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Fixed point theorems under weakly contractive conditions via auxiliary functions in ordered G-metric spaces

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Abstract

We present some fixed point results for a single mapping and a pair of compatible mappings via auxiliary functions which satisfy a generalized weakly contractive condition in partially ordered complete G–metric spaces. Some examples are furnished to illustrate the useability of our main results. At the end, an application is presented to the study of existence and uniqueness of solutions for a boundary value problem for certain second order ODE and the respective integral equation.

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1. Introduction and preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been at the center because of its wide range of applications in different areas such as variational inequalities, optimization and parameterize estimation and other problems in nonlinear analysis. A number of generalizations of metric spaces have been introduced and studied by many authors. One of such generalization was introduced and studied by Dhage [8] in 1992 and called it D–metric space. However, it was noticed in [17, 18] and elsewhere that most of the claims related to the fundamental

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topological structure of D–metric space are incorrect. To address the problem of structure of D–metric spaces, Z. Mustafa and B. Sims introduced a new class metric spaces known as G–metric spaces. In a G–metric space every triplet of elements is assigned to a non–negative real number. Analysis of the structure of these spaces was studied and presented in [17, 18, 19]. They also extended some well–known fixed point theorems from metric spaces to G–metric spaces including the Banach contraction principle. The notion of G–metric spaces received good attentions and many authors obtained fixed results in G–metric spaces. Abbas and Rhoades [1] initiated the study of coincidence and common fixed point theory in these spaces.

Recently, fixed point theory has been developed rapidly in partially ordered metric spaces, as well as in partially ordered G–metric spaces [6, 34]. The first result in this direction was given by Ran and Reurings [31, Theorem 2.1] who presented its applications to matrix equation. Subsequently, Nieto and Rodŕiguez–López [28] extended the result of Ran and Reurings for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Further, Harjani and Sadarangani [11] used the partial ordering and proved some fixed point theorems for weakly contractive operators on ordered metric spaces. Subsequently [12] they generalized these results by considering a pair of altering functions (ψ, φ). Nashine and Altun [24] and Nashine and Bessem [26] generalized the results of Harjani and Sadarangani for a pair of compatible mappings. Nashine et al. [27] also proved fixed point theorems for T–weakly isotone increasing mappings which satisfy a generalized nonlinear contractive condition in complete ordered metric spaces and gave an application to an existence theorem for a solution of some integral equations. In [15], Erdal and Salimi established fixed point theorems via auxiliary functions in the framework of partially ordered metric spaces, and improved the results of Choudhury and Kundu [7]).

On the other hand, Choudhury and Maity [6] and Agarwal et al. [2] proved coupled fixed point theorems in the framework of a partially ordered G–metric space. Thereafter many work has been done by many authors in G–metric space (see details in [33, 34]).

In the present paper, an attempt has been made to establish fixed point theorems under a generalized weakly contraction condition via auxiliary functions in the setting of ordered G–metric spaces. Some examples are furnished to demonstrate the validity of the obtained results. Our results generalizes the results in [6, 11, 12, 15, 21, 26] in the sense that considered contractive condition is more general in the framework of G–metric spaces. We present also some consequences of the obtained results to fixed points for a single mapping and a pair of mappings satisfying a general contractive condition of integral type in complete partially ordered G–metric spaces. In conclusion, we apply accomplished fixed point results for generalized weakly contraction type mappings to the study of existence and uniqueness of solutions for a boundary value problem for certain second order ODE and the respective integral equation.

2. Preliminaries

At first, we introduce some notations and definitions that will be used later. For more details on the following definitions and results, we refer the reader to [19, 6].

Definition 2.1. (Mustafa and Sims, [19]) Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangular inequality).

Then the function G is called a G–metric on X and the pair (X, G) is called a G–metric space.

Definition 2.2. (Mustafa and Sims, [19]) Let (X, G) be a G–metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if

$$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$$

and one says that the sequence $\{x_n\}$ is G–convergent to x. Thus, if $x_n \to x$ in a G–metric space (X, G), then for any $\epsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq N$.

It has been shown in [19] that the G–metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to one point.

Definition 2.3. (Mustafa and Sims, [19]) Let (X, G) be a G–metric space, a sequence $\{x_n\}$ is called G–Cauchy if for every $\epsilon > 0$, there is a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

We next state the following known auxiliary results.

Lemma 2.4. (Mustafa and Sims, [19]) If (X, G) is a G–metric space, then the following are equivalent:

(1) $\{x_n\}$ is G–convergent to x.
(2) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.
(3) $G(x_n, x, x) \to 0$, as $n \to \infty$.

Lemma 2.5. (Mustafa and Sims, [17]) If (X, G) is a G–metric space, then the following are equivalent:

(1) The sequence $\{x_n\}$ is G–Cauchy.
(2) For every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq N$.

Lemma 2.6. (Mustafa and Sims, [19]) If (X, G) is a G–metric space, then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$.

Combining Lemmas 2.5 and 2.6 we have the following result.

Lemma 2.7. If (X, G) is a G–metric space then $\{x_n\}$ is a G–Cauchy sequence if and only if for every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$ for all $m > n \geq N$.

Definition 2.8. Let (X, G), (X', G') be two G–metric spaces. Then a function $f : X \to X$ is G–continuous at a point $x \in X$ if and only if it is G sequentially continuous at x, that is, whenever $\{x_n\}$ is G–convergent to x, $\{f(x_n)\}$ is G–convergent to $f(x)$.

Definition 2.9. (Mustafa and Sims, [19]) A G–metric space (X, G) is called symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.10. (Mustafa and Sims, [19]) A G–metric space (X, G) is said to be G–complete (or a complete G–metric space) if every G–Cauchy sequence in (X, G) is convergent in X.

The following are examples of G–metric spaces.

Example 2.11. Let $(\mathbb{R}, d)$ be the standard metric space. Define G_s by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

for all $x, y, z \in \mathbb{R}$. Then it is clear that $(\mathbb{R}, G_s)$ is a G–metric space.

Example 2.12. Let $X = \{a, b\}$. Define G on $X \times X \times X$ by

$$G(a, a, a) = G(b, b, b) = 0, \;\; G(a, a, b) = 1, \;\; G(a, b, b) = 2,$$

and extend G to $X \times X \times X$ by using the symmetry in the variables. Then it is clear that (X, G) is an asymmetric G–metric space.

Throughout this paper $(X, \preceq)$ denotes a partially ordered set. By '$x \succeq y$ holds', we mean that '$y \preceq x$ holds' and by '$x \prec y$ holds' we mean that '$x \preceq y$ holds and $x \neq y$'.

Definition 2.13. Let X be a nonempty set. $(X, G, \preceq)$ is called an ordered G–metric space if:

(i) (X, G) is a G–metric space,

(ii) $(X, \preceq)$ is a partially ordered set.

Definition 2.14. Let $(X, \preceq)$ be a partially ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 2.15. Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be such that, for $x, y \in X$, $x \preceq y$ implies $Tx \preceq Ty$. Then the mapping T is said to be non–decreasing.

Definition 2.16. Let $(X, G, \preceq)$ be a partially ordered G–metric space. We say that X is regular if and only if the following hypothesis holds: if $\{z_n\}$ is a non–decreasing sequence in X with respect to $\preceq$ such that $z_n \to z \in X$ as $n \to \infty$, then $z_n \preceq z$ for all $n \in \mathbb{N}$.

Khan et al. [16] introduced the concept of altering distance function:

Definition 2.17. (Khan et al., [16]) A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is continuous and non–decreasing,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

One of the generalizations of the contraction principle was suggested by Alber and Guerre–Delabriere [3] in Hilbert spaces by introducing the concept of weakly contractive mappings.

A self–mapping T on a metric space X is called weakly contractive if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \tag{2.1}$$

where ϕ is an altering distance function.

Rhoades [32] showed that most of the results of [3] are still true for any Banach space. Subsequently, several authors (see, e.g., [7, 9, 10, 21]) introduced various generalizations and refinements of weakly contractive mappings concept, and established new results.

To complete the proofs of our results, we introduce a new weakly contraction condition in G–metric space, and for this we need the following classes of functions from $[0, +\infty)$ into itself, discussed in [14, 15]:

$$\begin{aligned}\Psi &= \{\, \psi : \psi \text{ is nondecreasing and lower semicontinuous } \},\\ \Phi_1 &= \{\, \alpha : \alpha \text{ is upper semicontinuous } \},\\ \Phi_2 &= \{\, \beta : \beta \text{ is lower semicontinuous } \}.\end{aligned}$$

3. Main results

This section is divided in two subsections. In the first one, we derive fixed point results in the case of a continuous nondecreasing self–mapping on complete G–metric space endowed with partial order, or in the case of a regular space. In the second subsection we are concerned with a pair of continuous self–mappings which is relatively weakly increasing and G–compatible.

3.1. Result–I

We will beginning with a case of a single mapping. The following theorem is a generalization of Theorems 2.1 and 2.2 of Harjani and Sadarangani [12] to G–metric space.

