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Stability for comprehensive classes of analytic functions

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Abstract

In this paper the problem of stability for comprehensive classes of analytic functions in T_δ- neighborhoods are studied. Also the lower and upper bounds of radius of stability are obtained.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disc $\mathbb{U} = \{z : \ |z| < 1\}$ in the complex plane $\mathbb{C}$. The subclass $\mathcal{S}$ of $\mathcal{A}$ is the class of univalent functions. Also Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the subclasses of $\mathcal{S}$ consisting of starlike and convex functions respectively.
A functions $f(z) \in \mathcal{A}$ is said to be in the class $R(\alpha, \beta, \gamma)$ if it satisfies

$$\mathcal{R}e\left(\frac{zf'(z) + \gamma z^2 f''(z)}{(1-\gamma)f(z) + \gamma z f'(z)}\right) > \alpha\left|\frac{zf'(z) + \gamma z^2 f''(z)}{(1-\gamma)f(z) + \gamma z f'(z)} - 1\right| + \beta$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$ and $0 \leq \gamma \leq 1$. For $\gamma = 0$ and $\gamma = 1$ we obtain the subclasses α-uniformly starlike of order β and α-uniformly convex of order β respectively. We note that if $f \in R(\alpha, \beta, \gamma)$ then for $\alpha > 1$, $\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma z f'(z)}$ lies in the region $G \equiv G(\alpha, \beta) = \{w = u + iv : \mathcal{R}ew < \alpha|w - 1| + \beta\}$, that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 + \frac{\alpha^2}{\alpha^2 - 1} v^2 = \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2},$$

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vertices at the points $(\frac{\alpha^2-\beta}{\alpha^2-1},\frac{\beta-1}{\sqrt{\alpha^2-1}})$, $(\frac{\alpha^2-\beta}{\alpha^2-1},\frac{1-\beta}{\sqrt{\alpha^2-1}})$, $(\frac{\alpha+\beta}{\alpha+1},0)$, $(\frac{\alpha-\beta}{\alpha-1},0)$.
For $\gamma=0$, the class $R(\alpha,\beta)=MD(\alpha,\beta)$ was studied earlier by J. Nishiwakiand S. Owa [11] . Many subclasses of the class $R(\alpha,\beta,\gamma)$ such as starlike functions and convex functions and uniformly starlike functions and uniformly convex functions were studied in earlier works[2, 13, 10, 12, 18].
Let the Hadamard product (or convolution) of two functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$, be given by $(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n$ and the integral convolution be given by

$$(f\otimes g)(z)=z+\sum_{n=2}^{\infty}\frac{a_nb_n}{n}z^n.$$

Also note that if I denotes $I(z)=z$ then $f*I=I$ and $f\otimes I=I$.
The convolution has the algebraic properties of ordinary multiplication. In convolution theory, the concept of duality is important. Many authors have used the powerful method of duality for study properties of analytic function (for example, see [7, 8, 17]). The concept of duality in geometric function theory was stated by Ruscheweyh in the book[16]. Denote the dual set of $\nu\subset\mathcal{A}$ by ν^* then according to definition in [15] we have

$$\nu^*=\{g\in\mathcal{A}:\frac{(f*g)(z)}{z}\neq 0\,,f\in\nu,z\in\mathbb{U}\}.$$

Let $D\subset\mathcal{A}$ be given such that $D^*=R(\alpha,\beta,\gamma)$. Then it is easy to see that

$$f\in\mathcal{R}(\alpha,\beta,\gamma)\qquad iff\qquad \frac{(f*g)(z)}{z}\neq 0,\quad (g\in D,z\in\mathbb{U}).$$

If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, then T_δ- neighborhood of the function f is defined as

$$TN_\delta(f)=\{g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{A}:\sum_{n=2}^{\infty}T_n|a_n-b_n|\leq\delta\},$$

where $\delta>0$ and $T=\{T_n\}_{n=2}^{\infty}$ is a sequence of positive numbers. In[17, 6] authors investigated T_δ- neighborhood for various subclasses of analytic functions. We also define $TN_\delta(A)=\bigcup_{f\in A}TN_\delta(f),(A\subset\mathcal{A})$. St. Ruscheweyh in [14] considered $T=\{T_n\}_{n=2}^{\infty}$ and showed that if $f\in\mathcal{K}$, then $TN_{1/4}(f)\subset S^*$.
Assume that A,B are subclasses of the class $\mathcal{A}$. Then the set of all functions $f*g$ and $f\otimes g$, where $f\in A$ and $g\in B$, will be denoted by $A*B$ and $A\otimes B$, respectively. Let $A*B\subset C$, the Hadamard product is called T-C-stable on the pair of classes (A,B) if there exists $\delta>0$ such that $TN_\delta(A)*TN_\delta(B)\subset C$. Stability of the integral convolution is defined in a similar way. The constant δ_T which characterizes the stability of Hadamard or integral convolution is called the radius of stability and it is defined as follows.

Definition 1.1. *Let A,B,C be the subclasses of the class $\mathcal{A}$ and $A*B\subset C$. Then the constant $\delta_T(A*B,C)$, defined by*

$$\delta_T(A*B,C)=sup\{\delta:TN_\delta(A)*TN_\delta(B)\subset C\},$$

is called the radius of stability of the convolution on the pair (A, B)*. The constant* $\delta_T(A \otimes B, C)$*, defined by*

$$\delta_T(A \otimes B, C) = sup\left\{\delta : TN_\delta(A) \otimes TN_\delta(B) \subset C\right\},$$

is called the radius of stability of the integral convolution on the pair (A, B)*.*

Bednarz in [3] studied T-C-stability for certain classes of analytic functions. Also, Bednarz, Kanas, Sokół, Aghalary and Shams et al. [4, 5, 1, 17] recently investigated the problem of stability for various subclasses of analytic functions. In this paper we investigate the problem of stability for the class $\mathcal{R}(\alpha, \beta, \gamma)$ in T_δ- neighborhoods and we find the lower and upper bounds for radius of stability.

2. Preliminaries

For obtaining our results we need the following definitions and lemmas to prove our main results.

