Int. J. Nonlinear Anal. Appl. 9 (2018) No. 2, 145-159
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2018.12688.1647

Ulam stabilities for nonlinear Volterra-Fredholm delay integrodifferential equations

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(Communicated by H. Khodaei)

Abstract

In the present research paper we derive results about existence and uniqueness of solutions and Ulam–Hyers and Rassias stabilities of nonlinear Volterra–Fredholm delay integrodifferential equations. Pachpatte's inequality and Picard operator theory are the main tools that are used to obtain our main results. We concluded this work with applications of obtained results and few illustrative examples.

Keywords: Volterra–Fredholm integrodifferential equations, Ulam–Hyers stability, Ulam–Hyers–Rassias stability, Integral inequality, Picard operator.
2010 MSC: Primary 45N05, 45M10; Secondary 34G20, 35A23.

1. Introduction

The Ulam's stability problem of functional equations [25] "Under what conditions there exist an additive mapping near an approximately additive mapping?" and its first attempt by Hyers [8] in the case of Banach spaces is well known. The concept of Ulam–Hyers stability is extended by Rassias [20]. Thereafter, many mathematicians have studied and extend the concept of Ulam–Hyers and Rassias stabilities for different kinds of equations. Literature survey shows that several techniques have been developed by mathematicians to investigate the Ulam–Hyers and Rassias stabilities of differential and integral equations. The most popular techniques that deals with Ulam–Hyers and Rassias stabilities of different kinds of differential and integral equations includes: fixed points technique [1, 2, 3, 9, 10, 17, 23, 24], successive approximations method [5, 6, 7, 15] and applying integral inequalities [16, 17, 22].

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Received: October 2017 *Revised:* April 2018

Inspired by the work of Rus [22] and Otrocol *et al.* [16, 17], in the present paper we derive results pertaining to existence and uniqueness of solutions and Ulam–Hyers and Rassias stabilities of nonlinear Volterra–Fredholm delay integrodifferential equation (VFDIE):

$$x'(t) = f\left(t, x(t), x(g(t)), \int_0^t h_1(t,s,x(s),x(g(s)))ds, \int_0^b h_2(t,s,x(s),x(g(s)))ds\right) \quad (t \in I), \tag{1.1}$$

where

(i) $I = [0,b] \ \ 0 < b < \infty$;
(ii) $f \in C\left([0,b] \times \mathbb{R}^4, \mathbb{R}\right)$, $h_i \in C\left([0,b] \times [0,b] \times \mathbb{R}^2, \mathbb{R}\right)$ for $i = 1,2$ and $g \in C\left([0,b],[-r,b]\right)$, $0 < r < \infty$ such that $g(t) \leq t$.

Picard operator theory, abstract Gronwall lemma and Pachpatte's inequality play leading role to obtain the sufficient conditions that guarantee existence and uniqueness of solutions and Ulam–Hyers and Rassias stabilities of nonlinear VFDIE (1.1). The considered nonlinear VFDIE (1.1) in the present is more general than that of considered in [3, 10, 16, 17, 22, 23, 24]. Therefore the results obtained in this papers can be regarded as generalisation of those which are obtained in [3, 10, 16, 17, 22, 23, 24]. Existence and uniqueness of solutions of nonlinear Volterra–Fredholm delay integrodifferential equations and their variants have been dealt in [4, 11, 12]. Recently, Kucche and Shikhare [13, 14] obtained results about existence and uniqueness of solutions and Ulam–Hyers and Rassias stabilities of nonlinear integrodifferential equations in Banach spaces.

The paper is arranged as follows. In Section 2, we define the Ulam–Hyers and Rassias type stability concepts for (1.1) and we state the theorems which are essential to obtain our main results. Section 3, deals with Ulam stability results for VFDIE (1.1). At last, we give applications and examples to illustrate the results.

2. Preliminaries

We follow the notations and definitions of [22] to deal with Ulam–Hyers and Rassias type stability of VFDIE (1.1). Consider the nonlinear Volterra–Fredholm delay integrodifferential equations

$$x'(t) = f\left(t, x(t), x(g(t)), \int_0^t h_1(t,s,x(s),x(g(s)))ds, \int_0^b h_2(t,s,x(s),x(g(s)))ds\right) \quad (t \in I), \tag{2.1}$$

$$x(t) = \phi(t), \ t \in [-r, 0], \tag{2.2}$$

where $\phi \in C\left([-r,0],\mathbb{R}\right)$.

Definition 2.1. A function $x \in C\left([-r,b],\mathbb{R}\right) \cap C^1\left([0,b],\mathbb{R}\right)$ that verifies the equations (2.1) and (2.2) is called the solution of initial value problem (2.1)–(2.2).

For $\epsilon > 0$ and a nondecreasing continuous function $\psi \in C\left([-r,b],\mathbb{R}_+\right)$, consider the following inequalities:

$$\left|y'(t) - f\left(t, y(t), y(g(t)), \int_0^t h_1(t,s,y(s),y(g(s)))ds, \int_0^b h_2(t,s,y(s),y(g(s)))ds\right)\right| \leq \epsilon, \ (t \in I), \tag{2.3}$$

$$\left|y'(t) - f\left(t, y(t), y(g(t)), \int_0^t h_1(t,s,y(s),y(g(s)))ds, \int_0^b h_2(t,s,y(s),y(g(s)))ds\right)\right| \leq \psi(t), \ (t \in I), \tag{2.4}$$

$$\left|y'(t) - f\left(t, y(t), y(g(t)), \int_0^t h_1(t,s,y(s),y(g(s)))ds, \int_0^b h_2(t,s,y(s),y(g(s)))ds\right)\right| \leq \epsilon\, \psi(t), \ (t \in I). \tag{2.5}$$

Definition 2.2. The equation (2.1) is said to be Ulam–Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C^1([-r, b], \mathbb{R})$ of (2.3) there exists a solution $x \in C^1([-r, b], \mathbb{R})$ of (2.1) with $|y(t) - x(t)| \le C_f\, \epsilon,\ t \in [-r, b]$.

Definition 2.3. The equation (2.1) is said to be generalized Ulam–Hyers stable if there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+),\ \theta_f(0) = 0$ such that for each solution $y \in C^1([-r, b], \mathbb{R})$ of (2.3) there exists a solution $x \in C^1([-r, b], \mathbb{R})$ of (2.1) with $|y(t) - x(t)| \le \theta_f(\epsilon),\ t \in [-r, b]$.

Definition 2.4. The equation (2.1) is said to be Ulam–Hyers–Rassias stable with respect to the positive nondecreasing continuous function $\psi : [-r, b] \to \mathbb{R}_+$ if there exists $C_{f,\psi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C^1([-r, b], \mathbb{R})$ of (2.5) there exists a solution $x \in C^1([-r, b], \mathbb{R})$ of (2.1) with $|y(t) - x(t)| \le C_{f,\psi}\, \epsilon\, \psi(t),\ t \in [-r, b]$.