Theorem 3.1. *Let $(X, G, \preceq)$ be a partially ordered complete G–metric space and let $T : X \to X$ be a nondecreasing self–mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $s, t \geq 0$,*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ imply } \psi(t) - \alpha(s) + \beta(s) > 0, \tag{3.1}$$

and

$$\psi(G(Tx, Ty, Tz)) \leq \alpha(\Theta(x, y, z)) - \beta(\Theta(x, y, z)) \tag{3.2}$$

where

$$\begin{aligned}\Theta(x, y, z) = \max\{&G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz),\\ &\frac{1}{3}[G(x, Ty, Ty) + G(y, Tz, Tz) + G(z, Tx, Tx)]\}\end{aligned}$$

for all $x, y, z \in X$ with $z \preceq y \preceq x$. We suppose the following hypotheses:

(i) T is continuous, or

(ii) X is regular.

If there exists $x_0 \in X$ such that $x_0 \prec Tx_0$, then T has a fixed point, that is, there exists $z \in X$ such that $z = Tz$.

Proof . Let x_0 be the given point in X. Suppose $Tx_0 \neq x_0$. we choose $x_1 \in X$ such that $Tx_0 = x_1$. Since T is a nondecreasing function, we have $x_0 \prec x_1 = Tx_0 \preceq Tx_1$. Again, let $x_2 = Tx_1$. Then we have

$$x_0 \prec x_1 = Tx_0 \preceq x_2 = Tx_1 \preceq Tx_2.$$

Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ with

$$x_0 \prec x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \{1, 2, \cdots\}$, then $x_{n_0+1} = x_{n_0} = Tx_{n_0}$ and so we are finished. Now we can suppose

$$G(x_{n+1}, x_{n+2}, x_{n+2}) > 0 \tag{3.3}$$

for all $n \geq 1$. First, we will prove that $\lim_{n\to\infty} G(x_{n+1}, x_{n+2}, x_{n+2}) = 0$.

Since $x_n \preceq x_{n+1}$, we can use (3.2) for these points, and we have for $n \geq 1$

$$\begin{aligned}
\Theta(x_n, x_{n+1}, x_{n+1}) &= \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_n, Tx_n, Tx_n), G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \\
&\quad \frac{1}{3}[G(x_n, Tx_{n+1}, Tx_{n+1}) + G(x_{n+1}, Tx_{n+1}, Tx_{n+1}) + G(x_{n+1}, Tx_n, Tx_n)]\} \\
&= \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2}), \\
&\quad \frac{1}{3}[G(x_n, x_{n+2}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+2})]\}.
\end{aligned}$$

By (G_5), we have

$$G(x_n, x_{n+2}, x_{n+2}) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}).$$

Thus

$$\frac{1}{3}[G(x_n, x_{n+2}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+2})] \leq \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\}.$$

Therefore we have

$$\Theta(x_n, x_{n+1}, x_{n+1}) = \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\}.$$

Now we claim that

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_n, x_{n+1}, x_{n+1}) \tag{3.4}$$

for all $n \geq 1$. Suppose this is not true, that is, there exists $n_0 \geq 1$ such that $G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2}) > G(x_{n_0}, x_{n_0+1}, x_{n_0+1})$. Now since $x_{n_0} \preceq x_{n_0+1}$, we can use the inequality (3.2) for these elements, and we have

$$\begin{aligned}
\psi(G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})) &= \psi(G(Tx_{n_0}, Tx_{n_0+1}, Tx_{n_0+1})) \\
&\leq \alpha(\max\{G(x_{n_0}, x_{n_0+1}, x_{n_0+1}), G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})\}) \\
&\quad -\beta(\max\{G(x_{n_0}, x_{n_0+1}, x_{n_0+1}), G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})\}).
\end{aligned}$$

If

$$\max\{G(x_{n_0}, x_{n_0+1}, x_{n_0+1}), G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})\} = G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2}),$$

then

$$\psi(G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})) \leq \alpha(G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})) - \beta(G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})).$$

By the assumption (3.1) it follows that $G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2}) = 0$, which contradicts the condition (3.3). Therefore

$$\max\{G(x_{n_0}, x_{n_0+1}, x_{n_0+1}), G(x_{n_0+1}, x_{n_0+2}, x_{n_0+2})\} = G(x_{n_0}, x_{n_0+1}, x_{n_0+1}).$$

Hence, we have

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_n, x_{n+1}, x_{n+1}) \quad \forall n \in \mathbb{N}.$$

Therefore, (3.4) is true and so the sequence $\{G(x_n, x_{n+1}, x_{n+1}) : n \in \mathbb{N}\}$ is nonincreasing and bounded below. Thus there exists $\rho \geq 0$ such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = \rho.$$

Now suppose that $\rho > 0$. It follows from (3.2) that

$$\psi(G(x_{n+1}, x_{n+2}, x_{n+2})) \quad \leq \quad \alpha(G(x_n, x_{n+1}, x_{n+1})) - \beta(G(x_n, x_{n+1}, x_{n+1}))$$

and using the properties of functions ψ, α, β we get

$$\begin{aligned}\psi(\rho) &\leq \liminf \psi(G(x_{n+1}, x_{n+2}, x_{n+2})) \leq \limsup \psi(G(x_{n+1}, x_{n+2}, x_{n+2})) \\ &\leq \limsup[\alpha(G(x_n, x_{n+1}, x_{n+1})) - \beta(G(x_n, x_{n+1}, x_{n+1}))] \\ &= \limsup \alpha(G(x_n, x_{n+1}, x_{n+1})) - \liminf \beta(G(x_n, x_{n+1}, x_{n+1}))] \\ &\leq \alpha(\rho) - \beta(\rho).\end{aligned}$$

Using again the condition (3.1), we get that it is only possible if $\rho = 0$. Thus

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3.5}$$

Next, we show that $\{x_n\}$ is a G–Cauchy sequence in X. Suppose to the contrary; that is, $\{x_n\}$ is G–Cauchy. Then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m(i)}\}$ and $\{x_{n(i)}\}$ of $\{x_n\}$ such that $n(i)$ is the smallest index for which

$$n(i) > m(i) > i, \;\; G(x_{m(i)}, x_{n(i)}, x_{n(i)}) \geq \epsilon. \tag{3.6}$$

This means that

$$G(x_{m(i)}, x_{n(i)-1}, x_{n(i)-1}) < \epsilon. \tag{3.7}$$

From (3.6), (3.7) and (G_5), we get

$$\begin{aligned}\epsilon &\leq G(x_{m(i)}, x_{n(i)}, x_{n(i)}) \\ &\leq G(x_{m(i)}, x_{n(i)-1}, x_{n(i)-1}) + G(x_{n(i)-1}, x_{n(i)}, x_{n(i)}) \\ &< \epsilon + G(x_{n(i)-1}, x_{n(i)}, x_{n(i)}).\end{aligned}$$

Passing to the limit as $i \to \infty$ and using (3.5), we get

$$\lim_{i\to\infty} G(x_{m(i)}, x_{n(i)}, x_{n(i)}) = \epsilon. \tag{3.8}$$

Using (G_5), we have

$$\begin{aligned}G(x_{m(i)}, x_{n(i)}, x_{n(i)}) &\leq G(x_{m(i)}, x_{m(i)+1}, x_{m(i)+1}) + G(x_{m(i)+1}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)+1}, x_{n(i)}, x_{n(i)}) \\ &\leq G(x_{m(i)}, x_{m(i)+1}, x_{m(i)+1}) + G(x_{m(i)+1}, x_{m(i)}, x_{m(i)}) + G(x_{m(i)}, x_{n(i)}, x_{n(i)}) \\ &\quad + G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)+1}, x_{n(i)}, x_{n(i)}).\end{aligned}$$

Passing to the limit as $i \to \infty$ in the above inequalities and using (3.5), (3.8), we get that

$$\lim_{i\to\infty} G(x_{m(i)+1}, x_{n(i)+1}, x_{n(i)+1}) = \epsilon. \tag{3.9}$$

Again using (G_5),

$$\begin{aligned}G(x_{m(i)}, x_{n(i)+1}, x_{n(i)+1}) &\le G(x_{m(i)}, x_{n(i)}, x_{n(i)}) + G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1})\\ &\le G(x_{m(i)}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)+1}, x_{n(i)}, x_{n(i)})\\ &\quad + G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}).\end{aligned}$$

Passing to the limit as $i \to \infty$ in the above inequalities and using (3.5), (3.8), we get that

$$\lim_{i\to\infty} G(x_{m(i)}, x_{n(i)+1}, x_{n(i)+1}) = \epsilon. \tag{3.10}$$

Using (G_5) to get

$$\begin{aligned}G(x_{m(i)+1}, x_{n(i)}, x_{n(i)}) &\le G(x_{m(i)+1}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)+1}, x_{n(i)}, x_{n(i)})\\ &\le G(x_{m(i)+1}, x_{n(i)}, x_{n(i)}) + G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1})\\ &\quad + G(x_{n(i)+1}, x_{n(i)}, x_{n(i)}).\end{aligned}$$

