diamond_\alpha t \leq \sqrt{\left(\int_a^b |h(t)||f(t)|^2 \Diamond_\alpha t\right) \left(\int_a^b |h(t)||g(t)|^2 \Diamond_\alpha t\right)}. \tag{1.5}$$

Theorem 1.12. (Chen et al. [9]) Let $a, b \in \mathbb{T}$ with $a < b$ and let $f_i \in C([a, b]_{\mathbb{T}}, \mathbb{R})$, $i = 1, \dots, n$ be $\Diamond_\alpha$–integrable functions and $p_i > 1$ such that $\sum\limits_{i=1}^{n} \frac{1}{p_i} = 1$. Then

$$\int_a^b \prod_{i=1}^{n} |f_i(t)| \Diamond_\alpha t \leq \prod_{i=1}^{n} \left(\int_a^b |f_i(t)|^{p_i} \Diamond_\alpha t\right)^{\frac{1}{p_i}}, \tag{1.6}$$

which is generalized Rogers–Hölder's Inequality.

2. Main Results

Our first main result is proved for different conditions imposed on p, q and r as $p > 0$, $q > 0$ but $r < 0$; $p > 0$, $r > 0$ but $q < 0$ or $q > 0$, $r > 0$ but $p < 0$ and is reversed if $q < 0$, $r < 0$ but $p > 0$; $p < 0$, $r < 0$ but $q > 0$ or $p < 0$, $q < 0$ but $r > 0$ in upcoming theorem.

Theorem 2.1. *Let $h, f_i \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\Diamond_\alpha$–integrable functions for $i = 1,2,3$ and p, q and r be three nonzero real numbers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$. Further assume that $\prod_{i=1}^{3} f_i(x) = 1$.*

(*i*) *If $p > 0$, $q > 0$ but $r < 0$, then*

$$\left(\int_a^b |h(x)||f_1(x)|^p \Diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |h(x)||f_2(x)|^q \Diamond_\alpha x\right)^{\frac{1}{q}} \left(\int_a^b |h(x)||f_3(x)|^r \Diamond_\alpha x\right)^{\frac{1}{r}} \geq 1. \tag{2.1}$$

(*ii*) *If $q < 0$, $r < 0$ but $p > 0$, then*

$$\left(\int_a^b |h(x)||f_1(x)|^p \Diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |h(x)||f_2(x)|^q \Diamond_\alpha x\right)^{\frac{1}{q}} \left(\int_a^b |h(x)||f_3(x)|^r \Diamond_\alpha x\right)^{\frac{1}{r}} \leq 1. \tag{2.2}$$

Proof . To prove (i), given condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$ can be rearranged as $\frac{1}{(-\frac{p}{r})} + \frac{1}{(-\frac{q}{r})} = 1$. Let $P = -\frac{p}{r} > 1$, $Q = -\frac{q}{r} > 1$. Applying Rogers–Hölder's inequality, we have

$$\int_a^b |h(x)||f_1(x)f_2(x)|\Diamond_\alpha x \leq \left(\int_a^b |h(x)||f_1(x)|^P \Diamond_\alpha x\right)^{\frac{1}{P}} \left(\int_a^b |h(x)||f_2(x)|^Q \Diamond_\alpha x\right)^{\frac{1}{Q}},$$

and

$$\int_a^b |h(x)||f_1(x)f_2(x)|\Diamond_\alpha x \leq \left(\int_a^b |h(x)||f_1(x)|^{-\frac{p}{r}} \Diamond_\alpha x\right)^{-\frac{r}{p}} \left(\int_a^b |h(x)||f_2(x)|^{-\frac{q}{r}} \Diamond_\alpha x\right)^{-\frac{r}{q}}. \tag{2.3}$$

