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Coupled fixed points of generalized Kanann contraction and its applications

Naser Ghafoori Adl[a], Davood Ebrahimi Bagha[b,*], Mohammad Sadegh Asgari[b]

[a]*Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*
[b]*Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University, Tehran, Iran*

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Abstract

The purpose of this paper is to find of the theoretical results of fixed point theorems for a mixed monotone mapping in a metric space endowed with partially order by using the generalized Kanann type contractivity of assumption. Also, as an application, we prove the existence and uniqueness of solution for a first–order ordinary differential equation with periodic boundary conditions admitting only the existence of a mixed $\leq$-solution.

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1. Introduction

Fixed point theory, the contraction mapping theorem and the abstract monotone iterative technique [7, 8] are a very useful tool in solving a variety of problems in control theory [3, 5], economic theory [2], nonlinear analysis [4, 6, 12] and global analysis [13, 14]. Recently, there is a trend to weaken the requirement on the contraction by considering metric spaces endowed with partial order. This approach was initiated in [11] where some applications to matrix equation are studied. This fixed point theorem was refined and extended in [9, 10] and was applied to the periodic boundary value problem considering both monotone cases.

In this paper, we extend the theoretical results of coupled fixed point for a mixed monotone mapping in partially ordered sets, using the generalized Kanann type contractivity and then apply

*Corresponding author
Email addresses: naser.ghafoori@gmail.com (Naser Ghafoori Adl), dav.ebrahimibagha@iauctb.ac.ir (Davood Ebrahimi Bagha), msasgari@yahoo.com (Mohammad Sadegh Asgari)

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them to study a problem of ordinary differential equations. We also will find results on existence of solution for the periodic boundary value problem (PBVP)

$$\begin{cases} u' = f(t,u) + g(t,u) \\ u(0) = u(T), \end{cases} \quad \forall t \in I = [0,T], \tag{1.1}$$

where $T > 0$ and $f, g \in C(I \times \mathbb{R}, \mathbb{R})$. Recall that a solution for (1.1) is a function $\alpha \in C^1(I, \mathbb{R})$ satisfying conditions in (1.1) and a lower solution for (1.1), is a function $\alpha \in C^1(I, \mathbb{R})$ such that

$$\alpha'(t) \le f(t, \alpha(t)) + g(t, \alpha(t)) \qquad \alpha(0) \le \alpha(T).$$

An upper solution for (1.1) satisfies the reversed inequalities. Also, an element $(\alpha, \beta) \in C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R})$ is called a mixed $\le$-solution of (1.1), if for every $t \in I$,

$$\begin{cases} \alpha'(t) \le f(t, \alpha(t)) + g(t, \beta(t)) & \alpha(0) \le \alpha(T), \\ \beta'(t) \ge f(t, \beta(t)) + g(t, \alpha(t)) & \beta(0) \ge \beta(T). \end{cases}$$

It is well know [8] that the existence of a lower solution α and an upper solution β with $\alpha \le \beta$ implies the existence of a solution of the periodic boundary value problem

$$\begin{cases} u' = f(t,u) \\ u(0) = u(T) \end{cases} \quad \forall t \in I = [0,T],$$

where $T > 0$ and $f \in C(I \times \mathbb{R}, \mathbb{R})$. In [1], Bhaskar and Lakshmikantham introduced the notions of mixed monotone property and coupled fixed point for the contractive mapping $f : X \times X \to X$, where X is a partially ordered metric space, and proved some coupled fixed point theorems with having a lower coupled fixed point. Then as an application of the coupled fixed point theorems, they determined the existence and uniqueness of the solution of (1.1). For the function f and under certain conditions, demonstrate that $x = y$. This implies the existence of a fixed point for the function f. This approach is also discussed by replacing the continuity assumption on the mapping f with an appropriate assumption on the metric space, as was done in [9, 10].