Passing to the limit in the above inequalities and using (3.5), (3.9), we get that

$$\lim_{i\to\infty} G(x_{m(i)+1}, x_{n(i)}, x_{n(i)}) = \epsilon. \tag{3.11}$$

Since $x_{m(i)} \preceq x_{n(i)}$, we have

$$\begin{aligned}&\Theta(x_{m(i)}, x_{n(i)}, x_{n(i)})\\ =\ &\max\{G(x_{m(i)}, x_{n(i)}, x_{n(i)}), G(x_{m(i)}, Tx_{m(i)}, Tx_{m(i)}), G(x_{n(i)}, Tx_{n(i)}, Tx_{n(i)}),\\ &\frac{1}{3}[G(x_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) + G(x_{n(i)}, Tx_{n(i)}, Tx_{n(i)}) + G(x_{n(i)}, Tx_{m(i)}, Tx_{m(i)})]\}\\ =\ &\max\{G(x_{m(i)}, x_{n(i)}, x_{n(i)}), G(x_{m(i)}, x_{m(i)+1}, x_{m(i)+1}), G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}),\\ &\frac{1}{3}[G(x_{m(i)}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)}, x_{m(i)+1}, x_{m(i)+1})]\}\\ \le\ &\max\{G(x_{m(i)}, x_{n(i)}, x_{n(i)}), G(x_{m(i)}, x_{m(i)+1}, x_{m(i)+1}), G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}),\\ &\frac{1}{3}[G(x_{m(i)}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + 2G(x_{n(i)}, x_{n(i)}, x_{m(i)+1})]\}.\end{aligned}$$

Therefore from (3.2), we have

$$\begin{aligned}&\psi(G(x_{m(i)+1}, x_{n(i)+1}, x_{n(i)+1}))\\ =\ &\psi(G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}))\\ \le\ &\alpha(\max\{G(x_{m(i)}, x_{n(i)}, x_{n(i)}), G(x_{m(i)}, x_{m(i)+1}, x_{m(i)+1}), G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}),\\ &\frac{1}{3}[G(x_{m(i)}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + 2G(x_{n(i)}, x_{n(i)}, x_{m(i)+1})]\})\\ &-\beta(\max\{G(x_{m(i)}, x_{n(i)}, x_{n(i)}), G(x_{m(i)}, x_{m(i)+1}, x_{m(i)+1}), G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}),\\ &\frac{1}{3}[G(x_{m(i)}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + 2G(x_{n(i)}, x_{n(i)}, x_{m(i)+1})]\}).\end{aligned}$$

Passing to the limit as $i \to \infty$, and using (3.5), (3.8), (3.9), (3.10), (3.11) and the properties of functions ψ, α, β we get

$$\begin{aligned}\psi(\epsilon) &\leq \limsup \alpha(\max\{G(x_{m(i)}, x_{n(i)}, x_{n(i)}), G(x_{m(i)}, x_{m(i)+1}, x_{m(i)+1}), G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}),\\ &\quad \frac{1}{3}[G(x_{m(i)}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + 2G(x_{n(i)}, x_{n(i)}, x_{m(i)+1})]\})\\ &\quad - \liminf \beta(\max\{G(x_{m(i)}, x_{n(i)}, x_{n(i)}), G(x_{m(i)}, x_{m(i)+1}, x_{m(i)+1}), G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}),\\ &\quad \frac{1}{3}[G(x_{m(i)}, x_{n(i)+1}, x_{n(i)+1}) + G(x_{n(i)}, x_{n(i)+1}, x_{n(i)+1}) + 2G(x_{n(i)}, x_{n(i)}, x_{m(i)+1})]\})\\ &\leq \alpha(\epsilon) - \beta(\epsilon).\end{aligned}$$

Using (3.1), we get $\epsilon = 0$, a contradiction. Thus $\{x_n\}$ is a G–Cauchy sequence in X. Thus there is $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{3.12}$$

Suppose that (i) holds. Since T is G–continuous, we get that $x_{n+1} = Tx_n$ is a G–convergent sequence to Tz. By the uniqueness of limit we get that $z = Tz$ and so z is a fixed point of T.

Suppose that (ii) holds. Since $\{x_n\}$ is a non–decreasing sequence such that $x_n \to z$ and X is regular, it follows that $x_n \preceq z$ for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}\Theta(x_n, z, z) &= \max\{G(x_n, z, z), G(x_n, Tx_n, Tx_n), G(z, Tz, Tz),\\ &\quad \frac{1}{3}[G(x_n, Tz, Tz) + G(u, Tz, Tz) + G(z, Tx_n, Tx_n)]\}\\ &= \max\{G(x_n, z, z), G(x_n, x_{n+1}, x_{n+1}), G(z, Tz, Tz),\\ &\quad \frac{1}{3}[G(x_n, Tz, Tz) + G(z, Tz, Tz) + G(z, x_{n+1}, x_{n+1})]\}.\end{aligned}$$

By (3.2), we obtain

$$\begin{aligned}\psi(G(x_{n+1}, Tz, Tz)) &= \psi(G(Tx_n, Tz, Tz))\\ &\leq \alpha(\max\{G(x_n, z, z), G(x_n, x_{n+1}, x_{n+1}), G(z, Tz, Tz),\\ &\quad \frac{1}{3}[G(x_n, Tz, Tz) + G(u, Tz, Tz) + G(z, x_{n+1}, x_{n+1})]\})\\ &\quad -\beta(\max\{G(x_n, z, z), G(x_n, x_{n+1}, x_{n+1}), G(z, Tz, Tz),\\ &\quad \frac{1}{3}[G(x_n, Tz, Tz) + G(u, Tz, Tz) + G(z, x_{n+1}, x_{n+1})]\}).\end{aligned}$$

Passing to the upper limit as $n \to \infty$ in the above inequalities and using (3.12) and the fact that G is continuous on its variables, we get that

$$\psi(G(z, Tz, Tz)) \leq \alpha(G(z, Tz, Tz)) - \beta(G(z, Tz, Tz)).$$

Using (3.1), we get $G(z, Tz, Tz) = 0$ and hence $z = Tz$. Thus z is a fixed point of T. $\square$

Now we give a sufficient condition for the uniqueness of fixed point.

Theorem 3.2. *Let all the conditions of Theorem 3.1 be fulfilled and, moreover, the space $(X, G, \preceq)$ satisfy the following condition: for all $x, y \in X$ there exists $z \in X$, $z \preceq Tz$, satisfying both $x \preceq z$ and $y \preceq z$ or there exists $z \in X$, $z \preceq Tz$, satisfying both $x \succeq z$ and $y \succeq z$. Then the fixed point of T is unique.*

Proof . Let x and y be two fixed points of T, i.e., $Tx = x$ and $Ty = y$. Consider the following two possible cases.

1°) x and y are comparable. Then we can apply the condition (3.1) and obtain that

$$\psi(G(x,y,z)) = \psi(G(Tx,Ty,Tz)) \le \alpha(\Theta(x,y,z)) - \beta(\Theta(x,y,z))$$

where

$$\begin{aligned}\Theta(x,y,z) &= \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz),\\ &\quad \frac{1}{3}[G(x,Ty,Ty) + G(y,Tz,Tz) + G(z,Tx,Tx)]\}\\ &\le \max\{G(x,y,z), \frac{1}{3}[G(x,y,z) + G(x,y,z) + G(x,y,z)]\}\\ &= G(x,y,z)\end{aligned}$$

and hence $\psi(G(x,y,z)) \le \alpha(G(x,y,z)) - \beta(G(x,y,z))$ which is possible only if $G(x,y,z) = 0$ and hence $x = y = z$. 2°) Suppose now that x and y are not comparable. Choose an element $z \in X$, $z \preceq Tz$ comparable with both of them. Then also $x = T^n x$ is comparable with $T^n z$ for each n (since T is nondecreasing). Applying (3.1) one obtains that

$$\begin{aligned}\psi(G(x,T^n z,T^n z)) &= \psi(G(TT^{n-1}x, TT^{n-1}z, TT^{n-1}z))\\ &\le \alpha(\Theta(T^{n-1}x, T^{n-1}z, T^{n-1}z)) - \beta(\Theta(T^{n-1}x, T^{n-1}z, T^{n-1}z)),\end{aligned}$$

where

$$\begin{aligned}&\Theta(T^{n-1}x, T^{n-1}z, T^{n-1}z)\\ &= \max\{G(T^{n-1}x, T^{n-1}z, T^{n-1}z), G(T^{n-1}x, TT^{n-1}x, TT^{n-1}x),\\ &\quad G(T^{n-1}z, TT^{n-1}z, TT^{n-1}z), G(T^{n-1}z, TT^{n-1}z, TT^{n-1}z),\\ &\quad \frac{1}{3}[G(T^{n-1}x, TT^{n-1}z, TT^{n-1}z) + G(T^{n-1}z, TT^{n-1}z, TT^{n-1}z) + G(T^{n-1}z, TT^{n-1}x, TT^{n-1}x)]\}\\ &= \max\{G(x, T^{n-1}z, T^{n-1}z), G(T^{n-1}z, T^n z, T^n z), G(T^{n-1}x, T^n z, T^n z)\}\end{aligned}$$