By replacing $|f_1(x)|$ by $|f_1(x)|^{-r}$ and $|f_2(x)|$ by $|f_2(x)|^{-r}$ and taking power $-\frac{1}{r} > 0$, (2.3) takes the form

$$\left(\int_a^b |h(x)||f_1(x)f_2(x)|^{-r} \Diamond_\alpha x\right)^{-\frac{1}{r}} \leq \left(\int_a^b |h(x)||f_1(x)|^p \Diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |h(x)||f_2(x)|^q \Diamond_\alpha x\right)^{\frac{1}{q}}. \tag{2.4}$$

As $f_1(x)f_2(x)f_3(x) = 1$, then (2.4) takes the form

$$\left(\int_a^b |h(x)||f_3(x)|^r \Diamond_\alpha x\right)^{-\frac{1}{r}} \leq \left(\int_a^b |h(x)||f_1(x)|^p \Diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |h(x)||f_2(x)|^q \Diamond_\alpha x\right)^{\frac{1}{q}}.$$

This follows inequality (2.1).
Now to prove (ii), the given condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$ can be rearranged as $\frac{1}{(-\frac{q}{p})} + \frac{1}{(-\frac{r}{p})} = 1$. Let $P = -\frac{q}{p} > 1$, $Q = -\frac{r}{p} > 1$. Applying Rogers–Hölder's inequality, we get

$$\int_a^b |h(x)||f_2(x)f_3(x)|\Diamond_\alpha x \leq \left(\int_a^b |h(x)||f_2(x)|^{-\frac{q}{p}} \Diamond_\alpha x\right)^{-\frac{p}{q}} \left(\int_a^b |h(x)||f_3(x)|^{-\frac{r}{p}} \Diamond_\alpha x\right)^{-\frac{p}{r}}$$

and then replacing $|f_2(x)|$ by $|f_2(x)|^{-p}$ and $|f_3(x)|$ by $|f_3(x)|^{-p}$ and taking power $-\frac{1}{p} < 0$, we get required inequality (2.2). This completes the proof. □

Now we generalize the inequalities (2.1) and (2.2) in upcoming theorem.

Theorem 2.2. *Let $h, f_i \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\Diamond_\alpha$–integrable functions for $i = 1, 2, \ldots, n$ and p_i be nonzero real numbers with $\sum\limits_{i=1}^{n} \frac{1}{p_i} = 0$. Further assume that $\prod\limits_{i=1}^{n} f_i(x) = 1$.*

(i) If all p_i for $i = 1, 2, \ldots, n$ but one are positive, then

$$\prod_{i=1}^{n} \left(\int_a^b |h(x)||f_i(x)|^{p_i} \Diamond_\alpha x \right)^{\frac{1}{p_i}} \geq 1. \tag{2.5}$$

(ii) If all p_i for $i = 1, 2, \ldots, n$ but one are negative, then

$$\prod_{i=1}^{n} \left(\int_a^b |h(x)||f_i(x)|^{p_i} \Diamond_\alpha x \right)^{\frac{1}{p_i}} \leq 1. \tag{2.6}$$

Proof . To prove (i), the given condition becomes $\sum\limits_{i=1}^{n-1} \frac{1}{p_i} + \frac{1}{p_n} = 0$ can be rearranged as $\sum\limits_{i=1}^{n-1} \frac{1}{(-\frac{p_i}{p_n})} = 1$, where all p_i are positive for $i = 1, \ldots, n-1$ but p_n is negative. Let $P_i = -\frac{p_i}{p_n} > 1$ for $i = 1, \ldots, n-1$. Applying generalized Rogers–Hölder's inequality, we have

$$\int_a^b |h(x)| \prod_{i=1}^{n-1} |f_i(x)| \Diamond_\alpha x \leq \prod_{i=1}^{n-1} \left(\int_a^b |h(x)||f_i(x)|^{P_i} \Diamond_\alpha x \right)^{\frac{1}{P_i}}$$

and

$$\int_a^b |h(x)| \prod_{i=1}^{n-1} |f_i(x)| \Diamond_\alpha x \leq \prod_{i=1}^{n-1} \left(\int_a^b |h(x)||f_i(x)|^{-\frac{p_i}{p_n}} \Diamond_\alpha x \right)^{-\frac{p_n}{p_i}}. \tag{2.7}$$