Definition 2.5. The equation (2.1) is said to be generalized Ulam–Hyers–Rassias stable with respect to the positive nondecreasing continuous function $\psi : [-r, b] \to \mathbb{R}_+$ if there exists $C_{f,\psi} > 0$ such that for each solution $y \in C^1([-r, b], \mathbb{R})$ of (2.4) there exists a solution $x \in C^1([-r, b], \mathbb{R})$ of (2.1) with $|y(t) - x(t)| \le C_{f,\psi}\, \psi(t),\ t \in [-r, b]$.

Remark 2.6. A function $y \in C^1(I, \mathbb{R})$ is a solution of inequation (2.3) if there exists a function $p_y \in C(I, \mathbb{R})$ (which depend on y) such that

(i) $|p_y(t)| \le \epsilon,\ t \in I$;

(ii) $y'(t) = f\left(t, y(t), y(g(t)), \int_0^t h_1(t, s, y(s), y(g(s)))ds, \int_0^b h_2(t, s, y(s), y(g(s)))ds\right) + p_y(t),\ t \in I.$

Remark 2.7. If $y \in C^1(I, \mathbb{R})$ satisfies inequation (2.3), then y is a solution of the integral inequation

$$\left|y(t) - y(0) - \int_0^t f\left(s, y(s), y(g(s)), \int_0^s h_1(s, \tau, y(\tau), y(g(\tau)))d\tau, \int_0^b h_2(s, \tau, y(\tau), y(g(\tau)))d\tau\right) ds\right| \le \epsilon t,\ t \in I. \tag{2.6}$$

Indeed, if $y \in C^1(I, \mathbb{R})$ satisfies inequation (2.3), by Remark 2.6, we have

$$y'(t) = f\left(t, y(t), y(g(t)), \int_0^t h_1(t, s, y(s), y(g(s)))ds, \int_0^b h_2(t, s, y(s), y(g(s)))ds\right) + p_y(t),\ t \in I.$$

This gives

$$\begin{aligned}&\left|y(t) - y(0) - \int_0^t f\left(s, y(s), y(g(s)), \int_0^s h_1(s, \tau, y(\tau), y(g(\tau)))d\tau, \int_0^b h_2(s, \tau, y(\tau), y(g(\tau)))d\tau\right) ds\right| \\ &\le \int_0^t |p_y(s)|ds \\ &\le \epsilon t,\ t \in I.\end{aligned}$$

Theorem 2.8. (Pachpatte, [18]) Let $z(t), u(t), v(t), w(t) \in C([\alpha, \beta], \mathbb{R}_+)$ and $k \ge 0$ be a real constant and

$$z(t) \le k + \int_\alpha^t u(s)\left[z(s) + \int_\alpha^s v(\sigma)z(\sigma)d\sigma + \int_\alpha^\beta w(\sigma)z(\sigma)d\sigma\right] ds \quad \text{for} \quad t \in [\alpha, \beta].$$

If

$$r^* = \int_\alpha^\beta w(\sigma)\exp\left(\int_\alpha^\sigma [u(\tau) + v(\tau)]d\tau\right) d\sigma < 1,$$

then

$$z(t) \le \frac{k}{1 - r^*}\exp\left(\int_\alpha^t [u(s) + v(s)]ds\right) \quad \text{for} \quad t \in [\alpha, \beta].$$

Remark 2.9. The constant k in the Theorem 2.8 can be replaced by a positive nondecreasing continuous function. Thus we have the following Corollary and the proof of same can be completed following the arguments in the Theorem 1.7.4 of [19, p. 39].

Corollary 2.10. Let $z(t), u(t), v(t), w(t) \in C([\alpha,\beta],\mathbb{R}_+)$ and $n(t)$ be a positive and nondecreasing continuous function defined on $[\alpha,\beta]$ for which inequality

$$z(t) \leq n(t) + \int_\alpha^t u(s)\left[z(s) + \int_\alpha^s v(\sigma)z(\sigma)d\sigma + \int_\alpha^\beta w(\sigma)z(\sigma)d\sigma\right] ds \text{ for } t \in [\alpha,\beta].$$

If

$$r^* = \int_\alpha^\beta w(\sigma)\exp\left(\int_\alpha^\sigma [u(\tau)+v(\tau)]d\tau\right) d\sigma < 1,$$

then

$$z(t) \leq \frac{n(t)}{1-r^*}\exp\left(\int_\alpha^t [u(s)+v(s)]ds\right) \text{ for } t \in [\alpha,\beta].$$

Definition 2.11. (Rus, [21]) Let (X,d) be a metric space. An operator $\mathcal{A}: X \to X$ is a Picard operator if there exists $x^* \in X$ such that

(i) $\mathcal{F}_\mathcal{A} = \{x^*\}$ where $\mathcal{F}_\mathcal{A} = \{x \in X : \mathcal{A}(x) = x\}$ is the fixed point set of $\mathcal{A}$;
(ii) the sequence $(\mathcal{A}^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Lemma 2.12. (Rus, [21]) Let $(X,d,\leq)$ be an ordered metric space and $\mathcal{A}: X \to X$ be an increasing Picard operator ($\mathcal{F}_\mathcal{A} = x^*_\mathcal{A}$). Then, for $x \in X$, $x \leq \mathcal{A}(x)$ implies $x \leq x^*_\mathcal{A}$ while $x \geq \mathcal{A}(x)$ implies $x \geq x^*_\mathcal{A}$.

3. Stability Results

First, we list the hypotheses that are needed to prove our main results.

(A1) There exist $L(\cdot), G_i(\cdot) \in C(J,\mathbb{R}_+), i = 1,2$ such that

$$|f(t,x_1,x_2,x_3,x_4) - f(t,y_1,y_2,y_3,y_4)| \leq L(t)\left(\sum_{j=1}^4 |x_j - y_j|\right),$$

$$|h_i(t,s,x_1,x_2) - h_i(t,s,y_1,y_2)| \leq G_i(t)\left(\sum_{j=1}^2 |x_j - y_j|\right)$$

for all $t,s \in [0,b]$ and $x_j, y_j \in \mathbb{R}, j = 1,2,3,4$.