for n sufficiently large, because $G(T^{n-1}z, T^n z, T^n z) \to 0$ when $n \to \infty$ (the last assertion can be proved, starting from the assumption $z \preceq Tz$, in the same way as the similar conclusion in the proof of Theorem 3.1). Similarly as in the proof of Theorem 3.1, it can be shown that $G(x, T^n z, T^n z) \le \Theta(x, T^{n-1}z, T^{n-1}z) \le G(x, T^{n-1}z, T^{n-1}z)$. It follows that the sequence $G(x, T^n z, T^n z)$ is nonincreasing and it has a limit $\ell \ge 0$. Assuming that $\ell > 0$ and passing to the limit in the relation

$$\psi(G(x,T^n z,T^n z)) \le \alpha(\Theta(x, T^{n-1}z, T^{n-1}z)) - \beta(\Theta(x, T^{n-1}z, T^{n-1}z)),$$

one obtains that $\ell = 0$, a contradiction. In the same way it can be deduced that $G(y, T^n z, T^n z) \to 0$ as $n \to \infty$. Now, passing to the limit in $G(x,y,y) \le G(x, T^n z, T^n z) + G(T^n z, y, y)$, it follows that $G(x,y,y) = 0$. Hence, $x = y$ and the uniqueness of the fixed point is proved. □

Remark 3.3. Theorem 3.1–Theorem 3.2 are also true even if we replace $\Theta(x,y,z)$ by the following:

$$\begin{aligned}\Theta_1(x,y,z) = \max\{&G(x,y,z), G(x,x,Tx), G(y,y,Ty), G(z,z,Tz),\\ &\frac{1}{3}[G(x,x,Ty) + G(y,y,Tz) + G(z,z,Tx)]\}.\end{aligned}$$

We furnish the following example to demonstrate the validity of the hypotheses of Theorem 3.1:

Example 3.4. Let $X = \{1, 2, 3\}$ and $G : X \times X \times X \to \mathbb{R}^+$ be defined as follows (the values of G are invariant under permutations of its arguments):

$$G(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \text{ are all distinct} \\ x+z, & \text{if } x = y \neq z \\ 0, & \text{if } x = y = z. \end{cases}$$

Then (X, G) is a complete G–metric space. Let a relation $\preceq$ on X be defined as follows:

$$\preceq := \{(1,1), (2,2), (3,3), (3,1)\}.$$

Clearly, $\preceq$ is a partial order on X. Consider the mapping $T : X \to X$ defined by

$$T = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}.$$

Now, we will show that $(X, G, \preceq)$ is regular.
Let $\{z_n\}$ be a non–decreasing sequence in X with respect to $\preceq$ such that $z_n \to z \in X$ as $n \to \infty$. We have $z_n \preceq z_{n+1}$ for all $n \in \mathbb{N}$.
• If $z_0 = 1$, then $z_0 = 1 \preceq z_1$. From the definition of $\preceq$, we have $z_1 = 1$. By induction, we get $z_n = 1$ for all $n \in \mathbb{N}$ and $z = 1$. Then, $z_n \preceq z$ for all $n \in \mathbb{N}$.
• If $z_0 = 2$, then $z_0 = 2 \preceq z_1$. From the definition of $\preceq$, we have $z_1 = 2$. By induction, we get $z_n = 2$ for all $n \in \mathbb{N}$ and $z = 2$. Then, $z_n \preceq z$ for all $n \in \mathbb{N}$.
• If $z_0 = 3$, then $z_0 = 3 \preceq z_1$. From the definition of $\preceq$, we have $z_1 \in \{3, 1\}$. By induction, we get $z_n \in \{3, 1\}$ for all $n \in \mathbb{N}$. Suppose that there exists $p \geq 1$ such that $z_p = 1$. From the definition of $\preceq$, we get $z_n = z_p = 1$ for all $n \geq p$. Thus, we have $z = 1$ and $z_n \preceq z$ for all $n \in \mathbb{N}$. Now, suppose that $z_n = 3$ for all $n \in \mathbb{N}$. In this case, we get $z = 3$ and $z_n \preceq z$ for all $n \in \mathbb{N}$.
Thus, we have proved that in all cases, we have $z_n \preceq z$ for all $n \in \mathbb{N}$. Then, $(X, G, \preceq)$ is regular.

We will check that T satisfies the condition (3.2) of Theorem 3.1 with functions ψ, α, β given as $\psi(t) = t$, $\alpha(t) = t$, $\beta(t) = \frac{1}{10}t$ (obviously belonging to the respective classes). Note that in this case $\alpha(t) - \beta(t) = \frac{9}{10}t$.

To verify (3.2), we divide all the cases for every $x, y, z \in X$ in the following manner:

- For $x = y = z$ (or $x = y = 1, z = 3$ or $x = y = 3, z = 1$). In this case (or cases) the left-hand side is equal to 0 and hence (3.2) is satisfied.

- If $x = 1, y = 2, z = 3$, then $\psi(Tx, Ty, Tz) = 4$ and

$$\begin{aligned} \Theta(x,y,z) &= \max\{G(1,2,3), G(1,T1,T1), G(2,T2,T2), G(3,T3,T3), \\ &\quad \frac{1}{3}[G(1,T2,T2) + G(2,T3,T3) + G(3,T1,T1)]\} = 6. \end{aligned}$$

Thus the relation (3.2) is satisfied as

$$4 = \psi(G(Tx,Ty,Tz)) \leq \alpha(\Theta(x,y,z)) - \beta(\Theta(x,y,z)) = (\frac{9}{10})6.$$

- If $x = 1, y = 1, z = 2$ (or $x = 2, y = 2, z = 3$ or $x = 2, y = 2, z = 3$), then $\psi(Tx, Ty, Tz) = 4$ and it is to show $\Theta(x, y, z) = 5$. Thus the relation (3.2) is satisfied as

$$4 = \psi(G(Tx, Ty, Tz)) \leq \alpha(\Theta(x, y, z)) - \beta(\Theta(x, y, z)) = (\frac{9}{10})5.$$

Then with $x_0 = 2$, Theorem 3.1 is applicable to this example. It is also observed that T has a unique fixed point $z = 1$.

3.2. Results-II

In this subsection, we present coincidence point results for compatible mappings in G–metric spaces.

Recall the following notions. Let X be a non–empty set and $R, T : X \to X$ be given self–mappings on X. If $w = Rx = Tx$ for some $x \in X$, then x is called a coincidence point of R and T, and w is called a point of coincidence of R and T. The pair $\{R, T\}$ of self–mappings in a metric space (X, d) is said to be compatible if $d(TRx_n, RTx_n) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X satisfying $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Rx_n$. The pair $\{R, T\}$ is said to be weakly compatible if $RTx = TRx$, whenever $Tx = Rx$ for some x in X. We will use the following version of compatibility of these mappings in a G–metric space.

Definition 3.5. Mappings $R, T : X \to X$ are said to be compatible in a G-metric space (X, G) if

$$G(TRx_n, RTx_n, RTx_n) \to 0 \quad \text{as} \quad n \to \infty,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Rx_n$ in (X, G).

The following notion has been recently introduced by Nashine and Samet [26]. Let X be a non–empty set and $R : X \to X$ be a given mapping. For every $x \in X$, we denote by $R^{-1}(x)$ the subset of X defined by

$$R^{-1}(x) := \{u \in X \mid Ru = x\}.$$

Definition 3.6. (Nashine and Samet, [26]) Let $(X, \preceq)$ be a partially ordered set and $T, R : X \to X$ be given mappings such that $TX \subseteq RX$. We say that T is weakly increasing with respect to R if for all $x \in X$, we have:

$$Tx \preceq Ty, \ \forall\, y \in R^{-1}(Tx).$$

Remark 3.7. If $R : X \to X$ is the identity mapping ($Rx = x$ for all $x \in X$), and T is weakly increasing with respect to R then T is a weakly increasing mapping. Note that the notion of weakly increasing mappings was introduced in [4].

The results of this subsection extend the results of Nashine and Samet [26] to generalized contractive conditions defined in generalized metric space.