By replacing $|f_i(x)|$ by $|f_i(x)|^{-p_n}$ for $i = 1, \ldots, n-1$ and taking power $-\frac{1}{p_n} > 0$, (2.7) takes the form

$$\left(\int_a^b |h(x)| \prod_{i=1}^{n-1} |f_i(x)|^{-p_n} \Diamond_\alpha x \right)^{-\frac{1}{p_n}} \leq \prod_{i=1}^{n-1} \left(\int_a^b |h(x)||f_i(x)|^{p_i} \Diamond_\alpha x \right)^{\frac{1}{p_i}}. \tag{2.8}$$

Applying condition $\prod\limits_{i=1}^{n} f_i(x) = 1$, (2.8) takes the form

$$\left(\int_a^b |h(x)||f_n(x)|^{p_n} \Diamond_\alpha x \right)^{-\frac{1}{p_n}} \leq \prod_{i=1}^{n-1} \left(\int_a^b |h(x)||f_i(x)|^{p_i} \Diamond_\alpha x \right)^{\frac{1}{p_i}}.$$

This follows inequality (2.5).

Now to prove (ii), the given condition $\sum\limits_{i=1}^{n-1} \frac{1}{p_i} + \frac{1}{p_n} = 0$ can be rearranged as $\sum\limits_{i=1}^{n-1} \frac{1}{(-\frac{p_i}{p_n})} = 1$, where all p_i for $i = 1, \ldots, n-1$ are negative but p_n is positive. Let $P_i = -\frac{p_i}{p_n} > 1$ for $i = 1, \ldots, n-1$. Applying Rogers–Hölder's inequality, we get

$$\int_a^b |h(x)| \prod_{i=1}^{n-1} |f_i(x)| \Diamond_\alpha x \leq \prod_{i=1}^{n-1} \left(\int_a^b |h(x)||f_i(x)|^{-\frac{p_i}{p_n}} \Diamond_\alpha x \right)^{-\frac{p_n}{p_i}}$$

and then replacing $|f_i(x)|$ by $|f_i(x)|^{-p_n}$ and taking power $-\frac{1}{p_n} < 0$, we get required inequality (2.6). $\square$

As diamond–α integral is the combination of delta and nabla integrals. Now we present our results on delta and nabla calculus.

Corollary 2.3. Let $h, f_i \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ for $i = 1, 2, 3$ and p, q and r be three nonzero real numbers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$. Further assume that $\prod_{i=1}^{3} f_i(x) = 1$.

(i) If $p > 0$, $q > 0$ but $r < 0$; $p > 0$, $r > 0$ but $q < 0$ or $q > 0$, $r > 0$ but $p < 0$, then

$$\left(\int_a^b |h(x)||f_1(x)|^p \,\Delta x\right)^{\frac{1}{p}} \left(\int_a^b |h(x)||f_2(x)|^q \,\Delta x\right)^{\frac{1}{q}} \left(\int_a^b |h(x)||f_3(x)|^r \,\Delta x\right)^{\frac{1}{r}} \geq 1. \quad (2.9)$$

(ii) If $q < 0$, $r < 0$ but $p > 0$; $p < 0$, $r < 0$ but $q > 0$ or $p < 0$, $q < 0$ but $r > 0$, then

$$\left(\int_a^b |h(x)||f_1(x)|^p \,\Delta x\right)^{\frac{1}{p}} \left(\int_a^b |h(x)||f_2(x)|^q \,\Delta x\right)^{\frac{1}{q}} \left(\int_a^b |h(x)||f_3(x)|^r \,\Delta x\right)^{\frac{1}{r}} \leq 1. \quad (2.10)$$

Remark 2.4. If $\mathbb{T} = \mathbb{Z}$ and $h(x) = 1$, then (2.9) takes its discrete form for positive values of sets of f_i, g_i, h_i with condition $f_i g_i h_i = 1$ for $i = 1, \ldots, n$, which is given in (1.1). Moreover, if $\mathbb{T} = \mathbb{Z}$, then (2.10) takes reverse discrete form of (1.1).