Lemma 2.1. *Let* $H_t(z)$ *be given by*

$$H_t(z) = \frac{z}{(w(t)-1)(1-z)}[(w(t)(1-\gamma) + (w(t)\gamma - 1 + \gamma)\frac{1}{1-z} - \frac{\gamma(1+z)}{(1-z)^2})],$$

where $0 \le \beta < 1, w(t) = t\alpha + \beta \pm i\sqrt{t^2 - (t\alpha + \beta - 1)^2}, t^2 - (t\alpha + \beta - 1)^2 \ge 0, \alpha > 1,$ *and* $\frac{1-\beta}{1+\alpha} \le t \le \frac{\beta-1}{1-\alpha}$*, then a function* $f \in \mathcal{A}$ *is in* $\mathcal{R}(\alpha, \beta, \gamma)$ *if and only if for all* z *in* $\mathbb{U}$

$$\frac{(f * H_t)(z)}{z} \neq 0.$$

Proof . Let us assume that for $f \in \mathcal{A}$, $\frac{(f*H_t)(z)}{z} \neq 0$, $(z \in \mathbb{U})$, then we have

$$\begin{aligned}\frac{(f * H_t)(z)}{z} &= \frac{1}{z(w(t)-1)}\{f(z) * [w(t)(1-\gamma)\frac{z}{1-z} + (w(t)\gamma - 1 + \gamma)\frac{z}{(1-z)^2} - \frac{\gamma z(1+z)}{(1-z)^3}]\} \\ &= \frac{1}{z(w(t)-1)}\{w(t)(1-\gamma)f(z) + (w(t)\gamma - 1 + \gamma)zf'(z) - \gamma(z^2f''(z) + zf'(z)\} \\ &\neq 0, \end{aligned} \tag{2.1}$$

or

$$\frac{zf'(z) + \gamma z^2 f''(z)}{(1-\gamma)f(z) + \gamma z f'(z)} \neq w(t).$$

since boundry of region G can be taken as $w(t) = t\alpha + \beta \pm i\sqrt{t^2 - (t\alpha + \beta - 1)^2}$ for any $\frac{1-\beta}{1+\alpha} \le t \le \frac{\beta-1}{1-\alpha}$, this means that $\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)}$ lies completely either inside G or complement of G for all $z \in \mathbb{U}$. At $z = 0$, $\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)} = 1 \in G$ so that $\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)} \in G$ for all $z \in \mathbb{U}$, which shows that $f \in \mathcal{R}(\alpha, \beta, \gamma)$. The converse part follows easily since all the steps can be retraced back. This completes the proof of Lemma 2.1 □

Using the definition of dual set and Lemma 2.1 we can easily obtain the following result.

Corollary 2.2. *Let*

$$D = \{h \in \mathcal{A} : h(z) = \frac{z}{(w(t)-1)(1-z)}[(w(t)(1-\gamma) + (w(t)\gamma - 1 + \gamma)\frac{1}{1-z} - \frac{\gamma(1+z)}{(1-z)^2})]\}$$

where $0 \le \beta < 1, w(t) = t\alpha + \beta \pm i\sqrt{t^2 - (t\alpha + \beta - 1)^2}, t^2 - (t\alpha + \beta - 1)^2 \ge 0, \alpha > 1,$ *and* $\frac{1-\beta}{1+\alpha} \le t \le \frac{\beta-1}{1-\alpha}$, *then* $D^* = \mathcal{R}(\alpha, \beta, \gamma)$.

Lemma 2.3. *Let* $\alpha > 1$ *and* $0 \le \beta < 1$. *If* $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in D$, *then*

$$|c_n| \le \frac{1 - \beta + (n-1)(1+\alpha)}{1-\beta}(1 + (n-1)\gamma).$$

Proof . From the power series expansion of the function $h(z) \in D$ in Corollary 2.2 we obtain

$$c_n = \frac{w(t) - n}{w(t) - 1}(1 + (n-1)\gamma),$$

and therefore

$$|c_n|^2 = \frac{t^2 + (n-1)(n + 1 - 2(t\alpha + \beta))}{t^2}(1 + (n-1))\gamma)^2.$$

Since $\frac{1-\beta}{1+\alpha} \le t \le \frac{\beta-1}{1-\alpha}$, then $-2(t\alpha + \beta) \le 2(t-1)$ and we get

$$\begin{aligned} |c_n|^2 &\le \frac{t^2 + (n-1)[n + 1 + 2(t-1)]}{t^2}(1 + (n-1)\gamma)^2 \\ &= \frac{t^2 + (n-1)[n - 1 + 2t]}{t^2}(1 + (n-1)\gamma)^2 \\ &= \frac{(n-1+t)^2(1 + (n-1)\gamma)^2}{t^2}, \end{aligned} \tag{2.2}$$

and so

$$|c_n| \le \frac{(n-1+t)(1 + (n-1)\gamma)}{t}.$$

Now, since $\frac{1-\beta}{1+\alpha} \le t$, we obtain

$$\begin{aligned} |c_n| &\le (1 + \frac{(n-1)}{t})(1 + (n-1)\gamma) \\ &\le (1 + \frac{(n-1)(1+\alpha)}{1-\beta})(1 + (n-1)\gamma) \\ &= \frac{1 - \beta + (n-1)(1+\alpha)}{1-\beta}(1 + (n-1)\gamma). \end{aligned} \tag{2.3}$$

□

Corollary 2.4. *Let* $\alpha > 1$ *and* $0 \le \beta < 1$ *then* $g(z) = z + Az^n \in \mathcal{R}(\alpha, \beta, \gamma)$ *if and only if*

$$|A| \le \frac{1-\beta}{[1 - \beta + (n-1)(1+\alpha)](1 + (n-1)\gamma)}. \tag{2.4}$$

Proof . First we prove the sufficient condition. Since for every $h \in D$ we have

$$\begin{aligned} |\frac{(g*h)(z)}{z}| &= |1+c_nAz^{n-1}| \\ &\geq 1-|c_nAz| \\ &\geq 1-|z| \\ &> 0, \end{aligned} \tag{2.5}$$

so Corollary 2.2 gives $g \in D^* = \mathcal{R}(\alpha,\beta,\gamma)$. Suppose that $g \in \mathcal{R}(\alpha,\beta,\gamma)$, and

$$h(z) = z + \sum_{n=2}^{\infty} \frac{(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)}{1-\beta} z^n \in D.$$

Then

$$\frac{(g*h)(z)}{z} = 1 + A\frac{(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)}{1-\beta} z^{n-1}.$$

Then, for $|A| > \frac{1-\beta}{(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)}$ there exist a point $\xi \in \mathbb{U}$ such that $\frac{(g*h)(\xi)}{\xi} = 0$, this shows that the inequality 2.4 must hold. □

Corollary 2.5. *Let $\alpha > 1$ and $0 \leq \beta < 1$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$. If*

$$\sum_{n=2}^{\infty} \frac{(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)}{1-\beta} |a_n| \leq 1,$$

then $f \in \mathcal{R}(\alpha,\beta,\gamma)$.