The structure of this paper is as follows: In the rest of this section we will review the necessary parts from a mixed monotone mapping in a metric space endowed with partially order. In Section 2, we prove the existence of a coupled fixed point for $f : X \times X \to X$ under the generalized Kanann type contractivity condition and establish the uniqueness under an additional assumption on the metric space. Further, we also establish that components of the coupled fixed point are equal. In Section 3, as an application of the theorems proved in Section 2, we discuss the existence of a unique solution of the periodic boundary value problem (1.1).

In the rest of this introduction we will briey recall the definitions and basic properties of coupled fixed point. For more information we refer to [1].

Definition 1.1. Let $(X, \le)$ be a partially ordered set and $f : X \times X \to X$ be a mapping. Then we say that f has the mixed monotone property if $f(x, y)$ is nondecreasing with respect to first component and is nonincreasing in second component that is

$$x_1 \le x_2 \Rightarrow f(x_1, y) \le f(x_2, y) \qquad \forall\, x_1, x_2, y \in X,$$
$$y_1 \le y_2 \Rightarrow f(x, y_1) \ge f(x, y_2) \qquad \forall\, y_1, y_2, x \in X.$$

Example 1.2. Let $X = \mathbb{R}$ and $\leq$ be the usual total order in $\mathbb{R}$. Then the mapping

$$\begin{aligned} &f : X \times X \to X \\ &f(x, y) = x^3 + x - y + 1 \quad \forall x, y \in X, \end{aligned}$$

has the mixed monotone property.

Definition 1.3. Let $(X, \leq)$ be a partially ordered set and $f : X \times X \to X$ be a mapping. Then:

(1) An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping f if

$$x = f(x, y) \qquad \text{and} \qquad y = f(y, x).$$

(2) An element $(x, y) \in X \times X$ is called a $\leq$-coupled fixed point of f if,

$$x \leq f(x, y) \quad \text{and} \quad y \geq f(y, x).$$

Example 1.4. Let $X = \mathbb{R}$ and $f : X \times X \to X$ defined by $f(x, y) = x^2 + x - y + 1$, for all $x, y \in X$. Then the elements $(0, -1), (-1, 0)$ are coupled fixed point of the mapping f.

Example 1.5. Suppose $X = [0, 1]$ and $\leq$ be the usual total order in $[0, 1]$. Suppose that $f : X \times X \to X$ defined by $f(x, y) = \frac{x+1}{y+5}$, for all $x, y \in X$. Then $(0, 1)$ is an $\leq$-coupled fixed point of f.

Definition 1.6. Let $(X, \leq)$ be a partially ordered set and let d be a complete metric on X. Then a mapping $f : X \times X \to X$ is called a generalized Kannan type mapping, if there exist real numbers $\alpha, \beta \in (0, \frac{1}{2})$ such that

$$d(f(x, y), f(u, v)) \leq \alpha d(x, f(x, y)) + \beta d(u, f(u, v)) \qquad \forall\, x \geq u,\ y \leq v.$$

Example 1.7. Let $X = [1, \frac{5}{4}]$ and $\leq$ be the usual total order in $[1, \frac{5}{4}]$ and $d(x, y) = |x - y|$. Now defined the map $f : X \times X \to X$ by $f(x, y) = \frac{1}{5}x - \frac{4}{5}y + 1$ for all $x, y \in X$. Suppose that $\alpha = \beta = \frac{1}{4}$, then for every $x \geq u$ and $y \leq v$ we have:

$$\begin{aligned} &\frac{1}{4}d\big(x, f(x, y)\big) + \frac{1}{4}d\big(u, f(u, v)\big) - d\big(f(x, y), f(u, v)\big) \\ &\qquad\qquad = y + \frac{2}{5}u - \frac{3}{5}v - \frac{1}{2} \geq \frac{3}{10} \geq 0. \end{aligned}$$

Therefore, f is a generalized Kannan type mapping with $\alpha = \beta = \frac{1}{4}$ and has the mixed monotone property.

2. Coupled fixed points of generalized Kanann type mapping

We start with the following theorem that proves the existence of coupled fixed point for a continuous generalized Kannan type mapping, with having an $\leq$-coupled fixed point.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and d be a complete metric on X. Suppose $f : X \times X \to X$ is a continuous mapping such that:*

(i) $f : X \times X \to X$ has the mixed monotone property on X.