Remark 2.5. As for $\mathbb{T} = \mathbb{R}$, we have

$$\int_a^b \textbf{.}\, \Delta x = \int_a^b \textbf{.}\, dx.$$

Therefore we can get continuous versions of (2.9) and (2.10), respectively in next corollary.

Corollary 2.6. Let $h, f_i \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R})$ for $i = 1, 2, 3$ and p, q and r be three nonzero real numbers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$. Further assume that $\prod_{i=1}^{3} f_i(x) = 1$.

(i) If $p > 0$, $q > 0$ but $r < 0$; $p > 0$, $r > 0$ but $q < 0$ or $q > 0$, $r > 0$ but $p < 0$, then

$$\left(\int_a^b |h(x)||f_1(x)|^p \,\nabla x\right)^{\frac{1}{p}} \left(\int_a^b |h(x)||f_2(x)|^q \,\nabla x\right)^{\frac{1}{q}} \left(\int_a^b |h(x)||f_3(x)|^r \,\nabla x\right)^{\frac{1}{r}} \geq 1. \quad (2.11)$$

(ii) If $q < 0$, $r < 0$ but $p > 0$; $p < 0$, $r < 0$ but $q > 0$ or $p < 0$, $q < 0$ but $r > 0$, then

$$\left(\int_a^b |h(x)||f_1(x)|^p \,\nabla x\right)^{\frac{1}{p}} \left(\int_a^b |h(x)||f_2(x)|^q \,\nabla x\right)^{\frac{1}{q}} \left(\int_a^b |h(x)||f_3(x)|^r \,\nabla x\right)^{\frac{1}{r}} \leq 1. \quad (2.12)$$

Now second part of our main results is started. The following inequality is a weighted symmetric form of Rogers–Holder's inequality.

Theorem 2.7. *Let $h, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\Diamond_\alpha$–integrable functions.*

(i) *Take p, q and r three nonzero positive real numbers. Then the following inequality holds*

$$\begin{aligned}\left(\int_a^b |h(x)||f(x)g(x)|\Diamond_\alpha x\right)^{\frac{r}{p}+\frac{r}{q}} &\leq \left(\int_a^b |h(x)||f(x)|^{\frac{p}{r}}\Diamond_\alpha x \int_a^b |h(x)||f(x)|^{2-\frac{p}{r}}|g(x)|^2\Diamond_\alpha x\right)^{\frac{r}{2p}} \\ &\times \left(\int_a^b |h(x)||g(x)|^{\frac{q}{r}}\Diamond_\alpha x \int_a^b |h(x)||g(x)|^{2-\frac{q}{r}}|f(x)|^2\Diamond_\alpha x\right)^{\frac{r}{2q}}; \quad (2.13)\end{aligned}$$

(*ii*) *Take p and q are positive real numbers and r be negative. Then the following reverse inequality holds*

$$\left(\int_a^b |h(x)||f(x)g(x)|\Diamond_\alpha x\right)^{\frac{r}{p}+\frac{r}{q}} \geq \left(\int_a^b |h(x)||f(x)|^{\frac{p}{r}}\Diamond_\alpha x \int_a^b |h(x)||f(x)|^{2-\frac{p}{r}}|g(x)|^2\Diamond_\alpha x\right)^{\frac{r}{2p}} \times \left(\int_a^b |h(x)||g(x)|^{\frac{q}{r}}\Diamond_\alpha x \int_a^b |h(x)||g(x)|^{2-\frac{q}{r}}|f(x)|^2\Diamond_\alpha x\right)^{\frac{r}{2q}} \quad (2.14)$$

and sign of equality holds in (2.13) *and* (2.14)*, if* $|f(x)|^{\frac{p}{r}-1} = |Cg(x)|$ *and* $|g(x)|^{\frac{q}{r}-1} = |Df(x)|$*, where* $|C|$ *and* $|D|$ *are two real numbers.*