Proof . Let $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in D$, since for all $n \geq 2$,

$$|c_n| \leq \frac{(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)}{1-\beta},$$

then we have

$$\begin{aligned} |\frac{(f*h)(z)}{z}| &= |1+\sum_{n=2}^{\infty} a_n c_n z^{n-1}| \\ &\geq 1-\sum_{n=2}^{\infty} |a_n||c_n||z| \\ &> 1-\sum_{n=2}^{\infty} |a_n||c_n| \\ &\geq 0. \end{aligned} \tag{2.6}$$

Then $\frac{(f*h)(z)}{z} \neq 0$ and from Corollary 2.2 we have $f \in D^* = \mathcal{R}(\alpha,\beta,\gamma)$.
□

Lemma 2.6. *Let $\alpha > 1$ and $0 \leq \beta < 1$, if for $f \in \mathcal{A}$ and every $\epsilon \in \mathbb{C}$ such that $|\epsilon| < \delta$, $F_\epsilon(z) = \frac{f(z)+\epsilon z}{1+\epsilon} \in \mathcal{R}(\alpha,\beta,\gamma)$, then for every $h \in D$, $|\frac{(f*h)(z)}{z}| \geq \delta$, $(z \in \mathbb{U})$.*

Proof . Let $F_\epsilon \in \mathcal{R}(\alpha,\beta,\gamma)$, then by Corollary 2.2 for every $h \in D$ we have $\frac{(F_\epsilon*h)(z)}{z} \neq 0, z \in \mathbb{U}$. Equivalently, $\frac{(f*h)(z)+\epsilon z}{(1+\epsilon)z} \neq 0$ in $\mathbb{U}$ or $\frac{(f*h)(z)}{z} \neq -\epsilon$ which shows that $|\frac{(f*h)(z)}{z}| \geq \delta$. □

Lemma 2.7. *[16] Let f(z) and g(z) be in the class $\mathcal{K}$ and $\mathcal{S}^*$ respectively. Then for every function $F(z)$ analytic in $\mathbb{U}$, we have*

$$\frac{f(z) * F(z)g(z)}{f(z) * g(z)} \in \overline{Co}(F(\mathbb{U}))\ ,\ z \in \mathbb{U},$$

where $\overline{Co}$ denote the closed convex hull.

Lemma 2.8. *Let $\alpha > 1$ and $0 \leq \beta < 1$. If $f \in \mathcal{R}(\alpha, \beta, \gamma)$ and $g \in \mathcal{K}$. Then $f * g \in \mathcal{R}(\alpha, \beta, \gamma)$.*

Proof . It is sufficient to prove that

$$\frac{z(f*g)'(z) + \gamma z^2 (f*g)''(z)}{(1-\gamma)(f*g)(z) + \gamma z(f*g)'(z)} \in G.$$

We have

$$\begin{aligned} \frac{z(f*g)'(z) + \gamma z^2 (f*g)''(z)}{(1-\gamma)(f*g)(z) + \gamma z(f*g)'(z)} &= \frac{g(z) * zf'(z) + \gamma(g(z) * z^2 f''(z))}{(1-\gamma)(g*f)(z) + \gamma(g(z)f'(z))} \\ &= \frac{g * \frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)}((1-\gamma)f(z) + \gamma zf'(z))}{g(z) * ((1-\gamma)f(z) + \gamma zf'(z)}. \end{aligned} \tag{2.7}$$

Since

$$\mathcal{R}e(\frac{z((1-\gamma)f(z) + \gamma zf'(z))'}{(1-\gamma)f(z) + \gamma zf'(z)}) = \mathcal{R}e(\frac{zf'(z) + \gamma z^2 f''(z)}{(1-\gamma)f(z) + \gamma zf'(z)}) > 0,$$

thus $(1-\gamma)f(z) + \gamma zf'(z) \in S^*$, so using Lemma2.7 we obtain

$$\frac{z(f*g)'(z) + \gamma z^2 (f*g)''(z)}{(1-\gamma)(f*g)(z) + \gamma z(f*g)'(z)} \in \overline{Co}(F(\mathbb{U})) \subset G,$$

where $F(z) = \frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)}$ and G is a convex domain. Then

$$f * g \in \mathcal{R}(\alpha, \beta, \gamma).$$

□

Definition 2.9. *A function $f \in \mathcal{A}$ is said to be in the class $N(\alpha, \beta)$ if for all $z \in \mathbb{U}$,*

$$zf'(z) \in R(\alpha, \beta, \gamma).$$

Let $f \in \mathcal{A}$ and $\varphi(z) = f'(z) + zf''(z)$, then a function f is said to be in the class $\mathcal{N}(\alpha, \beta)$ if for all $z \in \mathbb{U}$, $\mathcal{R}e(1 + \frac{z\varphi\prime(z)}{\varphi(z)}) > \alpha \left|\frac{z\varphi\prime(z)}{\varphi(z)}\right| + \beta$.

Lemma 2.10. *Let $\alpha > 0$ and $0 \leq \beta < 1$. If $f \in \mathcal{N}(\alpha, \beta)$, then for ϵ with $|\epsilon| < \frac{1}{4}$,*

$$F_\epsilon(z) = \frac{f(z) + \epsilon z}{1 + \epsilon} \in \mathcal{R}(\alpha, \beta, \gamma).$$