(A2) The function $\psi : [-r,b] \to \mathbb{R}_+$ is positive, nondecreasing and continuous and there exists $\lambda > 0$ such that

$$\int_0^t \psi(s)ds \leq \lambda\psi(t),\ t \in [0,b].$$

Theorem 3.1. *Suppose that the functions f and h_i $(i = 1,2)$ in (2.1) satisfy the condition (A1). Assume (A2) holds and let $N_f = 2\int_0^b L(s)\left[1 + \int_0^b G_1(\tau)d\tau + \int_0^b G_2(\tau)d\tau\right] ds < 1$. Then,*

(i) the problem (2.1)–(2.2) has a unique solution $x \in C([-r,b],\mathbb{R}) \cap C^1([0,b],\mathbb{R})$;

(ii) the equation (2.1) *is Ulam–Hyers–Rassias stable with respect the function ψ, provided*

$$q^* = \int_0^b G_2(\sigma)\exp\left(\int_0^\sigma [2L(\tau)+G_1(\tau)]d\tau\right)d\sigma < 1. \tag{3.1}$$

Proof . (i) In the view of assumptions, $f \in C([0,b]\times\mathbb{R}^4,\mathbb{R})$, $h_i \in C([0,b]\times[0,b]\times\mathbb{R}^2,\mathbb{R})$ for $i=1,2$ and $g \in C([0,b],[-r,b])\,, 0<r<\infty$, the initial value problem (2.1)–(2.2) is equivalent to the following integral equations

$$x(t) = \phi(0) + \int_0^t f\left(s, x(s), x(g(s)), \int_0^s h_1(s,\tau,x(\tau),x(g(\tau)))d\tau, \int_0^b h_2(s,\tau,x(\tau),x(g(\tau)))d\tau\right)ds \ (t\in I),$$
$$x(t) = \phi(t),\ t\in[-r,0].$$

Consider the Banach space $C = C([-r,b],\mathbb{R})$ endowed with Chebyshev norm $\|\cdot\|_C$ and define the operator $\mathcal{B}_f : C\to C$ by

$$\mathcal{B}_f(x)(t) = \phi(0) + \int_0^t f\left(s, x(s), x(g(s)), \int_0^s h_1(s,\tau,x(\tau),x(g(\tau)))d\tau, \int_0^b h_2(s,\tau,x(\tau),x(g(\tau)))d\tau\right)ds \ (t\in I),$$
$$\mathcal{B}_f(x)(t) = \phi(t),\ t\in[-r,0].$$

Note that,

$$|\mathcal{B}_f(x)(t) - \mathcal{B}_f(y)(t)| = 0, \quad x,y\in C([-r,b],\mathbb{R})\,, \quad t\in[-r,0]. \tag{3.2}$$

Next, for any $t\in I$,

$$\begin{aligned}
&|\mathcal{B}_f(x)(t) - \mathcal{B}_f(y)(t)| \\
&= \left|\int_0^t f\left(s, x(s), x(g(s)), \int_0^s h_1(s,\tau,x(\tau),x(g(\tau)))d\tau, \int_0^b h_2(s,\tau,x(\tau),x(g(\tau)))d\tau\right)ds\right. \\
&\qquad \left. - \int_0^t f\left(s, y(s), y(g(s)), \int_0^s h_1(s,\tau,y(\tau),y(g(\tau)))d\tau, \int_0^b h_2(s,\tau,y(\tau),y(g(\tau)))d\tau\right)ds\right| \\
&\le \int_0^t L(s)\Big\{|x(s)-y(s)| + |x(g(s)) - y(g(s))| \\
&\qquad + \int_0^s G_1(\tau)\left[|x(\tau)-y(\tau)| + |x(g(\tau))-y(g(\tau))|\right]d\tau \\
&\qquad\quad + \int_0^b G_2(\tau)\left[|x(\tau)-y(\tau)| + |x(g(\tau))-y(g(\tau))|\right]d\tau\Big\}ds \\
&\le \int_0^t L(s)\Big\{\max_{0\le\sigma_1\le s}|x(\sigma_1)-y(\sigma_1)| + \max_{0\le\sigma_1\le s}|x(g(\sigma_1))-y(g(\sigma_1))| \\
&\qquad + \int_0^s G_1(\tau)\left[\max_{0\le\sigma_2\le\tau}|x(\sigma_2)-y(\sigma_2)| + \max_{0\le\sigma_2\le\tau}|x(g(\sigma_2))-y(g(\sigma_2))|\right]d\tau \\
&\qquad\quad + \int_0^b G_2(\tau)\left[\max_{0\le\sigma_3\le\tau}|x(\sigma_3)-y(\sigma_3)| + \max_{0\le\sigma_3\le\tau}|x(g(\sigma_3))-y(g(\sigma_3))|\right]d\tau\Big\}ds \\
&\le \int_0^t L(s)\Big\{\max_{-r\le\sigma_1\le b}|x(\sigma_1)-y(\sigma_1)| + \max_{-r\le\tau_1\le b}|x(\tau_1)-y(\tau_1)| \\
&\qquad + \int_0^s G_1(\tau)\left[\max_{-r\le\sigma_2\le b}|x(\sigma_2)-y(\sigma_2)| + \max_{-r\le\tau_2\le b}|x(\tau_2)-y(\tau_2)|\right]d\tau \\
&\qquad\quad + \int_0^s G_2(\tau)\left[\max_{-r\le\sigma_3\le b}|x(\sigma_3)-y(\sigma_3)| + \max_{-r\le\tau_3\le b}|x(\tau_3)-y(\tau_3)|\right]d\tau\Big\}ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^b L(s)\left\{2\|x-y\|_C+2\int_0^b G_1(\tau)\|x-y\|_C\,d\tau+2\int_0^b G_2(\tau)\|x-y\|_C\,d\tau\right\}ds\\
&=2\left(\int_0^b L(s)\left[1+\int_0^b G_1(\tau)d\tau+\int_0^b G_2(\tau)d\tau\right]ds\right)\|x-y\|_C\\
&=N_f\|x-y\|_C\,. \qquad (3.3)
\end{aligned}
$$

From (3.2) and (3.3), we have

$$\|\mathcal{B}_f(x)-\mathcal{B}_f(y)\|_C \leq N_f\|x-y\|_C\,,\ x,y\in C\left([-r,b],\mathbb{R}\right).$$

Since $N_f<1$, the operator $\mathcal{B}_f$ is a contraction on complete space C. Hence by Banach contraction principle the operator $\mathcal{B}_f$ has fixed point $\tilde{x}:[-r,b]\to\mathbb{R}$ which is solution of the problem (2.1)–(2.2).
(ii) Let $y\in C\left([-r,b],\mathbb{R}\right)\cap C^1\left([0,b],\mathbb{R}\right)$ be a solution of inequation (2.5). Let $x\in C\left([-r,b],\mathbb{R}\right)\cap C^1\left([0,b],\mathbb{R}\right)$ is the unique solution of the problem

$$
\begin{aligned}
&x^{'}(t)=f\left(t,x(t),x(g(t)),\int_0^t h_1(t,s,x(s),x(g(s)))ds,\int_0^b h_2(t,s,x(s),x(g(s)))ds\right)\ (t\in I),\\
&x(t)=y(t),\ t\in[-r,0].
\end{aligned}
$$