Proof . For the proof of (i), we can write

$$\begin{aligned}
&\left(\int_a^b |h(x)||f(x)g(x)|\Diamond_\alpha x\right)^2 \\
= &\left(\int_a^b |h(x)||f(x)|^{\frac{p}{2r}}|f(x)|^{1-\frac{p}{2r}}|g(x)|\Diamond_\alpha x\right)^2 \\
\leq &\left(\int_a^b |h(x)||f(x)|^{\frac{p}{r}}\Diamond_\alpha x\right)\left(\int_a^b |h(x)||f(x)|^{2(1-\frac{p}{2r})}|g(x)|^2\Diamond_\alpha x\right) \\
= &\left(\int_a^b |h(x)||f(x)|^{\frac{p}{r}}\Diamond_\alpha x\right)\left(\int_a^b |h(x)||f(x)|^{2-\frac{p}{r}}|g^2(x)|\Diamond_\alpha x\right).
\end{aligned}$$

As $\frac{r}{p} > 0$, then by taking power $\frac{r}{p}$ on both sided we obtain

$$\left(\int_a^b |h(x)||f(x)g(x)|\Diamond_\alpha x\right)^{\frac{r}{p}} \leq \left(\int_a^b |h(x)||f(x)|^{\frac{p}{r}}\Diamond_\alpha x \int_a^b |h(x)||f(x)|^{2-\frac{p}{r}}|g(x)|^2\Diamond_\alpha x\right)^{\frac{r}{2p}}. \quad (2.15)$$

Similarly, we can write

$$\left(\int_a^b |h(x)||f(x)g(x)|\Diamond_\alpha x\right)^{\frac{r}{q}} \leq \left(\int_a^b |h(x)||g(x)|^{\frac{q}{r}}\Diamond_\alpha x \int_a^b |h(x)||g(x)|^{2-\frac{q}{r}}|f(x)|^2\Diamond_\alpha x\right)^{\frac{r}{2q}}. \quad (2.16)$$

Combining (2.15) and (2.16), we get (2.13). If p and q are positive real numbers and r be negative, then (2.14) is clear. Clearly the sign of equality holds in (2.13) and (2.14), if $|f(x)|^{\frac{p}{r}-1} = |Cg(x)|$ and $|g(x)|^{\frac{q}{r}-1} = |Df(x)|$, where $|C|$ and $|D|$ are two real numbers. □

Remark 2.8. If $\frac{p}{r} = \frac{q}{r} = 2$, then (2.13) reduces to (1.5).

Remark 2.9. Further by using GM–AM inequality, (2.13) can be written as

$$\left(\int_a^b |h(x)||f(x)g(x)|\Diamond_\alpha x\right)^{\frac{r}{p}+\frac{r}{q}} \leq \frac{1}{2}\left(\int_a^b |h(x)||f(x)|^{\frac{p}{r}}\Diamond_\alpha x\right)^{\frac{r}{p}}\left(\int_a^b |h(x)||g(x)|^{\frac{q}{r}}\Diamond_\alpha x\right)^{\frac{r}{q}} + \frac{1}{2}\left(\int_a^b |h(x)||f(x)|^{2-\frac{p}{r}}|g(x)|^2\Diamond_\alpha x\right)^{\frac{r}{p}}\left(\int_a^b |h(x)||g(x)|^{2-\frac{q}{r}}|f(x)|^2\Diamond_\alpha x\right)^{\frac{r}{q}}. \quad (2.17)$$

Remark 2.10. If $\alpha = 1$, $r = 1$, $h(x) = 1$, $\frac{1}{p} + \frac{1}{q} < 1$ and $\mathbb{T} = \mathbb{Z}$ and p, q, f_k and g_k $(k = 1, 2, \ldots, n)$ are positive real numbers, then discrete version of (2.17) can be written as

$$\left(\sum_{k=1}^{n} f_k g_k\right)^{\frac{1}{p}+\frac{1}{q}} \leq \frac{1}{2}\left(\left(\sum_{k=1}^{n} f_k^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n} g_k^q\right)^{\frac{1}{q}} + \left(\sum_{k=1}^{n} f_k^{2-p} g_k^2\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n} f_k^2 g_k^{2-q}\right)^{\frac{1}{q}}\right),$$

as given in [10, 13].