Proof . Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

$$\begin{aligned} F_\epsilon(z) &= \frac{f(z) + \epsilon z}{1+\epsilon} \\ &= \frac{z(1+\epsilon) + \sum_{n=2}^{\infty} a_n z^n}{1+\epsilon} \\ &= \frac{f(z) * [z(1+\epsilon) + \sum_{n=2}^{\infty} z^n]}{1+\epsilon} \\ &= f(z) * \frac{z - \frac{\epsilon}{1+\epsilon} z^2}{1-z} = f(z) * h(z), \end{aligned} \tag{2.8}$$

where $h(z) = \frac{z - \frac{\epsilon}{1+\epsilon} z^2}{1-z}$. Differentiating logarithmically we obtain,

$$\begin{aligned} \frac{zh'(z)}{h(z)} &= \frac{z - \frac{2\epsilon}{1+\epsilon} z^2}{z - \frac{\epsilon}{1+\epsilon} z^2} + \frac{z}{1-z} \\ &= \frac{-\rho z}{1-\rho z} + \frac{1}{1-z}, \end{aligned} \tag{2.9}$$

where $\rho = \frac{\epsilon}{1+\epsilon}$. Hence $\mathcal{R}e \frac{zh'(z)}{h(z)} > 0$ if $|\rho| < \frac{1}{3}$ then $|\epsilon| < \frac{1}{4}$. Therefore h is starlike in $\mathbb{U}$ if $|\epsilon| < \frac{1}{4}$and so

$$\int_0^z \frac{h(t)}{t} dt = z + \sum_{n=2}^{\infty} \frac{h_n z^n}{n} = h(z) * \log(\frac{1}{1-z}),$$

is convex for $|\epsilon| < \frac{1}{4}$. Also we have,

$$F_\epsilon(z) = (f * h)(z) = zf'(z) * (h(z) * \log \frac{1}{1-z}) \in \mathcal{R}(\alpha, \beta, \gamma).$$

□

Lemma 2.11. *Let $\alpha > 1$ and $0 \le \beta < 1$. If $f \in \mathcal{N}(\alpha, \beta)$ and $h \in D$, then* $|\frac{(f*h)(z)}{z}| \ge \frac{1}{4}$.

Proof . Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{N}(\alpha, \beta)$ and $h \in D$, then from Lemma 2.10 for $|\epsilon| < \frac{1}{4}$ we have $F_\epsilon(z) = \frac{f(z)+\epsilon z}{1+\epsilon} \in \mathcal{R}(\alpha, \beta, \gamma)$. Thus

$$\frac{1}{z}[h(z) * F_\epsilon(z)] \neq 0 \,, \; |\epsilon| < \frac{1}{4}.$$

Now from the properties of Hadamard product we obtain

$$\begin{aligned} \frac{1+\epsilon}{z}[h(z) * \frac{f(z) + \epsilon z}{1+\epsilon}] &= \frac{1}{z}[h(z) * (f(z) + \epsilon z)] \\ &= \frac{1}{z}[h(z) * f(z)] + \epsilon \neq 0. \end{aligned} \tag{2.10}$$

Hence for $|\epsilon| < \frac{1}{4}$, $\frac{1}{z}[h(z) * f(z)] \neq -\epsilon$, and so $|\frac{(f*h)(z)}{z}| \ge \frac{1}{4}$. □

Lemma 2.12. *Let $\alpha > 1$ and $0 \leq \beta < 1$. If $f \in \mathcal{R}(\alpha,\beta,\gamma)$, then*

$$|a_2| \leq \frac{2(1-\beta)}{(\alpha-1)(1+\gamma)},$$

$$|a_n| \leq \frac{2(1-\beta)}{(\gamma n^2 - 2n\gamma + n - 1 + \gamma)(\alpha-1)} \prod_{j=1}^{n-2} (1 + \frac{2(1-\beta)}{j(\alpha-1)}). \quad (n \geq 3)$$

Proof . Let $f \in \mathcal{R}(\alpha,\beta,\gamma)$, then

$$\begin{aligned}\mathcal{R}e(\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma z f'(z)}) &> \alpha|\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma z f'(z)} - 1| + \beta \\ &\geq \alpha \mathcal{R}e(\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma z f'(z)} - 1) + \beta,\end{aligned}$$

implies that

$$\alpha - \beta + (1-\alpha)\mathcal{R}e(\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma z f'(z)}) > 0.$$

Let us define the function

$$p(z) = \frac{\alpha - \beta + (1-\alpha)\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma z f'(z)}}{1-\beta}. \tag{2.11}$$

Then $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and $\mathcal{R}ep(z) > 0$ $(z \in \mathbb{U})$. Therefore, if we write

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2.12}$$

then $|p_n| \leq 2$ for $n \geq 1$. From 2.11 and 2.12, we obtain

$$(1-\alpha)\sum_{n=1}^{\infty}(\gamma n^2 - 2n\gamma + n - 1 + \gamma)a_n z^n = (\sum_{n=1}^{\infty}(1-\beta)p_n z^n)(z + \sum_{n=1}^{\infty}(n\gamma + 1 - \gamma)a_n z^n).$$

Therefore we have

$$\begin{aligned}a_n = {} & \frac{1-\beta}{(\gamma n^2 - 2n\gamma + n - 1 + \gamma)(1-\alpha)}[p_{n-1} + p_{n-2}(\gamma+1)a_2 + p_{n-3}(2\gamma+1)a_3 + ... \\ & + p_2((n-3)\gamma+1)a_{n-2} + p_1((n-2)\gamma+1)a_{n-1}],\end{aligned}$$

for all $n \geq 2$. When $n = 2$,

$$|a_2| \leq \frac{1-\beta}{(\alpha-1)(1+\gamma)}|p_1| \leq 2\frac{1-\beta}{(\alpha-1)(1+\gamma)}.$$

And when $n = 3$,

$$\begin{aligned}|a_3| &\leq \frac{1-\beta}{2(\alpha-1)(2\gamma+1)}(|p_2| + |p_1||a_2|(1+\gamma)) \\ &\leq \frac{2(1-\beta)}{2(\alpha-1)(2\gamma+1)}(1 + (\gamma+1)\,|a_2|).\end{aligned} \tag{2.13}$$

Let us suppose that

$$\begin{aligned}|a_k| \le & \frac{2(1-\beta)}{(\gamma k^2-2k\gamma+k-1+\gamma)(\alpha-1)}(1+(\gamma+1)|a_2|+...+((k-3)\gamma+1)\,|a_{k-2}| \\ & +((k-2)\gamma+1)\,|a_{k-1}|) \\ \le & \frac{2(1-\beta)}{(\gamma k^2-2k\gamma+k-1+\gamma)(\alpha-1)}\prod_{j=1}^{k-2}(1+\frac{2(1-\beta)}{j(\alpha-1)}) \qquad (k\ge 3). \end{aligned} \tag{2.14}$$