Then x satisfies the integral equations

$$x(t)=y(0)+\int_0^t f\left(s,x(s),x(g(s)),\int_0^s h_1(s,\tau,x(\tau),x(g(\tau)))d\tau,\int_0^b h_2(s,\tau,x(\tau),x(g(\tau)))d\tau\right)ds\ (t\in I), \tag{3.4}$$

$$x(t)=y(t),\ t\in[-r,0]. \tag{3.5}$$

Let $y\in C\left([-r,b],\mathbb{R}\right)\cap C^1\left([0,b],\mathbb{R}\right)$ satisfies the inequation (2.5). Then using the hypothesis (A2) and Remark 2.6 and 2.7, we obtain

$$
\begin{aligned}
&\left|y(t)-y(0)-\int_0^t f\left(s,y(s),y(g(s)),\int_0^s h_1(s,\tau,y(\tau),y(g(\tau)))d\tau,\int_0^b h_2(s,\tau,y(\tau),y(g(\tau)))d\tau\right)ds\right|\\
&\qquad\leq\int_0^t|p_y(s)|\,ds\leq\int_0^t\epsilon\psi(s)ds\leq\lambda\epsilon\psi(t)\ (t\in I). \qquad (3.6)
\end{aligned}
$$

Clearly

$$|y(t)-x(t)|=0,\quad t\in[-r,0].$$

Next, using hypothesis (A1), the equation (3.4) and the estimation in (3.6), for any $t\in I$,

$$
\begin{aligned}
&|y(t)-x(t)|\\
&=\left|y(t)-y(0)-\int_0^t f\left(s,x(s),x(g(s)),\int_0^s h_1(s,\tau,x(\tau),x(g(\tau)))d\tau,\int_0^b h_2(s,\tau,x(\tau),x(g(\tau)))d\tau\right)ds\right|\\
&\leq\left|y(t)-y(0)-\int_0^t f\left(s,y(s),y(g(s)),\int_0^s h_1(s,\tau,y(\tau),y(g(\tau)))d\tau,\int_0^b h_2(s,\tau,y(\tau),y(g(\tau)))d\tau\right)ds\right|\\
&\quad+\int_0^t\left|f\left(s,y(s),y(g(s)),\int_0^s h_1(s,\tau,y(\tau),y(g(\tau)))d\tau,\int_0^b h_2(s,\tau,y(\tau),y(g(\tau)))d\tau\right)\right.\\
&\qquad\left.-f\left(s,x(s),x(g(s)),\int_0^s h_1(s,\tau,x(\tau),x(g(\tau)))d\tau,\int_0^b h_2(s,\tau,x(\tau),x(g(\tau)))d\tau\right)\right|ds\\
&\leq\epsilon\lambda\psi(t)+\int_0^t L(s)\Big\{\,|y(s)-x(s)|+|y(g(s))-x(g(s))|
\end{aligned}
$$

$$+\int_0^s G_1(\tau)\left[|y(\tau)-x(\tau)|+|y(g(\tau))-x(g(\tau))|\right]d\tau\Bigg|$$
$$+\int_0^b G_2(\tau)\left[|y(\tau)-x(\tau)|+|y(g(\tau))-x(g(\tau))|\right]d\tau\Bigg\}\,ds. \tag{3.7}$$

In the view of inequality (3.7), we consider the operator $\mathcal{A}: C([-r,b],\mathbb{R}_+)\to C([-r,b],\mathbb{R}_+)$ defined by

$$\begin{aligned}
&\mathcal{A}(\eta_1)(t)=0,\quad t\in[-r,0],\\
&\mathcal{A}(\eta_1)(t)=\epsilon\lambda\psi(t)+\int_0^t L(s)\Bigg\{\eta_1(s)+\eta_1(g(s))+\int_0^s G_1(\tau)\left[\eta_1(\tau)+\eta_1(g(\tau))\right]d\tau\\
&\qquad\qquad+\int_0^b G_2(\tau)\left[\eta_1(\tau)+\eta_1(g(\tau))\right]d\tau\Bigg\}\,ds,\quad t\in[0,b].
\end{aligned}$$

Next, we prove that $\mathcal{A}$ is a Picard operator (see Definition 2.11). Let any $\eta_1,\eta_2\in C([-r,b],\mathbb{R}_+)$. Clearly,

$$|\mathcal{A}(\eta_1)(t)-\mathcal{A}(\eta_2)(t)|=0,\ t\in[-r,0].$$

Using hypothesis (A1), for all $t\in I$,

$$\begin{aligned}
&|\mathcal{A}(\eta_1)(t)-\mathcal{A}(\eta_2)(t)|\\
&\le\int_0^t L(s)\{|\eta_1(s)-\eta_2(s)|+|\eta_1(g(s))-\eta_2(g(s))|\\
&\quad+\int_0^s G_1(\tau)\left[|\eta_1(\tau)-\eta_2(\tau)|+|\eta_1(g(\tau))-\eta_2(g(\tau))|\right]d\tau\\
&\qquad+\int_0^b G_2(\tau)\left[|\eta_1(\tau)-\eta_2(\tau)|+|\eta_1(g(\tau))-\eta_2(g(\tau))|\right]d\tau\Bigg\}\,ds\\
&\le\int_0^t L(s)\Bigg\{\max_{0\le\sigma_1\le s}|\eta_1(\sigma_1)-\eta_2(\sigma_1)|+\max_{0\le\sigma_1\le s}|\eta_1(g(\sigma_1))-\eta_2(g(\sigma_1))|\\
&\quad+\int_0^s G_1(\tau)\left[\max_{0\le\sigma_2\le\tau}|\eta_1(\sigma_2)-\eta_2(\sigma_2)|+\max_{0\le\sigma_2\le\tau}|\eta_1(g(\sigma_2))-\eta_2(g(\sigma_2))|\right]d\tau\\
&\qquad+\int_0^b G_2(\tau)\left[\max_{0\le\sigma_3\le\tau}|\eta_1(\sigma_3)-\eta_2(\sigma_3)|+\max_{0\le\sigma_3\le\tau}|\eta_1(g(\sigma_3))-\eta_2(g(\sigma_3))|\right]d\tau\Bigg\}\,ds\\
&\le\int_0^t L(s)\Bigg\{\max_{-r\le\sigma_1\le b}|\eta_1(\sigma_1)-\eta_2(\sigma_1)|+\max_{-r\le\tau_1\le b}|\eta_1(\tau_1)-\eta_2(\tau_1)|\\
&\quad+\int_0^s G_1(\tau)\left[\max_{-r\le\sigma_2\le b}|\eta_1(\sigma_2)-\eta_2(\sigma_2)|+\max_{-r\le\tau_2\le b}|\eta_1(\tau_2)-\eta_2(\tau_2)|\right]d\tau\\
&\qquad+\int_0^b G_2(\tau)\left[\max_{-r\le\sigma_3\le b}|\eta_1(\sigma_3)-\eta_2(\sigma_3)|+\max_{-r\le\tau_3\le b}|\eta_1(\tau_3)-\eta_2(\tau_3)|\right]d\tau\Bigg\}\,ds\\
&\le\int_0^t L(s)\Bigg\{2\|\eta_1-\eta_2\|_C+2\int_0^s G_1(\tau)\|\eta_1-\eta_2\|_C\,d\tau+2\int_0^b G_2(\tau)\|\eta_1-\eta_2\|_C\,d\tau\Bigg\}\,ds\\
&\le\int_0^b L(s)\Bigg\{2\|\eta_1-\eta_2\|_C+2\int_0^b G_1(\tau)\|\eta_1-\eta_2\|_C\,d\tau+2\int_0^b G_2(\tau)\|\eta_1-\eta_2\|_C\,d\tau\Bigg\}\,ds\\
&=2\left(\int_0^b L(s)[1+\int_0^b G_1(\tau)d\tau+\int_0^b G_2(\tau)d\tau]ds\right)\|\eta_1-\eta_2\|_C\\
&=N_f\|\eta_1-\eta_1\|_C\,.
\end{aligned}$$