Theorem 3.8. *Let $(X, G, \preceq)$ be a partially ordered complete G–metric space and let $T, R : X \to X$ be self–mappings. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $s, t \geq 0$,*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ imply } \psi(t) - \alpha(s) + \beta(s) > 0, \tag{3.13}$$

and

$$\psi(G(Tx, Ty, Tz)) \leq \alpha(\Delta(x, y, z)) - \beta(\Delta(x, y, z)) \tag{3.14}$$

where

$$\Delta(x,y,z) = \max\{G(Rx,Ry,Rz), G(Rx,Tx,Tx), G(Ry,Ty,Ty), G(Rz,Tz,Tz), \frac{1}{3}[G(Rx,Ty,Ty)+G(Ry,Tz,Tz)+G(Rz,Tx,Tx)]\}$$

for all $x,y,z \in X$ with $Rz \preceq Ry \preceq Rx$. We suppose the following hypotheses:

(i) T and R are continuous,

(ii) $TX \subseteq RX$,

(iii) T is weakly increasing with respect to R,

(iv) the pair $\{T,R\}$ is compatible.

Then, T and R have a coincidence point, that is, there exists $u \in X$ such that $Ru = Tu$.

Remark 3.9. *Note that, in this theorem, we do not need the condition "there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$" of Theorem 3.1.*

Proof . Let x_0 be an arbitrary point in X. Since $TX \subseteq RX$, we can construct a sequence $\{x_n\}$ in X defined by:

$$Rx_{n+1} = Tx_n,\ \forall\, n \in \mathbb{N}. \tag{3.15}$$

Now, since $x_1 \in R^{-1}(Tx_0)$ and $x_2 \in R^{-1}(Tx_1)$, using that T is weakly increasing with respect to R, we obtain:

$$Rx_1 = Tx_0 \preceq Tx_1 = Rx_2 \preceq Tx_2 = Rx_3.$$

Continuing this process, we get:

$$Rx_1 \preceq Rx_2 \preceq x_3 \preceq \cdots \preceq Rx_n \preceq Rx_{n+1} \preceq \cdots .$$

Now we can suppose

$$G(Rx_{n+1}, Rx_{n+2}, Rx_{n+2}) > 0 \tag{3.16}$$

for all $n \geq 1$. We complete the proof in three steps.

Step I. We will prove that $\lim_{n\to\infty} G(Rx_{n+1}, Rx_{n+2}, Rx_{n+2}) = 0$. Since $Rx_n \preceq Rx_{n+1}$, we can use (3.15) for these points, and we have for $n \geq 1$

$$\begin{aligned}\Delta(x_n,x_{n+1},x_{n+1}) &= \max\{G(Rx_n,Rx_{n+1},Rx_{n+1}), G(Rx_n,Tx_n,Tx_n), G(Rx_{n+1},Tx_{n+1},Tx_{n+1}),\\ &\quad \frac{1}{3}[G(Rx_n,Tx_{n+1},Tx_{n+1})+G(Rx_{n+1},Tx_{n+1},Tx_{n+1})+G(Rx_{n+1},Tx_n,Tx_n)]\}\\ &= \max\{G(Rx_n,Rx_{n+1},Rx_{n+1}), G(Rx_{n+1},Rx_{n+2},Rx_{n+2}),\\ &\quad \frac{1}{3}[G(Rx_n,Rx_{n+2},Rx_{n+2})+G(Rx_{n+1},Rx_{n+2},Rx_{n+2})]\}.\end{aligned}$$

By (G_5), we have

$$G(Rx_n,Rx_{n+2},Rx_{n+2}) \leq G(Rx_n,Rx_{n+1},Rx_{n+1}) + G(Rx_{n+1},Rx_{n+2},Rx_{n+2}).$$

Thus

$$\begin{aligned}&\frac{1}{3}[G(Rx_n, Rx_{n+2}, Rx_{n+2}) + G(Rx_{n+1}, Rx_{n+2}, Rx_{n+2})]\\&\leq \max\{G(Rx_n, Rx_{n+1}, Rx_{n+1}), G(Rx_{n+1}, Rx_{n+2}, Rx_{n+2})\}.\end{aligned}$$

Therefore we have

$$\Delta(x_n, x_{n+1}, x_{n+1}) = \max\{G(Rx_n, Rx_{n+1}, Rx_{n+1}), G(Rx_{n+1}, Rx_{n+2}, Rx_{n+2})\}.$$

Now we claim that

$$G(Rx_{n+1}, Rx_{n+2}, Rx_{n+2}) \leq G(Rx_n, Rx_{n+1}, Rx_{n+1}) \tag{3.17}$$

for all $n \geq 1$. Suppose this is not true, that is, there exists $n_0 \geq 1$ such that

$$G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2}) > G(Rx_{n_0}, Rx_{n_0+1}, Rx_{n_0+1}).$$

Now since $Rx_{n_0} \preceq Rx_{n_0+1}$, we can use the inequality (3.14) for these elements, and we have

$$\begin{aligned}\psi(G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2})) &= \psi(G(Tx_{n_0}, Tx_{n_0+1}, Tx_{n_0+1}))\\&\leq \alpha(\max\{G(Rx_{n_0}, Rx_{n_0+1}, Rx_{n_0+1}), G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2})\})\\&\quad - \beta(\max\{G(Rx_{n_0}, Rx_{n_0+1}, Rx_{n_0+1}), G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2})\}).\end{aligned}$$

If

$$\max\{G(Rx_{n_0}, Rx_{n_0+1}, Rx_{n_0+1}), G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2})\} = G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2}),$$

then

$$\psi(G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2})) \leq \alpha(G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2})) - \beta(G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2})).$$

By the assumption (3.13) it follows that $G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2}) = 0$, which contradicts the condition (3.16). Therefore

$$\max\{G(Rx_{n_0}, Rx_{n_0+1}, Rx_{n_0+1}), G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2})\} = G(Rx_{n_0}, Rx_{n_0+1}, Rx_{n_0+1}).$$

Hence, we have

$$\psi(G(Rx_{n_0+1}, Rx_{n_0+2}, Rx_{n_0+2})) \leq \alpha(G(Rx_{n_0}, Rx_{n_0+1}, Rx_{n_0+1})) - \beta(G(Rx_{n_0}, Rx_{n_0+1}, Rx_{n_0+1})). \tag{3.18}$$

Thus

$$G(Rx_{n+1}, Rx_{n+2}, Rx_{n+2}) \leq G(Rx_n, Rx_{n+1}, Rx_{n+1}) \quad \forall n \in \mathbb{N}.$$

Therefore, (3.17) is true and so the sequence $\{G(Rx_n, Rx_{n+1}, Rx_{n+1}) : n \in \mathbb{N}\}$ is nonincreasing and bounded below. Thus there exists $\rho \geq 0$ such that

$$\lim_{n\to\infty} G(Rx_n, Rx_{n+1}, Rx_{n+1}) = \rho.$$

Now suppose that $\rho > 0$. It follows from (3.18) that

$$\psi(G(Rx_{n+1}, Rx_{n+2}, Rx_{n+2})) \leq \alpha(G(Rx_n, Rx_{n+1}, Rx_{n+1})) - \beta(G(Rx_n, Rx_{n_0+1}, Rx_{n_0+1}))$$

and using the properties of functions ψ, α, β we get that

$$\begin{aligned}\psi(\rho) &\le \liminf \psi(G(Rx_{n+1}, Rx_{n+2}, Rx_{n+2}))\\ &\le \limsup \psi(G(Rx_{n+1}, Rx_{n+2}, Rx_{n+2}))\\ &\le \limsup[\alpha(G(Rx_n, Rx_{n+1}, Rx_{n+1})) - \beta(G(Rx_n, Rx_{n_0+1}, Rx_{n_0+1}))]\\ &= \limsup \alpha(G(Rx_n, Rx_{n+1}, Rx_{n+1})) - \liminf \beta(G(Rx_n, Rx_{n_0+1}, Rx_{n_0+1}))\\ &\le \alpha(\rho) - \beta(\rho).\end{aligned}$$

Using again the condition (3.13), we get that it is only possible if $\rho = 0$. Thus

$$\lim_{n\to\infty} G(Rx_n, Rx_{n+1}, Rx_{n+1}) = 0. \tag{3.19}$$

Step II. We will prove that $\{Rx_n\}$ is a G–Cauchy sequence in X. Suppose the contrary, that is, $\{Rx_n\}$ is not G-Cauchy. Then there exists $\epsilon > 0$ for which we can find two subsequences $\{m(i)\}$ and $\{n(i)\}$ such that $n(i)$ is the smallest index for which

$$n(i) > m(i) > i, \;\; G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)}) \ge \epsilon. \tag{3.20}$$

This means that

$$G(Rx_{m(i)}, Rx_{n(i)-1}, Rx_{n(i)-1}) < \epsilon. \tag{3.21}$$

From (3.20), (3.21) and (G_5), we get

$$\begin{aligned}\epsilon &\le G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)})\\ &\le G(Rx_{m(i)}, Rx_{n(i)-1}, Rx_{n(i)-1}) + G(Rx_{n(i)-1}, Rx_{n(i)}, Rx_{n(i)})\\ &< \epsilon + G(Rx_{n(i)-1}, Rx_{n(i)}, Rx_{n(i)}).\end{aligned}$$