Then we see

$$1+(\gamma+1)|a_2|+...+((k-3)\gamma+1)|a_{k-2}|+((k-2)\gamma+1)|a_{k-1}| \le \prod_{j=1}^{k-2}(1+\frac{2(1-\beta)}{j(\alpha-1)}). \tag{2.15}$$

By using 2.14 and 2.15, we obtain that

$$\begin{aligned}|a_{k+1}| \le & \frac{2(1-\beta)}{(\gamma(k+1)^2-2(k+1)\gamma+(k+1)-1+\gamma)(\alpha-1)}(1+(\gamma+1)|a_2|+... \\ & +((k-2)\gamma+1)|a_{k-1}|+((k-1)\gamma+1)|a_k|) \\ \le & \frac{2(1-\beta)}{(\gamma(k+1)^2-2(k+1)\gamma+(k+1)-1+\gamma)(\alpha-1)}(\prod_{j=1}^{k-2}(1+\frac{2(1-\beta)}{j(\alpha-1)}) \\ & +\frac{2(1-\beta)((k-1)\gamma+1)}{(\gamma k^2-2k\gamma+k-1+\gamma)(\alpha-1)}\prod_{j=1}^{k-2}(1+\frac{2(1-\beta)}{j(\alpha-1)})) \\ \le & \frac{2(1-\beta)}{(\gamma(k+1)^2-2(k+1)\gamma+(k+1)-1+\gamma)(\alpha-1)} \\ & \prod_{j=1}^{k-2}(1+\frac{2(1-\beta)}{j(\alpha-1)})[1+\frac{2(1-\beta)}{(k-1)(\alpha-1)}] \\ \le & \frac{2(1-\beta)}{(\gamma(k+1)^2-2(k+1)\gamma+(k+1)-1+\gamma)(\alpha-1)}\prod_{j=1}^{k-1}(1+\frac{2(1-\beta)}{j(\alpha-1)}).\end{aligned}$$

This completes the proof of Lemma 2.12. □

Corollary 2.13. *Let $\alpha \ge 2$ and $0 \le \beta < 1$. If $f \in \mathcal{R}(\alpha,\beta,\gamma)$, then*

$$|a_n| \le \frac{n(1-\beta)}{(n-1)\gamma+1}. \quad (n\ge 3) \tag{2.16}$$

Proof . Since $\alpha \ge 2$, then $\alpha-1 \ge 1$ and $\frac{1}{\alpha-1} \le 1$ so from Lemma 2.12 for $n \ge 3$ we have

$$\begin{aligned}|a_n| \;\le\; & \frac{2(1-\beta)}{(\gamma n^2-2n\gamma+n-1+\gamma)(\alpha-1)}\prod_{j=1}^{n-2}(1+\frac{2(1-\beta)}{j(\alpha-1)}) \\ \le\; & \frac{2(1-\beta)}{\gamma n^2-2n\gamma+n-1+\gamma}\prod_{j=1}^{n-2}(\frac{j+2}{j}) \\ =\; & \frac{2(1-\beta)}{(n-1)((n-1)\gamma+1)}(\frac{1+2}{1})(\frac{2+2}{2})(\frac{3+2}{3})...(\frac{n-2+2}{n-2}) \\ =\; & \frac{(1-\beta)}{((n-1)\gamma+1)}\frac{2\times 3\times 4\times ...(n-1)\times n}{(n-1)!}.\end{aligned} \tag{2.17}$$

□

Remark 2.14. *According to Lemma 2.12 we have* $|a_2| \leq \frac{2(1-\beta)}{(\gamma+1)(\alpha-1)}$ *then by the condition of Corollary 2.13 we obtain* $|a_2| \leq \frac{2(1-\beta)}{\gamma+1}$*, this shows that the inequality 2.16 holds for all* $n \geq 2$.

3. Main Results

Throughout this section $T = \{T_n\}_{n=2}^{\infty}$ will always be the sequence given by

$$T_n = \frac{1-\beta+(n-1)(1+\alpha)}{1-\beta}(1+(n-1)\gamma),$$

unless otherwise mentioned.

Theorem 3.1. *Let* $\alpha \geq 2$ *and* $0 \leq \beta < 1$*, then for*

$$0 \leq \delta < \sqrt{[\frac{(2-\beta+\alpha)(3-2\beta+\gamma)}{2(1-\beta)}]^2 + \frac{(2-\beta+\alpha)(1+\gamma)}{2(1-\beta)}} - \frac{(2-\beta+\alpha)(3-2\beta+\gamma)}{2(1-\beta)},$$

we have

$$TN_\delta(\mathcal{R}(\alpha,\beta,\gamma)) \otimes TN_\delta(\mathcal{K}) \subset \mathcal{R}(\alpha,\beta,\gamma).$$

Proof . Let $f_0(z) = z + \sum_{n=2}^{\infty} a_0nz^n \in \mathcal{R}(\alpha,\beta,\gamma)$ and $g_0(z) = z + \sum_{n=2}^{\infty} b_0nz^n \in \mathcal{K}$. Also suppose that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in TN_\delta(f_0)$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in TN_\delta(g_0).$$

We want to show that

$$\frac{(f \otimes g * h)(z)}{z} \neq 0, \quad (h \in D).$$

By the identity

$f \otimes g * h = f_0 \otimes g_0 * h + f_0 \otimes (g-g_0) * h + (f-f_0) \otimes g_0 * h + (f-f_0) \otimes (g-g_0) * h,$

we obtain

$$\begin{aligned} |\frac{(f \otimes g * h)(z)}{z}| \geq |\frac{(f_0 \otimes g_0 * h)(z)}{z}| - |\frac{(f_0 \otimes (g-g_0) * h)(z)}{z}| \\ - |\frac{((f-f_0) \otimes g_0 * h)(z)}{z}| - |\frac{((f-f_0) \otimes (g-g_0) * h)(z)}{z}|. \end{aligned} \tag{3.1}$$