Therefore,
$$\|\mathcal{A}(\eta_1)-\mathcal{A}(\eta_2)\|_C \leq N_f \|\eta_1-\eta_2\|_C, \text{ for all } \eta_1,\eta_2 \in C([-r,b],\mathbb{R}_+).$$
Since $N_f<1$, $\mathcal{A}$ is a contraction on $C([-r,b],\mathbb{R}_+)$. Using Banach contraction principle, $\mathcal{A}$ is Picard operator and $F_{\mathcal{A}}=\{\eta^*\}$. Then, for $t\in I$,

$$\begin{aligned}\eta^*(t) = \epsilon\lambda\psi(t) + \int_0^t L(s)\Big\{\eta(s)+\eta(g(s)) + \int_0^s G_1(\tau)\left[\eta(\tau)+\eta(g(\tau))\right]d\tau \\ + \int_0^b G_2(\tau)\left[\eta(\tau)+\eta(g(\tau))\right]d\tau\Big\}\,ds, \quad t\in[0,b].\end{aligned}$$

Observe that η^* is increasing and $(\eta^*)' \geq 0$ on I. Therefore $\eta^*(g(t)) \leq \eta^*(t)$ as $g(t)\leq t, t\in I$ and hence

$$\begin{aligned}\eta^*(t) \leq \epsilon\lambda\psi(t) + \int_0^t 2L(s)\Big\{\eta^*(s) + \int_0^s G_1(\tau)\eta^*(\tau)d\tau \\ + \int_0^b G_2(\tau)\eta^*(\tau)d\tau\Big\}\,ds,\ t\in I.\end{aligned} \tag{3.8}$$

Applying variant of Pachpatte's inequality given in the Corollary 2.10 to inequation (3.8) with

$$z(t)=\eta^*(t),\ n(t)=\epsilon\lambda\psi(t),\ u(t)=2L(t),\ v(t)=G_1(t) \text{ and } w(t)=G_2(t),$$

we get

$$\begin{aligned}\eta^*(t) &\leq \frac{\epsilon\lambda\psi(t)}{1-q^*}\exp\left(\int_0^t\left[2L(s)+G_1(s)\right]ds\right)\\ &\leq \frac{\epsilon\lambda\psi(t)}{1-q^*}\exp\left(\int_0^b\left[2L(s)+G_1(s)\right]ds\right).\end{aligned}$$

Therefore
$$\eta^*(t) \leq C_{f,\psi}\,\epsilon\,\psi(t),$$
where
$$C_{f,\psi} = \frac{\lambda}{1-q^*}\exp\left(\int_0^b\left[2L(s)+G_1(s)\right]ds\right).$$
For $\eta(t)=|y(t)-x(t)|$ the inequation (3.7) gives $\eta(t)\leq A(\eta)(t)$. By applying abstract Gronwall lemma we obtain
$$\eta(t)\leq\eta^*(t),\ t\in[-r,b],$$
and hence

$$|y(t)-x(t)| \leq C_{f,\psi}\,\epsilon\,\psi(t), \quad t\in[-r,b]. \tag{3.9}$$

This proves that the equation (2.1) is Ulam–Hyers–Rassias stable with respect to the function ψ. □

Corollary 3.2. *Suppose that the functions f and $h_i\,(i=1,2)$ in* (2.1) *satisfy the condition (A1). Assume (A2) holds and let $N_f = 2\int_0^b L(s)\left[1+\int_0^b G_1(\tau)d\tau + \int_0^b G_2(\tau)d\tau\right]ds < 1$. Then, the equations* (2.1)–(2.2) *has a unique solution and the equation* (2.1) *is generalized Ulam–Hyers–Rassias stable with respect to function the ψ, provided that the condition* (3.1) *is satisfied.*

Proof . By taking $\epsilon = 1$ in the proof of Theorem 3.1, we get $|y(t) - x(t)| \leq C_{f,\psi}\, \psi(t),\ t \in [-r, b]$. Therefore (2.1) is generalized Ulam–Hyers–Rassias stable with respect to $\psi : [-r, b] \to \mathbb{R}_+$. □

Corollary 3.3. *Suppose that the functions f and $h_i\,(i = 1, 2)$ in (2.1) satisfy the condition (A1). Assume (A2) holds and let $N_f = 2\int_0^b L(s)\left[1 + \int_0^b G_1(\tau)d\tau + \int_0^b G_2(\tau)d\tau\right] ds < 1$. Then, the equations (2.1)–(2.2) has a unique solution and the equation (2.1) is Ulam–Hyers stable, provided that the condition (3.1) is satisfied.*

Proof . By taking $\psi : [-r, b] \to \mathbb{R}_+$ defined by $\psi(t) = 1$, $t \in [-r, b]$ in the proof of Theorem 3.1, we get $|y(t) - x(t)| \leq C_f\, \epsilon$, $t \in [-r, b]$. Therefore equation (2.1) is Ulam–Hyers stable. □

Corollary 3.4. *Suppose that the functions f and $h_i\,(i = 1, 2)$ in (2.1) satisfy the condition (A1). Assume (A2) holds and let $N_f = 2\int_0^b L(s)\left[1 + \int_0^b G_1(\tau)d\tau + \int_0^b G_2(\tau)d\tau\right] ds < 1$. Then, the equations (2.1)–(2.2) has a unique solution and the equation (2.1) is generalized Ulam–Hyers stable, provided that the condition (3.1) is satisfied.*

Proof . Consider the function $\theta_f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\theta_f(\epsilon) = \epsilon\, C_f$. Then $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$ and $|y(t) - x(t)| \leq \theta_f(\epsilon),\ t \in [-r, b]$. Therefore equation (2.1) is generalized Ulam–Hyers stable. □

4. Applications

Fix any $r > 0$ and define $g(t) = t - r,\ t \in [0, b]$. Clearly $g \in C([0, b], [-r, b])$ and $g(t) \leq t$. In this case VFDIE (2.1)–(2.2) reduces to a difference equation. Consider the nonlinear difference equations:

$$x'(t) = \tilde{f}\left(t, x(t), x(t-r), \int_0^t \tilde{h}_1(t, s, x(s), x(s-r))ds, \int_0^b \tilde{h}_2(t, s, x(s), x(s-r))ds\right),\ t \in [0, b], \tag{4.1}$$

$$x(t) = \phi(t),\ t \in [-r, 0], \tag{4.2}$$

which is special case of VFDIE (2.1)–(2.2). Consider the inequality

$$\left|y'(t) - \tilde{f}\left(t, y(t), y(t-r), \int_0^t \tilde{h}_1(t, s, y(s), y(s-r))ds, \int_0^b \tilde{h}_2(t, s, y(s), y(s-r))ds\right)\right| \leq \epsilon\, \psi(t),\ t \in [0, b],$$

where ϵ, ψ and ϕ are as specified in the preliminaries section.