Passing to the limit as $i \to \infty$ and using (3.19), we get

$$\lim_{i\to\infty} G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)}) = \epsilon.$$

Using (G_5), we have

$$\begin{aligned}&G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)})\\ &\le G(Rx_{m(i)}, Rx_{m(i)+1}, Rx_{m(i)+1}) + G(Rx_{m(i)+1}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)+1}, Rx_{n(i)}, Rx_{n(i)})\\ &\le G(Rx_{m(i)}, Rx_{m(i)+1}, Rx_{m(i)+1}) + G(Rx_{m(i)+1}, Rx_{m(i)}, Rx_{m(i)}) + G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)})\\ &\quad + G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)+1}, Rx_{n(i)}, Rx_{n(i)}).\end{aligned}$$

Passing to the limit as $i \to \infty$ in the above inequalities and using (3.19), (3.2), we get that

$$\lim_{i\to\infty} G(Rx_{m(i)+1}, Rx_{n(i)+1}, Rx_{n(i)+1}) = \epsilon. \tag{3.22}$$

Again using (G_5),

$$\begin{aligned}G(Rx_{m(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) &\le G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)}) + G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1})\\ &\le G(Rx_{m(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)+1}, Rx_{n(i)}, Rx_{n(i)})\\ &\quad + G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}).\end{aligned}$$

Passing to the limit as $i \to \infty$ in the above inequalities and using (3.19), (3.2), we get that

$$\lim_{i\to\infty} G(Rx_{m(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) = \epsilon. \tag{3.23}$$

Using (G_5) to get

$$\begin{aligned} G(Rx_{m(i)+1}, Rx_{n(i)}, Rx_{n(i)}) &\leq G(Rx_{m(i)+1}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)+1}, Rx_{n(i)}, Rx_{n(i)}) \\ &\leq G(Rx_{m(i)+1}, Rx_{n(i)}, Rx_{n(i)}) + G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) \\ &\quad + G(Rx_{n(i)+1}, Rx_{n(i)}, Rx_{n(i)}). \end{aligned}$$

Passing to the limit in the above inequalities and using (3.19), (3.22), we get that

$$\lim_{i\to\infty} G(Rx_{m(i)+1}, Rx_{n(i)}, Rx_{n(i)}) = \epsilon. \tag{3.24}$$

Since $Rx_{m(i)} \preceq Rx_{n(i)}$, we have

$$\begin{aligned} &\Delta(x_{m(i)}, x_{n(i)}, x_{n(i)}) \\ &= \max\{G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)}), G(Rx_{m(i)}, Tx_{m(i)}, Tx_{m(i)}), G(Rx_{n(i)}, Tx_{n(i)}, Tx_{n(i)}), \\ &\quad \frac{1}{3}[G(Rx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) + G(Sx_{n(i)}, Tx_{n(i)}, Tx_{n(i)}) + G(Rx_{n(i)}, Tx_{m(i)}, Tx_{m(i)})]\} \\ &= \max\{G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)}), G(Rx_{m(i)}, Rx_{m(i)+1}, Rx_{m(i)+1}), G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}), \\ &\quad \frac{1}{3}[G(Rx_{m(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)}, Rx_{m(i)+1}, Rx_{m(i)+1})]\} \\ &\leq \max\{G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)}), G(Rx_{m(i)}, Rx_{m(i)+1}, Rx_{m(i)+1}), G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}), \\ &\quad \frac{1}{3}[G(Rx_{m(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + 2G(Rx_{n(i)}, Rx_{n(i)}, Rx_{m(i)+1})]\}. \end{aligned}$$

Therefore, passing to the upper limit in (3.14) and using properties of the functions ψ, α, β with (3.19), (3.2), (3.22), (3.23), (3.24), we get that

$$\begin{aligned} &\psi(\epsilon) \\ &\leq \liminf \psi(G(Rx_{m(i)+1}, Rx_{n(i)+1}, Rx_{n(i)+1})) \\ &= \limsup \psi(G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)})) \\ &\leq \limsup[(\max\{G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)}), G(Rx_{m(i)}, Rx_{m(i)+1}, Rx_{m(i)+1}), G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}), \\ &\quad \frac{1}{3}[G(Rx_{m(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + 2G(Rx_{n(i)}, Rx_{n(i)}, Rx_{m(i)+1})]\}) \\ &\quad - \varphi(\max\{G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)}), G(Rx_{m(i)}, Rx_{m(i)+1}, Rx_{m(i)+1}), G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}), \\ &\quad \frac{1}{3}[G(Rx_{m(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + 2G(Rx_{n(i)}, Rx_{n(i)}, Rx_{m(i)+1})]\})] \\ &\leq \limsup[\alpha \max\{G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)}), G(Rx_{m(i)}, Rx_{m(i)+1}, Rx_{m(i)+1}), G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}), \\ &\quad \frac{1}{3}[G(Rx_{m(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + 2G(Rx_{n(i)}, Rx_{n(i)}, Rx_{m(i)+1})]\}] \\ &\quad - \liminf[\beta(\max\{G(Rx_{m(i)}, Rx_{n(i)}, Rx_{n(i)}), G(Rx_{m(i)}, Rx_{m(i)+1}, Rx_{m(i)+1}), G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}), \\ &\quad \frac{1}{3}[G(Rx_{m(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + G(Rx_{n(i)}, Rx_{n(i)+1}, Rx_{n(i)+1}) + 2G(Rx_{n(i)}, Rx_{n(i)}, Rx_{m(i)+1})]\})] \\ &\leq \alpha(\epsilon) - \beta(\epsilon). \end{aligned}$$

This is (because of $\varepsilon > 0$) a contradiction with (3.13). Thus $\{Rx_n\}$ is a G–Cauchy sequence in X.

Step III. Existence of a coincidence point. Since $\{Rx_n\}$ is a G–Cauchy sequence in the complete G-metric space, there exists $u \in X$ such that:

$$\lim_{n\to\infty} Rx_n = u. \tag{3.25}$$

From (3.25) and the continuity of R, we get:

$$\lim_{n\to\infty} R(Rx_n) = Ru. \tag{3.26}$$

By the rectangular inequality, we have:

$$G(Ru, Tu, Tu) \leq G(Ru, R(Rx_{n+1}), R(Rx_{n+1})) + G(R(Tx_n), T(Rx_n), T(Rx_n)) + G(T(Rx_n), Tu, Tu). \tag{3.27}$$

On the other hand, we have:

$$Rx_n \to u, \quad Tx_n \to u \quad \text{as } n \to \infty.$$

Since the pair $\{T, R\}$ is G-compatible, this implies that

$$\lim_{n\to\infty} G(R(Tx_n), T(Rx_n), T(Rx_n)) = 0. \tag{3.28}$$

Now, from the continuity of T and (3.25), we have:

$$\lim_{n\to\infty} G(T(Rx_n), Tu, Tu) = 0. \tag{3.29}$$

Combining (3.26), (3.28) and (3.29), and letting $n \to \infty$ in (3.27), we obtain:

$$G(Ru, Tu, Tu) \leq 0,$$

which implies that

$$Ru = Tu,$$

that is, u is a coincidence point of T and R. This makes the end to the proof. □

In the next theorem, we replace the continuity hypothesis by the hypothesis of regularity of the space X.

Theorem 3.10. *Let $(X, G, \preceq)$ be a partially ordered complete G–metric space and let $T, R : X \to X$ be self–mappings. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $s, t \geq 0$,*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ imply } \psi(t) - \alpha(s) + \beta(s) > 0, \tag{3.30}$$

and

$$\psi(G(Tx, Ty, Tz)) \leq \psi(\Delta(x, y, z)) - \varphi(\Delta(x, y, z)),$$

where

$$\Delta(x, y, z) = \max\{G(Rx, Ry, Rz), G(Rx, Tx, Tx), G(Ry, Ty, Ty), G(Rz, Tz, Tz), \frac{1}{3}[G(Rx, Ty, Ty) + G(Ry, Tz, Tz) + G(Rz, Tx, Tx)]\}$$

for all $x, y, z \in X$ with $Rz \preceq Ry \preceq Rx$. We suppose the following hypotheses:

(i) X is regular,

(ii) T is weakly increasing with respect to R,

(iii) RX is a closed subset of (X, d),

(ii) $TX \subseteq RX$.

Then, T and R have a coincidence point.