From Lemma 2.8 it follows that $f_0 * g_0 \in \mathcal{R}(\alpha,\beta,\gamma)$, also we have

$$z(f_0 \otimes g_0)'(z) = z + \sum_{n=2}^{\infty} a_{0n} b_{0n} z^n = (f_0 * g_0)(z) \in \mathcal{R}(\alpha,\beta,\gamma),$$

thus $f_0 \otimes g_0 \in \mathcal{N}(\alpha,\beta)$. Now from Lemma2.11 for all $h \in D$ we obtain

$$\left|\frac{(f_0 \otimes g_0 * h)(z)}{z}\right| \geq \frac{1}{4}. \tag{3.2}$$

Making use of Corollary 2.13 and Lemma 2.3 for $f_0(z) = z + \sum_{n=2}^{\infty} a_{0n} z^n$ and $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$ we obtain $|c_n| \leq \frac{1-\beta+(n-1)(1+\alpha)}{1-\beta}(1+(n-1)\gamma)$ and $|a_{0n}| \leq \frac{n(1-\beta)}{2((n-1)\gamma+1)}$. Now from the definition of $TN_\delta(f_0)$ and $TN_\delta(g_0)$ we have

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{|a_{0n}||b_n - b_{0n}||c_n|}{n} &\leq \frac{1-\beta}{1+\gamma} \sum_{n=2}^{\infty} \frac{1-\beta+(n-1)(1+\alpha)}{1-\beta}(1+(n-1)\gamma)|b_n - b_{0n}| \\ &= \frac{1-\beta}{1+\gamma} \sum_{n=2}^{\infty} T_n |b_n - b_{0n}| \\ &\leq \frac{\delta(1-\beta)}{1+\gamma}.\end{aligned} \tag{3.3}$$

Similarly, we get

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{|b_{0n}||a_n - a_{0n}||c_n|}{n} &\leq \frac{1}{2} \sum_{n=2}^{\infty} T_n |a_n - a_{0n}| \\ &\leq \frac{\delta}{2}.\end{aligned} \tag{3.4}$$

Finally we have

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{|a_n - a_{0n}||b_n - b_{0n}||c_n|}{n} &\leq \frac{\delta(1-\beta)}{2(2-\beta+\alpha)(1+\gamma)} \sum_{n=2}^{\infty} T_n |b_n - b_{0n}| \\ &\leq \frac{\delta^2(1-\beta)}{2(2-\beta+\alpha)(1+\gamma)}.\end{aligned} \tag{3.5}$$

By virtue of 3.2, 3.3, 3.4 and 3.5, inequality 3.1 gives

$$|\frac{(f \otimes g * h)(z)}{z}| \geq \frac{1}{4} - \frac{\delta(1-\beta)}{1+\gamma} - \frac{\delta}{2} - \frac{\delta^2(1-\beta)}{2(2-\beta+\alpha)(1+\gamma)}. \tag{3.6}$$

The right side of 3.6 is positive whenever

$$0 \leq \delta < \sqrt{[\frac{(2-\beta+\alpha)(3-2\beta+\gamma)}{2(1-\beta)}]^2 + \frac{(2-\beta+\alpha)(1+\gamma)}{2(1-\beta)}} - \frac{(2-\beta+\alpha)(3-2\beta+\gamma)}{2(1-\beta)}.$$

□

Corollary 3.2. . *Let $\alpha \geq 2$ and $0 \leq \beta < 1$, then we have*

$\delta_T(\mathcal{R}(\alpha,\beta,\gamma) \otimes \mathcal{K}, \mathcal{R}(\alpha,\beta,\gamma)) \geq$

$$\sqrt{[\frac{(2-\beta+\alpha)(3-2\beta+\gamma)}{2(1-\beta)}]^2 + \frac{(2-\beta+\alpha)(1+\gamma)}{2(1-\beta)}} - \frac{(2-\beta+\alpha)(3-2\beta+\gamma)}{2(1-\beta)}.$$

Theorem 3.3. *Let $\alpha \geq 2$ and $0 \leq \beta < 1$, then for*

$$0 \leq \delta < \sqrt{(2-\beta+\alpha)^2 + \frac{2(2-\beta+\alpha)(1+\gamma)}{1-\beta}} - (2-\beta+\alpha),$$

we have

$$TN_\delta(\{I\}) \otimes TN_\delta(\mathcal{R}(\alpha,\beta,\gamma)) \subset \mathcal{R}(\alpha,\beta,\gamma).$$

Proof .. Let $f_0(z) = I(z) = z$ and $g_0(z) = z + \sum_{n=2}^{\infty} b_{0n} z^n \in \mathcal{R}(\alpha,\beta,\gamma)$. Also suppose that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in TN_\delta(f_0)$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in TN_\delta(g_0),$$

then we have $\sum_{n=2}^{\infty} T_n|a_n| \leq \delta$ and $\sum_{n=2}^{\infty} T_n|b_n - b_{0n}| \leq \delta$. We want to show

$$\frac{(f \otimes g) * h)(z)}{z} \neq 0, \qquad (h \in D).$$

We have

$$\begin{aligned}|\frac{(f \otimes g * h)(z)}{z}| \geq |\frac{(f_0 \otimes g_0 * h)(z)}{z}| - |\frac{(f_0 \otimes (g-g_0) * h)(z)}{z}| \\ - |\frac{((f-f_0) \otimes g_0 * h)(z)}{z}| - |\frac{((f-f_0) \otimes (g-g_0) * h)(z)}{z}|.\end{aligned}$$

observe that, $(f_0 \otimes g_0 * h)(z) = z$ and $(f_0 \otimes (g-g_0) * h)(z) = 0$. Moreover we have

$$\begin{aligned}|\frac{((f-f_0) \otimes g_0 * h)(z)}{z}| &\leq \sum_{n=2}^{\infty} \frac{|a_n||b_{0n}||c_n|}{n} \\ &\leq \frac{1-\beta}{1+\gamma} \sum_{n=2}^{\infty} \frac{1-\beta+(n-1)(1+\alpha)}{1-\beta}(1+(n-1)\gamma)|a_n| \\ &= \frac{1-\beta}{1+\gamma} \sum_{n=2}^{\infty} T_n|a_n| \\ &\leq \frac{\delta(1-\beta)}{1+\gamma}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}|\frac{((f-f_0) \otimes (g-g_0) * h)(z)}{z}| &\leq \sum_{n=2}^{\infty} \frac{|a_n||b_n - b_{0n}||c_n|}{n} \\ &\leq \frac{1}{2} \sum_{n=2}^{\infty} T_n|a_n||b_n - b_{0n}| \\ &\leq \frac{\delta(1-\beta)}{2(2-\beta+\alpha)(1+\gamma)} \sum_{n=2}^{\infty} T_n|b_n - b_{0n}| \\ &\leq \frac{\delta^2(1-\beta)}{2(2-\beta+\alpha)(1+\gamma)}. \end{aligned} \tag{3.8}$$