As an application of results obtained in section 3, we have the following theorem for the difference equations (4.1)–(4.2).

Theorem 4.1. Let the functions $\tilde{f},\ \tilde{h}_i\ (i = 1, 2)$ in (4.1) satisfy the following conditions:

(B1) Let $\tilde{f} \in C([0, b] \times \mathbb{R}^4, \mathbb{R})$, $\tilde{h}_i \in C([0, b] \times [0, b] \times \mathbb{R}^2, \mathbb{R})$, and suppose there exist constants $L_{\tilde{f}},\ \tilde{G}_i\ (i = 1, 2) > 0$ such that

$$|\tilde{f}(t, u_1, u_2, u_3, u_4) - \tilde{f}(t, v_1, v_2, v_3, v_4)| \leq L_{\tilde{f}}(t)\left(\sum_{j=1}^{4} |u_j - v_j|\right),$$

$$|\tilde{h}_i(t, s, u_1, u_2) - \tilde{h}_i(t, s, v_1, v_2)| \leq \tilde{G}_i(t)\left(\sum_{j=1}^{2} |u_j - v_j|\right)$$

for all $t, s \in [0, b],\ u_j, v_j \in \mathbb{R}\ (j = 1, 2, 3, 4)$.

(B2) the function $\psi : [-r, b] \to \mathbb{R}_+$ is positive, nondecreasing and continuous and there exists $\lambda > 0$ such that
$$\int_0^t \psi(s)ds \leq \lambda\psi(t),\ t \in [0, b].$$

(B3) $N_{\tilde{f}} = 2\int_0^b L_{\tilde{f}}(s)\left[1 + \int_0^b \tilde{G}_1(\tau)d\tau + \int_0^b \tilde{G}_2(\tau)d\tau\right] ds < 1.$

Then the problem (4.1)–(4.2) has a unique solution x in $C([-r, b], \mathbb{R}) \cap C^1([0, b], \mathbb{R})$ and the equation (4.1) is Ulam–Hyers–Rassias stable with respect to the function ψ, provided
$$\tilde{q^*} = \int_0^b \tilde{G}_2(\sigma)\exp\left(\int_0^\sigma [2L_{\tilde{f}}(\tau) + \tilde{G}_1(\tau)]d\tau\right) d\sigma < 1.$$

Next, consider the nonlinear Volterra–Fredholm integrodifferential equations of the form:
$$x'(t) = \bar{f}\left(t, x(t), x(t^2), \int_0^t \bar{h_1}(t, s, x(s), x(s^2))ds, \int_0^1 \bar{h_2}(t, s, x(s), x(s^2))ds\right),\ t \in I = [0, 1], \tag{4.3}$$
$$x(t) = \phi(t), \quad t \in [-r, 0]. \tag{4.4}$$

Note that, the nonlinear Volterra–Fredholm integrodifferential equations (4.3)–(4.4) is the special case of (2.1)–(2.2) with $g(t) = t^2,\ t \in I = [0, 1]$. Clearly $g \in C([0, 1], [-r, 1])$ for any $r > 0$ and $g(t) \leq t,\ t \in I = [0, 1]$. Consider the following inequality
$$\left|y'(t) - \bar{f}\left(t, y(t), y(s^2), \int_0^t \bar{h_1}(t, s, y(s), y(s^2)ds, \int_0^1 \bar{h_2}(t, s, y(s), y(s^2)ds\right)\right| \\ \leq \epsilon\,\psi(t),\ t \in [0, 1].$$

where ϵ, ψ and ϕ are as specified in the preliminaries section.

Theorem 4.2. Let the functions $\bar{f},\ \bar{h_i}\ (i = 1, 2)$ in (4.3) satisfy the following conditions:

(D1) Let $\bar{f} \in C([0, 1] \times \mathbb{R}^4, \mathbb{R}),\ \bar{h_i} \in C([0, 1] \times [0, 1] \times \mathbb{R}^2, \mathbb{R})$ and there exist constants $L_{\bar{f}},\ \bar{G}_i\ (i = 1, 2) > 0$ such that
$$|\bar{f}(t, u_1, u_2, u_3, u_4) - \bar{f}(t, v_1, v_2, v_3, v_4)| \leq L_{\bar{f}}(t)\left(\sum_{j=1}^4 |u_j - v_j|\right),$$
$$|\bar{h_i}(t, s, u_1, u_2) - \bar{h_i}(t, s, v_1, v_2)| \leq \bar{G}_i(t)\left(\sum_{j=1}^2 |u_j - v_j|\right)$$
for all $t, s \in [0, 1],\ u_j, v_j \in \mathbb{R}\ (j = 1, 2, 3, 4)$.

(D2) the function $\psi : [-r, 1] \to \mathbb{R}_+$ is positive, nondecreasing and continuous and there exists $\lambda > 0$ such that
$$\int_0^t \psi(s)ds \leq \lambda\psi(t),\ t \in [0, 1],$$

(D3) $N_{\bar{f}} = 2\int_0^1 L_{\bar{f}}(s)\left[1 + \int_0^1 \bar{G}_1(\tau)d\tau + \int_0^1 \bar{G}_2(\tau)d\tau\right] ds < 1.$

Then the problem (4.3)–(4.4) has a unique solution x in $C([-r,1],\mathbb{R})\cap C([0,1],\mathbb{R})$ and the equation (4.3) is Ulam–Hyers–Rassias stable with respect to the function ψ, provided

$$\bar{q^*} = \int_0^1 \bar{G_2}(\sigma)\exp\left(\int_0^\sigma [2L_{\bar{f}}(\tau)+\bar{G_1}(\tau)]d\tau\right)d\sigma < 1.$$

Remark 4.3. Ulam–Hyers stability, generalized Ulam–Hyers stability and generalized Ulam–Hyers–Rassias stability of the equations (4.1) and (4.3) can be discussed on similar line as discussed in the section 3.