Proof . Following the proof of Theorem 3.8, we have that $\{Rx_n\}$ is a Cauchy sequence in (RX, G) which is complete since RX is a closed subspace of (X, G). Then, there exists $u = Rv$, $v \in X$ such that

$$\lim_{n\to\infty} Rx_n = u = Rv. \tag{3.31}$$

Since $\{Rx_n\}$ is a non-decreasing sequence and X is regular, it follows from (3.31) that $Rx_n \preceq Rv$ for all $n \in \mathbb{N}$. Hence, we can apply the considered contractive condition. Then, for $x = x_n$ and $y = z = v$, we obtain

$$\begin{aligned}\Theta(x_n, v, v) &= \max\{G(Rx_n, Rv, Rv), G(Rx_n, Tx_n, Tx_n), G(Rv, Tv, Tv),\\ &\quad \frac{1}{3}[G(Rx_n, Tv, Tv) + G(Rv, Tv, Tv) + G(Rv, Tx_n, Tx_n)]\}\\ &= \max\{G(Rx_n, Rv, Rv), G(Rx_n, Rx_{n+1}, Rx_{n+1}), G(Rv, Tv, Tv),\\ &\quad \frac{1}{3}[G(Rx_n, Tv, Tv) + G(Rv, Tv, Tv) + G(Rv, Rx_{n+1}, Rx_{n+1})]\}.\end{aligned}$$

By (3.14), we obtain

$$\begin{aligned}\psi(G(Rx_{n+1}, Tv, Tv)) &= \psi(G(Tx_n, Tv, Tv))\\ &\le \alpha(\max\{G(Rx_n, Rv, Rv), G(Rx_n, Rx_{n+1}, Rx_{n+1}), G(Rv, Tv, Tv),\\ &\quad \frac{1}{3}[G(Rx_n, Tv, Tv) + G(Rv, Tv, Tv) + G(Rv, Rx_{n+1}, Rx_{n+1})]\})\\ &\quad - \beta(\max\{G(Rx_n, Rv, Rv), G(Rx_n, Rx_{n+1}, Rx_{n+1}), G(Rv, Tv, Tv),\\ &\quad \frac{1}{3}[G(Rx_n, Tv, Tv) + G(Rv, Tv, Tv) + G(Rv, Rx_{n+1}, Rx_{n+1})]\}).\end{aligned}$$

Passing to the upper limit as $n \to \infty$ in the above inequalities as in the previous case, using (3.31) and the fact that G is continuous in its variables, we get that

$$\psi(G(Rv, Tv, Tv)) \le \alpha(G(Rv, Tv, Tv)) - \beta(G(Rv, Tv, Tv)).$$

Using again the condition (3.30), we get that it is only possible if $G(Rv, Tv, Tv) = 0$ and hence $Rv = Tv$. Hence, v is a coincidence point of T and R. This finishes the proof. □

Theorem 3.11. *Under the hypotheses of Theorem 3.10, and assuming that the order $\preceq$ is total, the mappings T and R have a unique point of coincidence, that is, there exists a unique $w \in X$ such that $Rz = Tz = w$ for some $z \in X$. In particular, if T is injective, they have also a unique coincidence point. If, moreover, T and R are weakly compatible, then T and R have a unique common fixed point.*

Proof . The existence of points $z, w \in X$ satisfying $Tz = Rz = w$ was proved in Theorem 3.1. Suppose that there exists another point $w_1 \in X$, $w_1 \neq w$ such that $Tz_1 = Rz_1 = w_1$ for some $z_1 \in X$. Without loss of generality, assume that $G(w_1, w, w) \geq G(w, w_1, w_1)$. Then

$$\begin{aligned}\Theta(z_1, z, z) &= \max\Big\{G(Rz_1, Rz, Rz), G(Rz_1, Tz_1, Tz_1), G(Rz, Tz, Tz), G(Rz, Tz, Tz),\\ &\qquad \frac{1}{3}[G(Rz_1, Tz, Tz) + G(Rz, Tz, Tz) + G(Rz, Tz_1, Tz_1)]\Big\}\\ &= \max\Big\{G(w_1, w, w), \frac{1}{3}[G(w_1, w, w) + G(w, w_1, w_1)]\Big\}\\ &= G(w_1, w, w) = \theta(z_1, z, z).\end{aligned}$$

Applying condition (3.2) to points z_1, z, z, we get that

$$\psi(G(w_1, w, w)) = \psi(G(Tz_1, Tz, Tz)) \leq \alpha(G(w_1, w, w)) - \beta(G(w_1, w, w)).$$

Using again the condition (3.30), we get that it is only possible if $G(w_1, w, w) = 0$ and $w_1 = w$, a contradiction. Thus, the point of coincidence w of R and T is unique. If T is injective, then also $z = z_1$ and the coincidence point is also unique. If these two mappings are weakly compatible, it follows that they have a unique common fixed point by a well known result of Jungck. □

Now, we present an example to illustrate our obtained result given by the Theorem 3.8 (inspired by [14]) showing the usage of Theorem 3.8 with (at least some of) functions ψ, α, β not being continuous.

Example 3.12. Let $X = [0, 1]$ and (X, G) be the complete G–metric space defined by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \text{ for all } x, y, z \in X.$$

We define an ordering $\preceq$ on X as follows:

$$x \preceq y \iff y \leq x, \quad \forall\, x, y \in X.$$

Define self–mappings $T, R : X \to X$ by $Rx = \frac{x}{2}$ and $Tx = \frac{x}{16}$.

If $\{x_n\}$ is a sequence in X such that $Rx_n = \frac{1}{2}x_n \to t$ and $Tx_n = \frac{1}{16}x_n \to t$ for some $t \in X$ then it is easy to see that $\lim_{n\to\infty} G(TRx_n, RTx_n, RTx_n) = 0$. Hence the pair of mappings $\{T, R\}$ is compatible. Also we can easily show that the mappings T and R satisfy the contractive condition (3.14). For instance,

$$G(Tx, Ty, Tz) = G(\frac{x}{16}, \frac{y}{16}, \frac{z}{16}) = \frac{1}{16}G(x, y, z) \leq \frac{3}{4}G(Rx, Ry, Rz) \leq \frac{3}{4}\Delta(x, y, z).$$

Now, by considering the control functions $\psi, \alpha, \beta : [0, +\infty) \to [0, +\infty)$ defined by:

$$\psi(t) = t + \frac{3}{2}, \quad \alpha(t) = t + \frac{5}{2} \quad \text{and} \quad \beta(t) = \frac{t}{4} + 1, \text{ for } t > 0,$$

we get:

$$\psi(G(Tx, Ty, Tz)) \leq \alpha(\Delta(x, y, z)) - \beta(\Delta(x, y, z)),$$

for all $x, y, z \in X$ such that $Rz \preceq Ry \preceq Rx$. Hence (3.14) is satisfied. Theorem 3.8 is applicable since all its hypotheses are satisfied. It may be observed that in this example the common fixed point of T and R (0) is unique.

Next, we demonstrate that if T and R do not satisfy the condition of compatibility then they may not have any common fixed point.

Example 3.13. Let $X = \mathbb{R}$, and (X, G) be the G–metric space defined by $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all x, y, z in X and R, T be self mappings on X defined as $Rx = [x]$ for all $x \in X$,

$$Tx = \begin{cases} -3, & \text{if } x \leq 0, \\ 0, & \text{if } 0 < x < 2, \\ 3, & \text{if } x \geq 2. \end{cases}$$

where $[x]$ denotes the integral part of x.

Take a sequence $\{x_n\}$ of points in X as $x_n = \frac{1}{n}$, then $Rx_n = 0 = Tx_n$ and $TRx_n = -3$, $RTx_n = 0$. Therefore, $\lim_{n\to\infty} G(RTx_n, TRx_n, TRx_n) = G(0, -3, -3) = 6 \neq 0$.

If $0 < x < 2$, then the left–hand side of (3.14) is 0 and hence (3.14) is satisfied. Thus, Theorem 3.8 is not applicable. It is observed that there is no common fixed point of T and R.

If $R : X \to X$ is the identity mapping, we can deduce easily the following fixed point result which improves the results of previous section.

Corollary 3.14. Let $(X, G, \preceq)$ be a partially ordered complete G–metric space and let $T : X \to X$ be a nondecreasing self–mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $s, t \geq 0$,

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ imply } \psi(t) - \alpha(s) + \beta(s) > 0,$$

and

$$\psi(G(Tx, Ty, Tz)) \leq \alpha(\Delta(x, y, z)) - \beta(\Delta(x, y, z))$$

where

$$\begin{aligned}\Delta(x, y, z) = \max\{&G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \\ &\frac{1}{3}[G(x, Ty, Ty) + G(y, Tz, Tz) + G(z, Tx, Tx)]\}\end{aligned}$$

for all $x, y, z \in X$ with $z \preceq y \preceq x$. We assume the following hypotheses:

(i) T is continuous or X is regular,

(ii) $Tx \preceq T(Tx)$ for all $x \in X$.

Then T has a fixed point, that is, there exists $u \in X$ such that $u = Tu$.

Remark 3.15. All the results of this section are valid if we replace $\Delta(x, y, z)$ by the following:

$$\begin{aligned}\Theta_2(x, y, z) = \max\{&G(Rx, Ry, Rz), G(Rx, Rx, Tx), G(Ry, Ry, Ty), G(Rz, Rz, Tz), \\ &\frac{1}{3}[G(Rx, Rx, Ty) + G(Ry, Ry, Tz) + G(Rz, Rz, Tx)]\}\end{aligned}$$

As a consequence of our results, we will deduce a fixed and coincidence point result for mappings satisfying a contraction of integral type in a partially ordered complete G–metric space. At first, let us introduce some notations.