Now, following the same techniques as in the proof of theorem 3.1 we conclude the result and we omit details. □

Corollary 3.4. . *Let* $0 \leq \beta < 1$ *and* $\alpha \geq 2$. *Then we have*

$$\delta_T((\{I\}) \otimes \mathcal{R}(\alpha,\beta,\gamma), \mathcal{R}(\alpha,\beta,\gamma)) \leq \sqrt{(2-\beta+\alpha)^2 + \frac{2(2-\beta+\alpha)(1+\gamma)}{1-\beta}} - (2-\beta+\alpha).$$

Corollary 3.5. *Let* $0 \leq \beta < 1$ *and* $\alpha > 1$. *Then we have*

$$\delta_T(\mathcal{R}(\alpha,\beta,\gamma) \otimes \mathcal{K}, \mathcal{R}(\alpha,\beta,\gamma)) \leq \mu_1 = \sqrt{\left[\frac{1}{2} + \frac{(2-\beta+\alpha)(1+\gamma)}{2(1-\beta)}\right]^2 + \frac{(2-\beta+\alpha)(1+\gamma)}{1-\beta}} - \left[\frac{1}{2} + \frac{(2-\beta+\alpha)(1+\gamma)}{2(1-\beta)}\right]. \tag{3.9}$$

and

$$\delta_T(\{I\} \otimes \mathcal{R}(\alpha,\beta,\gamma), \mathcal{R}(\alpha,\beta,\gamma)) \leq \mu_2 = \sqrt{\frac{1}{4} + \frac{2(2-\beta+\alpha)(1+\gamma)}{1-\beta}} - \frac{1}{2} \tag{3.10}$$

Proof . Let $g_0(z) = \frac{z}{1-z} = z + z^2 + z^3 + \dots$, $f_0(z) = z + \frac{1-\beta}{(2-\beta+\alpha)(1+\gamma)} z^2$, and $h_0(z) = I(z) = z$. From Corollary 2.4 we have $f_0 \in \mathcal{R}(\alpha,\beta,\gamma)$. Since $g_0 \in \mathcal{K}$, we have

$$g(z) = z + (1 + \frac{\delta(1-\beta)}{(2-\beta+\alpha)(1+\gamma)})z^2 + z^3 + \dots \in TN_\delta(g_0) \subset TN_\delta(\mathcal{K}),$$

$$f(z) = z + (\frac{1-\beta}{(2-\beta+\alpha)(1+\gamma)}(1+\delta))z^2 \in TN_\delta(f_0) \subset TN_\delta(\mathcal{R}(\alpha,\beta,\gamma)),$$

$$h(z) = z + \frac{\delta(1-\beta)}{(2-\beta+\alpha)(1+\gamma)} z^2 \in TN_\delta(h_0) \subset TN_\delta(\{I\}).$$

To show 3.9 and 3.10 it is sufficient to prove that

$$f \otimes g \notin \mathcal{R}(\alpha,\beta,\gamma) \qquad \textit{when} \qquad \delta > \mu_1$$

and

$$h \otimes g \notin \mathcal{R}(\alpha,\beta,\gamma) \qquad \textit{when} \qquad \delta > \mu_2.$$

We have

$$(f \otimes g)(z) = z + \frac{1-\beta}{2(2-\beta+\alpha)(1+\gamma)}(1+\delta)(1 + \frac{\delta(1-\beta)}{(2-\beta+\alpha)(1+\gamma)})z^2.$$

Let

$$\varphi(\delta) = \frac{1-\beta}{2(2-\beta+\alpha)(1+\gamma)}(1+\delta)(1 + \frac{\delta(1-\beta)}{(2-\beta+\alpha)(1+\gamma)}).$$

Then we have $\varphi(\mu_1) = \frac{1-\beta}{(2-\beta+\alpha)(1+\gamma)}$ and $\varphi(\delta) > \varphi(\mu_1)$ for $\delta > \mu_1$, therefore by Corollary2.4 we have $(f \otimes g)(z) \notin \mathcal{R}(\alpha,\beta,\gamma)$ when $\delta > \mu_1$. Also we have

$$(h \otimes f)(z) = z + \frac{(1-\beta)^2}{2(2-\beta+\alpha)^2(1+\gamma)^2}(\delta + \delta^2)z^2,$$

and

$$\frac{(1-\beta)^2}{2(2-\beta+\alpha)^2(1+\gamma)^2}(\mu_2+{\mu_2}^2)=\frac{(1-\beta)}{(2-\beta+\alpha)(1+\gamma)},$$

thus similarly $h\otimes f\notin\mathcal{R}(\alpha,\beta,\gamma)$ when $\delta>\mu_2$. □

Theorem 3.6. *Let $0\leq\beta<1$ and $\alpha>1$. Then we have*

(i) for $\delta_1=\sqrt{\frac{(2-\beta+\alpha)(1+\gamma)}{1-\beta}}$, $TN_{\delta_1}(\{I\})*TN_{\delta_1}(\{I\})\subset\mathcal{R}(\alpha,\beta,\gamma)$,

(ii) for $\delta_2=\sqrt{\frac{(2(2-\beta+\alpha))(1+\gamma)}{1-\beta}}$, $TN_{\delta_2}(\{I\})\otimes TN_{\delta_2}(\{I\})\subset\mathcal{R}(\alpha,\beta,\gamma)$.

The result is the best possible in each case.

Proof . (i) Let

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in TN_{\delta_1}(\{I\})$$

and

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in TN_{\delta_1}(\{I\}).$$

By making use of definition $TN_{\delta_1}(\{I\})$ we have

$$\sum_{n=2}^{\infty}T_n|a_n|\leq\delta_1, \tag{3.11}$$

and

$$\sum_{n=2}^{\infty}T_n|b_n|\leq\delta_1. \tag{3.12}$$

Since $0\leq\beta<1$ and $\alpha>1$, $T_n=\frac{1-\beta+(n-1)(1+\alpha)}{1-\beta}(1+(n-1)\gamma)$ is an increasing function of n so that from 3.11 we get

$$\sum_{n=2}^{\infty}|a_n|\leq\frac{(1-\beta)\delta_1}{(2-\beta+\alpha)(1+\gamma)},$$

which implies that

$$|a_n|\leq\frac{(1-\beta)\delta_1}{(2-\beta+\alpha)(1+\gamma)}\ (n\geq2).$$

Using the above inequality, it follows that

$$\sum_{n=2}^{\infty}\frac{(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)}{1-\beta}|a_n||b_n|\leq\frac{(1-\beta){\delta_1}^2}{(2-\beta+\alpha)(1+\gamma)}=1,$$

which in view of Corollary 2.5,$(f*g)(z)\in\mathcal{R}(\alpha,\beta,\gamma)$. The proof of (ii) is similar to part (i) and we omit the details.