5. Examples

We now present examples to illustrate the stability results we obtained.

Example 5.1. *Consider the nonlinear delay Volterra–Fredholm integrodifferential equations*

$$x'(t) = 0.506775 - \frac{2x(t)}{500} + \frac{\sin(x(g(t)))}{25} + \frac{\sin(2x(t))}{500} + \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2(x(s)) - \cos(x(g(s)))\right]ds$$
$$+\frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2(x(s)) - \cos(x(g(s)))\right]ds, \quad t\in[0,\pi], \tag{5.1}$$
$$x(t) = 0, \ t\in[-2,0], \tag{5.2}$$

where $g(t)=\frac{t}{5}$, $t\in[0,\pi]$, *then* $g(t)\le t$, $t\in[0,\pi]$.

Note that:

(i) Define $h_1 : [0,\pi]\times[0,\pi]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$h_1(t,s,x(s),x(g(s))) = \frac{1}{20}\left[\sin^2(x(s)) - \cos(x(g(s)))\right].$$

Then for any $t,s\in[0,\pi]$ and $x_1,x_2,y_1,y_2\in\mathbb{R}$ we have

$$|h_1(t,s,x_1,x_2) - h_1(t,s,y_1,y_2)| \le \frac{1}{20}\left|\sin^2 x_1 - \sin^2 y_1\right| + \frac{1}{20}\left|\cos x_2 - \cos y_2\right|.$$

Note that for any $\alpha,\beta\in\mathbb{R}$ with $\alpha<\beta$, applying mean value theorem, there exists $\gamma\in(\alpha,\beta)$ such that $\frac{\sin^2\alpha-\sin^2\beta}{\alpha-\beta} = -2\sin\gamma\cos\gamma \Rightarrow |\sin^2\alpha - \sin^2\beta| \le 2|\alpha-\beta|$. Thus

$$\begin{aligned}|h_1(t,s,x_1,x_2) - h_1(t,s,y_1,y_2)| &\le \frac{2}{20}|x_1-y_1| + \frac{1}{20}|x_2-y_2| \\ &\le \frac{1}{10}\{|x_1-y_1| + |x_2-y_2|\}.\end{aligned}$$

(ii) Define $h_2 : [0,\pi]\times[0,\pi]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$h_2(t,s,x(s),x(g(s))) = \frac{1}{12.5}\left[\sin^2(x(s)) - \cos(x(g(s)))\right].$$

Then for any $t,s\in[0,\pi]$ and $x_1,x_2,y_1,y_2\in\mathbb{R}$ we have

$$\begin{aligned}|h_2(t,s,x_1,x_2) - h_2(t,s,y_1,y_2)| &\le \frac{1}{12.5}\left|\sin^2 x_1 - \sin^2 y_1\right| + \frac{1}{12.5}\left|\cos x_2 - \cos y_2\right| \\ &\le \frac{2}{12.5}\{|x_1-y_1| + |x_2-y_2|\}.\end{aligned}$$

(iii) Define $f : [0, \pi] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$
\begin{aligned}
& f\left(t, x(t), x(g(t)), \int_0^t h_1(t, s, x(s), x(g(s)))ds, \int_0^\pi h_2(t, s, x(s), x(g(s)))ds\right) \\
&= 0.506775 - \frac{2x(t)}{500} + \frac{\sin(x(g(t)))}{25} + \frac{\sin(2x(t))}{500} + \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2(x(s)) - \cos(x(g(s)))\right] ds \\
&\quad + \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2(x(s)) - \cos(x(g(s)))\right] ds \\
&= 0.506775 - \frac{2x(t)}{500} + \frac{\sin(x(g(t)))}{25} + \frac{\sin(2x(t))}{500} + \frac{1}{12.5}\int_0^t h_1\left(t, s, x(s), x(g(s))\right) ds \\
&\quad \frac{1}{20}\int_0^\pi h_2\left(t, s, x(s), x(g(s))\right) ds.
\end{aligned}
$$

Then for any $t \in [0, \pi]$ and $x_j, y_j \in \mathbb{R}$, $j = 1, 2, 3, 4$, we have

$$
\begin{aligned}
& |f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \\
& \leq \left\{\frac{1}{500}|2x_1 - 2y_1| + \frac{1}{500}|\sin 2x_1 - \sin 2y_1|\right\} + \frac{1}{25}|\sin x_2 - \sin y_2| + \frac{1}{12.5}|x_3 - y_3| + \frac{1}{20}|x_4 - y_4| \\
& \leq \frac{1}{12.5}\left\{|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4|\right\}.
\end{aligned}
$$

The functions f, h in the equation (5.1) verifies the assumption (A1) with $L(t) = \frac{1}{12.5}$, $G_1(t) = \frac{1}{10}$, $G_2(t) = \frac{2}{12.5}$. Further,

$$
\begin{aligned}
N_f &= 2\int_0^b L(s)\left[1 + \int_0^b G_1(\tau)d\tau + \int_0^b G_2(\tau)d\tau\right] ds \\
&= 2\int_0^\pi \frac{1}{12.5}\left[1 + \int_0^\pi \frac{1}{10}d\tau + \int_0^\pi \frac{2}{12.5}d\tau\right] ds = 0.9132 < 1
\end{aligned}
$$

and

$$
\begin{aligned}
q^* &= \int_0^b G_2(\sigma)\exp\left(\int_0^\sigma [2L(\tau) + G_1(\tau)]d\tau\right) d\sigma \\
&= \int_0^\pi \frac{2}{12.5}\exp\left(\int_0^\sigma [\frac{2}{12.5} + \frac{1}{10}]d\tau\right) d\sigma = 0.2021 < 1.
\end{aligned}
$$

Hence by Corollary 3.3, the initial value problem (5.1)–(5.2) has unique solution on $[-2, \pi]$ and the equation (5.1) is Ulam–Hyres stable on $[0, \pi]$.

In fact, we see that

$$
x(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0, \pi], \\ 0 & \text{if } t \in [-2, 0], \end{cases}
$$

is the unique solution to the problem (5.1)–(5.2). The verification of the same is given below. For $x(t) = \frac{t}{2}$, $t \in [0, \pi]$ and $g(t) = \frac{t}{5}$, $t \in [0, \pi]$, we have

$$
\begin{aligned}
& 0.506775 - \frac{2x(t)}{500} + \frac{\sin(x(g(t)))}{25} + \frac{\sin(2x(t))}{500} + \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2(x(s)) - \cos(x(g(s)))\right] ds \\
&\quad + \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2(x(s)) - \cos(x(g(s)))\right] ds
\end{aligned}
$$

$$
\begin{aligned}
&= 0.506775 - \frac{2\left(\frac{t}{2}\right)}{500} + \frac{\sin(\frac{t}{10})}{25} + \frac{\sin(2\frac{t}{2})}{500} + \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2\left(\frac{s}{2}\right) - \cos\left(\frac{s}{10}\right)\right] ds \\
&\quad + \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2\left(\frac{s}{2}\right) - \cos\left(\frac{s}{10}\right)\right] ds \\
&= \frac{1}{2} = x'(t).
\end{aligned}
$$

Next, we discuss the Ulam–Hyres stability of the equation (5.1) with fixed delay $g(t) = \frac{t}{5}$, $t \in [0, \pi]$ by finding the exact solution $x(t)$ of equation (5.1) corresponding to given values of ϵ and given solutions $y(t)$ of the inequations.