We denote by Υ the set of functions $\Psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(i) Ψ is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$,

(ii) For all $\varepsilon > 0$, we have:

$$\int_0^{\varepsilon} \Psi(s)\, ds > 0.$$

Let $N \in \mathbb{N}$ be fixed. Let $\{\Psi_i\}_{1\le i\le N}$ be a family of N functions that belong to Υ. For all $t \ge 0$, we denote:

$$I_1(t) = \int_0^t \Psi_1(s)\ ds,$$

$$I_2(t) = \int_0^{I_1(t)} \Psi_2(s)\ ds = \int_0^{\int_0^t \Psi_1(s)\ ds} \Psi_2(s)\ ds$$

$$\vdots$$

$$I_N(t) = \int_0^{I_{N-1}(t)} \Psi_N(s)\ ds.$$

We have the following fixed point theorem of integral type.

Theorem 3.16. Let $(X, G, \preceq)$ be a partially ordered complete G–metric space and let $T : X \to X$ be a nondecreasing self–mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $s, t \ge 0$,

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ imply } \psi(t) - \alpha(s) + \beta(s) > 0,$$

and

$$I_N(\psi(G(Tx, Ty, Tz))) \le I_N(\alpha(\Delta(x, y, z))) - I_N(\beta(\Delta(x, y, z)))$$

where

$$\Delta(x, y, z) = \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), \frac{1}{3}[G(x, Ty, Ty) + G(y, Tz, Tz) + G(z, Tx, Tx)]\}$$

for all $x, y, z \in X$ with $z \preceq y \preceq x$. We assume the following hypotheses:

(i) T is continuous or regular,

(ii) $Tx \preceq T(Tx)$ for all $x \in X$.

Then T admits a fixed point.

Remark 3.17. Similar fixed and coincidence point results for mappings satisfying a contraction of integral type can be obtained from Theorems 3.1, 3.8 and 3.10.

4. An application to boundary value problems for ODE's

In this section we present another example where Theorem 3.1 and its corollaries can be applied. The example is inspired by [12].

We study the existence of solution for the following two–point boundary value problem for second order differential equation

$$\begin{cases} \dfrac{d^2u}{dt^2} = K(t, u(t)),\ t \in [0, 1],\ u \in [0, \infty) \\[2ex] u(0) = u(1) = 0. \end{cases}$$

This problem is equivalent to the integral equation

$$u(t) = \int_0^1 \mathcal{G}(t,s)K(s,u(s))\,ds, \qquad \text{for } t \in [0,1], \tag{4.1}$$

where $K : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ and $\mathcal{G}(t,s)$ is the Green function

$$\mathcal{G}(t,s) = \begin{cases} t(1-s), & 0 \le s < t \le 1 \\ s(1-t), & 0 \le t < s \le 1. \end{cases}$$

Denote $X = C([0,1], \mathbb{R}^+)$–the set of real non–negative continuous functions on $[0,1]$. We endow X with the G–metric

$$G(u,v,w) = \max_{t\in[0,1]} |u(t)-v(t)| + \max_{t\in[0,1]} |u(t)-w(t)| + \max_{t\in[0,1]} |v(t)-w(t)|,$$

for $u, v, w, \in X$. Then (X, G) is a G-complete metric space.

X can also be equipped with the partial order $\preceq$ given by:

$$u, v \in X, \qquad u \preceq v \Leftrightarrow u(t) \le v(t),\ \forall\, t \in [0,1].$$

Moreover, in [28], it is proved that $(X, G, \preceq)$ is regular (see Definition 2.16).

Consider the self-mapping $T : X \to X$ defined by

$$Tu(t) = \int_0^1 \mathcal{G}(t,s)K(s,u(s))\,ds, \text{ for } t \in [0,1]. \tag{4.2}$$

Clearly, u is a solution of (4.1) if and only if u is a fixed point of T.

We will prove the existence and uniqueness of the fixed point of T under the following conditions.

Theorem 4.1. *Suppose that the following hypotheses hold:*

(i) $K(s,\cdot)$ *is a non–decreasing continuous function for any fixed* $s \in [0,1]$*, that is,*

$$x, y \in \mathbb{R}^+,\ x \le y \Longrightarrow K(s,x) \le K(s,y).$$

(ii) for all $t, s \in [0,1]$*,* $u \in X$*, we have:*

$$K(t,u(s)) \le K\left(t, \int_0^1 K(s,u(\tau))\,d\tau\right),$$

(iii) for all $t \in [0,1]$ *and* $x, y \in \mathbb{R}$ *such that* $x \ge y$*,*

$$|K(t,x) - K(t,y)| \le 6|x-y|,$$

(iv) there exists $u_0 \in C([0,1])$ *such that* $u_0(t) \le Tu_0(t)$ *for* $t \in [0,1]$*.*

Then the integral equation (4.1) has a solution $u^* \in C([0,1])$*.*

Proof . Consider $T : X \to X$ given by (4.2). Let us prove that

$$Tu \preceq T(Tu) \text{ for all } u \in X. \tag{4.3}$$

Let $u \in C([0,1])$. From (ii), for all $t, s \in [0,1]$, we have:

$$\begin{aligned} Tu(t) &= \int_0^1 \mathcal{G}(t,s)K(s,u(s))\,ds \\ &\leq \int_0^1 \mathcal{G}(t,s)K\left(s, \int_0^1 \mathcal{G}(t,s)K(s,u(\tau))\,d\tau\right)\,ds \\ &= \int_0^1 \mathcal{G}(t,s)K(s,Tu(s))\,ds \\ &= T(Tu)(t). \end{aligned}$$

Then (4.3) holds. Since $\mathcal{G}(t,s) \geq 0$ for all $t, s \in [0,1]$, we deduce from (i) that for $u \leq v$

$$\int_0^1 \mathcal{G}(t,s)K(s,u(s))\,ds \leq \int_0^1 \mathcal{G}(t,s)K(s,v(s))\,ds,$$

i.e., $Tu(t) \leq Tv(t)$ holds for all $t \in [0,1]$. Thus T is a non–decreasing mapping. Also from (iv), $u_0 \preceq Tu_0$ for $u_0 \in X$. Since $\mathcal{G}(t,s) \geq 0$ for all $t, s \in [0,1]$, therefore for $u, v, w \in X$ such that $w \preceq v \preceq u$, by (iii), we have:

$$\begin{aligned} &G(Tu,Tv,Tw) \qquad\qquad (4.4) \\ &= \max_{t\in[0,1]} |Tu(t) - Tv(t)| + \max_{t\in[0,1]} |Tu(t) - Tw(t)| + \max_{t\in[0,1]} |Tv(t) - Tw(t)| \\ &= \max_{t\in[0,1]} \int_0^1 \mathcal{G}(t,s)\,|K(s,u(s)) - K(s,v(s))|\,ds \\ &\quad + \max_{t\in[0,1]} \int_0^1 \mathcal{G}(t,s)\,|K(s,u(s)) - K(s,w(s))|\,ds \\ &\quad + \max_{t\in[0,1]} \int_0^1 \mathcal{G}(t,s)\,|K(s,v(s)) - K(s,w(s))|\,ds \\ &\leq \max_{t\in[0,1]} \int_0^1 \mathcal{G}(t,s)(|u(s)-v(s)| + |u(s)-w(s)| + |v(s)-w(s)|)\,ds \\ &\leq G(u,v,w) \max_{t\in[0,1]} \int_0^1 \mathcal{G}(t,s)\,ds. \end{aligned}$$

It is easy to verify that

$$\int_0^1 \mathcal{G}(t,s)\,ds = \frac{-t^2}{2} + \frac{t}{2}$$

and that

$$\sup_{t\in[0,1]} \int_0^1 \mathcal{G}(t,s)\,ds = \frac{1}{8}.$$

With these facts, the inequality (4.4) give us

$$G(Tu,Tv,Tw) \leq \frac{3}{4}G(u,v,w) \leq \frac{3}{4}\Theta(u,v,w).$$

Now, by considering the control functions $\psi, \alpha, \beta : [0, +\infty) \to [0, +\infty)$ defined by:

$$\psi(t) = t + \frac{3}{2}, \quad \alpha(t) = t + \frac{5}{2} \quad \text{and } \beta(t) = \frac{t}{4} + 1, \text{ for } t > 0,$$

we get:

$$\psi(G(Tu, Tv, Tw)) \leq \alpha(\Theta(u, v, w)) - \beta(\Theta(u, v, w)),$$

for all $u, v, w \in C(I)$ such that $w \preceq v \preceq u$. Now, all the required hypotheses of Theorem 3.1 are satisfied. Then T admits a fixed point $u^* \in C([0, 1])$, that is, u^* is a solution to (4.1) (and so also of the given boundary value problem). □

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