To see that the containment relation in (i) is the best possible, we consider the function f and g defined in U by

$$f(z)=g(z)=z+\sqrt{\frac{1-\beta}{(2-\beta+\alpha)(1+\gamma)}}z^2,$$

clearly, $f, g \in TN_{\delta_1}(\{I\})$ and $(f * g) \in \mathcal{R}(\alpha, \beta, \gamma)$. Also considering the function f and g defined in U by

$$f(z) = g(z) = z + \sqrt{\frac{2(1-\beta)}{(2-\beta+\alpha)(1+\gamma)}}z^2,$$

it is easily seen that the result in (ii) is the best possible. This evidently completes proof. □

Corollary 3.7. *Let $0 \leq \beta < 1$ and $\alpha > 1$. Then we have*

$$(i) \qquad \delta_T(\{I\} * \{I\}, \mathcal{R}(\alpha, \beta, \gamma)) = \sqrt{\frac{(2-\beta+\alpha)(1+\gamma)}{1-\beta}},$$

$$(ii) \qquad \delta_T(\{I\} \otimes \{I\}, \mathcal{R}(\alpha, \beta, \gamma)) = \sqrt{\frac{2(2-\beta+\alpha)(1+\gamma)}{1-\beta}}.$$

Proof . (i) From Theorem 3.6 we have

$$\delta_T(\{I\} * \{I\}, \mathcal{R}(\alpha, \beta, \gamma)) \geq \delta_1 = \sqrt{\frac{(2-\beta+\alpha)(1+\gamma)}{1-\beta}}. \tag{3.13}$$

Moreover, Let

$$f(z) = g(z) = z + \frac{\delta(1-\beta)}{(2-\beta+\alpha)(1+\gamma)}z^2 \in TN_\delta(\{I\}).$$

Then we have

$$(f * g)(z) = z + (\frac{\delta(1-\beta)}{(2-\beta+\alpha)(1+\gamma)})^2 z^2.$$

Let $\varphi(\delta) = (\frac{\delta(1-\beta)}{(2-\beta+\alpha)(1+\gamma)})^2$, then $\varphi(\delta) > \varphi(\delta_1) = \frac{1-\beta}{(2-\beta+\alpha)(1+\gamma)}$ for $\delta > \delta_1$. Therefore, by Corollary 2.4, $f * g \notin \mathcal{R}(\alpha, \beta, \gamma)$ when $\delta > \delta_1$. This means that

$$\delta_T(\{I\} * \{I\}, \mathcal{R}(\alpha, \beta, \gamma)) \leq \delta_1. \tag{3.14}$$

The relation 3.13 and 3.14 give the result. The proof of part (ii) is similar to part (i) and we omit the details. □

Theorem 3.8. *let $a, b \in \mathbb{C}$, $c > |a| + |b|$ and*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1 + \frac{\gamma(1-\beta)+(1+\alpha)(\gamma+1)}{1-\beta}\frac{|ab|}{c-|a|-|b|-1} + \frac{\gamma(1+\alpha)}{1-\beta}\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}\right] \leq 2,$$

then

$$zF(a, b, c; z) \in \mathcal{R}(\alpha, \beta, \gamma).$$

Proof . Since $zF(a, b, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n$, we see from Corollary 2.5 that our conclusion is equivalent to

$$\sum_{n=2}^{\infty} \frac{(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)}{1-\beta}\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leq 1. \tag{3.15}$$

Let

$$S = \sum_{n=2}^{\infty}(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right|,$$

Note that the left side of 3.15 converges if $c > |a| + |b|$. Since $|(a)|_n \leq (|a|)_n$ we have

$$S \leq \sum_{n=2}^{\infty}(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}.$$

It suffices to prove that

$$\sum_{n=2}^{\infty}(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq 1-\beta.$$

We have

$$\begin{aligned}
&\textstyle\sum_{n=2}^{\infty}(1-\beta+(n-1)(1+\alpha))(1+(n-1)\gamma)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\
&= \textstyle\sum_{n=0}^{\infty}(1-\beta+(n+1)(1+\alpha))(1+(n+1)\gamma)\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}}\\
&= (1-\beta)\textstyle\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \gamma(1-\beta)\sum_{n=0}^{\infty}(n+1)\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}}\\
&+(1+\alpha)\textstyle\sum_{n=0}^{\infty}(n+1)\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \gamma(1+\alpha)\sum_{n=0}^{\infty}(n+1)^2\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}}\\
&= (1-\beta)\textstyle\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \gamma(1-\beta)\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n}}\\
&+(1+\alpha)\textstyle\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n}} + \gamma(1+\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n}}\\
&= (1-\beta)\textstyle\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \gamma(1+\alpha)\sum_{n=1}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n-1}}\\
&+[\gamma(1-\beta)+(1+\alpha)(\gamma+1)]\textstyle\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n}}\\
&= (1-\beta)\textstyle\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \gamma(1+\alpha)\sum_{n=0}^{\infty}\frac{(|a|)_{n+2}(|b|)_{n+2}}{(c)_{n+2}(1)_{n}}\\
&+[\gamma(1-\beta)+(1+\alpha)(\gamma+1)]\textstyle\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n}}\\
&= (1-\beta)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right] + \gamma(1+\alpha)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}\right]\\
&+[\gamma(1-\beta)+(1+\alpha)(\gamma+1)]\left[\frac{|ab|}{c-|a|-|b|-1}\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right].
\end{aligned}$$

The last expression is bounded above by $1-\beta$ if and only if
$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{\gamma(1-\beta)+(1+\alpha)(\gamma+1)}{1-\beta}\frac{|ab|}{c-|a|-|b|-1}+\frac{\gamma(1+\alpha)}{1-\beta}\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}\right] \leq 2.$ □

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