(i) Choose $\epsilon = 0.8$ and

$$
y_1(t) = \begin{cases} t & \text{if } t \in [0, \pi], \\ 0 & \text{if } t \in [-2, 0]. \end{cases}
$$

Then for $t \in [0, \pi]$, we have

$$
\begin{aligned}
&\Bigg| y_1'(t) - \Bigg(0.506775 - \frac{2y_1(t)}{500} + \frac{\sin(y_1(g(t)))}{25} + \frac{\sin(2y_1(t))}{500} + \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2(y_1(s)) - \cos(y_1(g(s)))\right] ds \\
&\qquad + \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2(y_1(s)) - \cos(y_1(g(s)))\right] ds\Bigg)\Bigg| \\
&= \Bigg| y_1'(t) - 0.506775 + \frac{2y_1(t)}{500} - \frac{\sin(y_1(g(t)))}{25} - \frac{\sin(2y_1(t))}{500} - \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2(y_1(s)) - \cos(y_1(g(s)))\right] ds \\
&\qquad - \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2(y_1(s)) - \cos(y_1(g(s)))\right] ds\Bigg| \\
&= \Bigg| 1 - 0.506775 + \frac{2t}{500} - \frac{\sin(\frac{t}{5})}{25} - \frac{\sin(2t)}{500} - \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2(s) - \cos\left(\frac{s}{5}\right)\right] ds \\
&\qquad - \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2(s) - \cos\left(\frac{s}{5}\right)\right] ds\Bigg| \leq 0.5322357 < \epsilon.
\end{aligned}
$$

For the solution $x(t)$ of (5.1)–(5.2) and constant $C = 2$ we have

$$
|y_1(t) - x(t)| = \left| t - \frac{t}{2} \right| \leq \frac{\pi}{2} < C\epsilon, \ t \in [0, \pi],
$$

and

$$
|y_1(t) - x(t)| = 0, \ t \in [-2, 0].
$$

Therefore

$$
|y_1(t) - x(t)| < C\epsilon, \ t \in [-2, \pi].
$$

(ii) Choose $\epsilon = 0.4$ and

$$
y_2(t) = \begin{cases} t & \text{if } \frac{t}{3} \in [0, \pi], \\ 0 & \text{if } t \in [-2, 0]. \end{cases}
$$

Then for $t \in [0, \pi]$, we have

$$
\begin{aligned}
&\Bigg| y_2'(t) - \Bigg(0.506775 - \frac{2y_2(t)}{500} + \frac{\sin(y_2(g(t)))}{25} + \frac{\sin(2y_2(t))}{500} + \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2(y_2(s)) - \cos(y_2(g(s)))\right] ds \\
&\qquad + \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2(y_2(s)) - \cos(y_2(g(s)))\right] ds\Bigg)\Bigg|
\end{aligned}
$$

$$= \left| y_2'(t) - 0.506775 + \frac{2y_2(t)}{500} - \frac{\sin(y_2(g(t)))}{25} - \frac{\sin(2y_2(t))}{500} - \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2(y_2(s)) - \cos(y_2(g(s)))\right] ds \right.$$
$$\left. - \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2(y_2(s)) - \cos(y_2(g(s)))\right] ds \right|$$
$$= \left| \frac{1}{3} - 0.506775 + \frac{2(\frac{t}{3})}{500} - \frac{\sin(\frac{t}{15})}{25} - \frac{\sin(2(\frac{t}{3}))}{500} - \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2\left(\frac{s}{3}\right) - \cos\left(\frac{s}{15}\right)\right] ds \right.$$
$$\left. - \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2\left(\frac{s}{3}\right) - \cos\left(\frac{s}{15}\right)\right] ds \right| \le 0.179402 < \epsilon.$$

For the solution $x(t)$ of (5.1)–(5.2) and constant $C = 1.5$ we have

$$|y_2(t) - x(t)| = \left|\frac{t}{3} - \frac{t}{2}\right| \le \frac{\pi}{6} < C\epsilon,\ t \in [0, \pi],$$

and

$$|y_2(t) - x(t)| = 0,\ t \in [-2, 0].$$

Therefore

$$|y_2(t) - x(t)| < C\epsilon,\ t \in [-2, \pi].$$

(iii) Choose $\epsilon = 0.7$ and $y_3(t) = 0$ $t \in [-2, \pi]$. Then for $t \in [0, \pi]$, we have

$$\left| y_3'(t) - \left(0.506775 - \frac{2y_3(t)}{500} + \frac{\sin(y_3(g(t)))}{25} + \frac{\sin(2y_3(t))}{500} + \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2(y_3(s)) - \cos(y_3(g(s)))\right] ds \right.\right.$$
$$\left.\left. + \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2(y_3(s)) - \cos(y_3(g(s)))\right] ds \right) \right|$$
$$= \left| y_3'(t) - 0.506775 + \frac{2y_3(t)}{500} - \frac{\sin(y_3(g(t)))}{25} - \frac{\sin(2y_3(t))}{500} - \frac{1}{12.5}\int_0^t \frac{1}{20}\left[\sin^2(y_3(s)) - \cos(y_3(g(s)))\right] ds \right.$$
$$\left. - \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[\sin^2(y_3(s)) - \cos(y_3(g(s)))\right] ds \right|$$
$$= \left| -0.506775 + 0 - 0 - 0 - 0 - \frac{1}{20}\int_0^\pi \frac{1}{12.5}\left[-1\right] ds \right|$$
$$= 0.5193 < \epsilon.$$

For the solution $x(t)$ of (5.1)–(5.2) and constant $C = 1.3$ we have

$$|y_3(t) - x(t)| = \left|0 - \frac{t}{2}\right| \le \frac{\pi}{2} < C\epsilon,\ t \in [0, \pi],$$

and

$$|y_2(t) - x(t)| = 0,\ t \in [-2, 0].$$

Therefore

$$|y_2(t) - x(t)| < C\epsilon,\ t \in [-2, \pi].$$

It is observed that corresponding to given values of ϵ and given solutions $y(t)$ of the inequations, we are able to find the the exact solution $x(t)$ of equation (5.1) satisfying $|y(t) - x(t)| \le C\,\epsilon,\ t \in [-2, \pi]$. Hence (5.1) is Ulam–Hyres stable on $[0, \pi]$.

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