Now we present delta and nabla versions of Theorem 2.7.

Corollary 2.11. Let $h, f, g \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$. Take p, q and r three positive real numbers. Then the following inequality holds

$$\begin{aligned}\left(\int_a^b |h(x)||f(x)g(x)|\Delta x\right)^{\frac{r}{p}+\frac{r}{q}} &\leq \left(\int_a^b |h(x)||f(x)|^{\frac{p}{r}}\Delta x \int_a^b |h(x)||f(x)|^{2-\frac{p}{r}}|g(x)|^2\Delta x\right)^{\frac{r}{2p}} \\ &\times \left(\int_a^b |h(x)||g(x)|^{\frac{q}{r}}\Delta x \int_a^b |h(x)||g(x)|^{2-\frac{q}{r}}|f(x)|^2\Delta x\right)^{\frac{r}{2q}}, \quad (2.18)\end{aligned}$$

and sign of equality holds if $|f(x)|^{\frac{p}{r}-1} = |Cg(x)|$ and $|g(x)|^{\frac{q}{r}-1} = |Df(x)|$, where $|C|$ and $|D|$ are two real numbers.

Remark 2.12. If $\mathbb{T} = \mathbb{Z}$, $r = 1$ and $h(x) = 1$, then we get discrete version of (2.18) as given in (1.2). And if $\mathbb{T} = \mathbb{R}$, then we get continuous version of (2.18).

Corollary 2.13. Let $h, f, g \in C_{ld}([a, b]_{\mathbb{T}}, \mathbb{R})$. Take p, q and r three positive real numbers. Then the following inequality holds

$$\begin{aligned}\left(\int_a^b |h(x)||f(x)g(x)|\nabla x\right)^{\frac{r}{p}+\frac{r}{q}} &\leq \left(\int_a^b |h(x)||f(x)|^{\frac{p}{r}}\nabla x \int_a^b |h(x)||f(x)|^{2-\frac{p}{r}}|g(x)|^2\nabla x\right)^{\frac{r}{2p}} \\ &\times \left(\int_a^b |h(x)||g(x)|^{\frac{q}{r}}\nabla x \int_a^b |h(x)||g(x)|^{2-\frac{q}{r}}|f(x)|^2\nabla x\right)^{\frac{r}{2q}}, \quad (2.19)\end{aligned}$$

and sign of equality holds if $|f(x)|^{\frac{p}{r}-1} = |Cg(x)|$ and $|g(x)|^{\frac{q}{r}-1} = |Df(x)|$, where $|C|$ and $|D|$ are two real numbers.

Remark 2.14. Let $h, f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R})$. Let p, q be positive real numbers and r be negative. Then inequalities (2.18) and (2.19) will be reversed.

To conclude this paper, we present two dimensional inequalities. Our results in two dimensional case are given in next section.

3. Two Dimensional Symmetric Rogers-Hölder's Weighted Inequalities

Theorem 3.1. *Let $h, f_i \in C([a, b]_{\mathbb{T}} \times [a, b]_{\mathbb{T}}, \mathbb{R})$ be $\diamondsuit_\alpha$–integrable functions for $i = 1, 2, 3$ and p, q and r be three nonzero real numbers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$. Further assume that $\prod_{i=1}^{3} f_i(x, y) = 1$.*

(i) *If $p>0$, $q>0$ but $r<0$; $p>0$, $r>0$ but $q<0$ or $q>0$, $r>0$ but $p<0$, then*

$$\left(\int_a^b\int_a^b |h(x,y)||f_1(x,y)|^p\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{1}{p}} \times\left(\int_a^b\int_a^b |h(x,y)||f_2(x,y)|^q\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{1}{q}}\left(\int_a^b\int_a^b |h(x,y)||f_3(x,y)|^r\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{1}{r}}\geq 1. \tag{3.1}$$

(ii) *If $q<0$, $r<0$ but $p>0$; $p<0$, $r<0$ but $q>0$ or $p<0$, $q<0$ but $r>0$, then*

$$\left(\int_a^b\int_a^b |h(x,y)||f_1(x,y)|^p\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{1}{p}} \times\left(\int_a^b\int_a^b |h(x,y)||f_2(x,y)|^q\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{1}{q}}\left(\int_a^b\int_a^b |h(x,y)||f_3(x,y)|^r\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{1}{r}}\leq 1. \tag{3.2}$$

Proof . Similar to proof of Theorem 2.1. □

Theorem 3.2. *Let $h,f,g\in C([a,b]_{\mathbb{T}}\times[a,b]_{\mathbb{T}},\mathbb{R})$ be $\lozenge_\alpha$–integrable functions.*

(i) *Let p, q and r be positive real numbers. Then the following inequality holds*

$$\left(\int_a^b\int_a^b |h(x,y)||f(x,y)g(x,y)|\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{r}{p}+\frac{r}{q}}$$
$$\leq\left(\int_a^b\int_a^b |h(x,y)||f(x,y)|^{\frac{p}{r}}\lozenge_\alpha x\lozenge_\alpha y\int_a^b\int_a^b |h(x,y)||f(x,y)|^{2-\frac{p}{r}}|g(x,y)|^2\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{r}{2p}}$$
$$\times\left(\int_a^b\int_a^b |h(x,y)||g(x,y)|^{\frac{q}{r}}\lozenge_\alpha x\lozenge_\alpha y\int_a^b\int_a^b |h(x,y)||g(x,y)|^{2-\frac{q}{r}}|f(x,y)|^2\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{r}{2q}} \tag{3.3}$$

and sign of equality holds if $|f(x,y)|^{\frac{p}{r}-1}=|Cg(x,y)|$ and $|g(x,y)|^{\frac{q}{r}-1}=|Df(x,y)|$, where $|C|$ and $|D|$ are two real numbers.

(ii) *Let p and q be positive real numbers and r be negative. Then the following reverse inequality holds*

$$\left(\int_a^b\int_a^b |h(x,y)||f(x,y)g(x,y)|\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{r}{p}+\frac{r}{q}}$$
$$\geq\left(\int_a^b\int_a^b |h(x,y)||f(x,y)|^{\frac{p}{r}}\lozenge_\alpha x\lozenge_\alpha y\int_a^b\int_a^b |h(x,y)||f(x,y)|^{2-\frac{p}{r}}|g(x,y)|^2\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{r}{2p}}$$
$$\times\left(\int_a^b\int_a^b |h(x,y)||g(x,y)|^{\frac{q}{r}}\lozenge_\alpha x\lozenge_\alpha y\int_a^b\int_a^b |h(x,y)||g(x,y)|^{2-\frac{q}{r}}|f(x,y)|^2\lozenge_\alpha x\lozenge_\alpha y\right)^{\frac{r}{2q}}$$

and sign of equality holds if $|f(x,y)|^{\frac{p}{r}-1}=|Cg(x,y)|$ and $|g|^{\frac{q}{r}-1}(x,y)=|Df(x,y)|$, where $|C|$ and $|D|$ are two real numbers.

Proof . Similar to the proof of Theorem 2.7. □

Remark 3.3. If $\frac{p}{r}=\frac{q}{r}=2$ and $h(x,y)=1$, then (3.3) reduces to

$$\int_a^b\int_a^b |f(x,y)g(x,y)|\lozenge_\alpha x\lozenge_\alpha y \quad\leq\quad \sqrt{\left(\int_a^b\int_a^b |f(x,y)|^2\lozenge_\alpha x\lozenge_\alpha y\right)\left(\int_a^b\int_a^b |g(x,y)|^2\lozenge_\alpha x\lozenge_\alpha y\right)},$$

which is given in [5].

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