(ii) (x_0, y_0) *is an* $\leq$*-coupled fixed point of* f.

(iii) *There exist* $\alpha, \beta \in (0, \frac{1}{2})$ *with*

$$d(f(x,y), f(u,v)) \leq \alpha d(x, f(x,y)) + \beta d(u, f(u,v)) \quad \forall\, x \geq u,\ y \leq v.$$

Then, there exists $(x,y) \in X \times X$ *such that* $f(x,y) = x$ *and* $f(y,x) = y$.

Proof . Define sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X, by $x_n = f(x_{n-1}, y_{n-1})$ and $y_n = f(y_{n-1}, x_{n-1})$. Let $(x_0, y_0) \in X \times X$ be an $\leq$-coupled fixed point of f. The mixed monotonicity of f implies that

$$x_1 = f(x_0, y_0) \leq f(x_1, y_1) = x_2, \quad \text{and} \quad y_1 = f(y_0, x_0) \geq f(y_1, x_1) = y_2.$$

Hence, induction gives

$$\begin{aligned} &x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots \\ &y_0 \geq y_1 \geq y_2 \geq \cdots \geq y_n \geq y_{n+1} \geq \cdots \end{aligned}$$

Now, we claim that for any $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) \leq (\frac{\beta}{1-\alpha})^n d(x_1, x_0) \tag{2.1}$$

$$d(y_{n+1}, y_n) \leq (\frac{\alpha}{1-\beta})^n d(y_1, y_0). \tag{2.2}$$

Indeed, for $n = 1$, using $x_1 \geq x_0$ and $y_1 \leq y_0$, we get

$$\begin{aligned} d(x_2, x_1) &= d(f(x_1, y_1), f(x_0, y_0)) \\ &\leq \alpha d(x_1, f(x_1, y_1)) + \beta d(x_0, f(x_0, y_0)) \\ &= \alpha d(x_2, x_1) + \beta d(x_1, x_0). \end{aligned}$$

Therefore $d(x_2, x_1) \leq \frac{\beta}{1-\alpha} d(x_1, x_0)$. Similarly, we have

$$\begin{aligned} d(y_2, y_1) &= d(f(y_0, x_0), f(y_1, x_1)) \\ &\leq \alpha d(y_0, f(y_0, x_0)) + \beta d(y_1, f(y_1, x_1)) \\ &= \beta d(y_2, y_1) + \alpha d(y_1, y_0). \end{aligned}$$

This implies that $d(y_2, y_1) \leq \frac{\alpha}{1-\beta} d(y_1, y_0)$. Now, assume that (2.1) and (2.2) hold. Using $x_{n+1} \geq x_n$ and $y_{n+1} \leq y_n$, we get

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &= d(f(x_{n+1}, y_{n+1}), f(x_n, y_n)) \\ &\leq \alpha d(x_{n+1}, f(x_{n+1}, y_{n+1})) + \beta d(x_n, f(x_n, y_n)) \\ &= \alpha d(x_{n+2}, x_{n+1}) + \beta d(x_{n+1}, x_n). \end{aligned}$$

Therefore

$$d(x_{n+2}, x_{n+1}) \leq \frac{\beta}{1-\alpha} d(x_{n+1}, x_n) \leq (\frac{\beta}{1-\alpha})^{n+1} d(x_1, x_0).$$

Similarly, we can show that $d(y_{n+2}, y_{n+1}) \leq (\frac{\alpha}{1-\beta})^{n+1} d(y_1, y_0)$. This implies that $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in X and for all $m \geq n$ we have

$$d(x_m, x_n) \leq (\frac{\beta}{1-\alpha})^n d(x_1, x_0)$$
$$d(y_m, y_n) \leq (\frac{\alpha}{1-\beta})^n d(y_1, y_0).$$

Since X is a complete metric space, there exist $x^*, y^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. Finally, we prove that $(x^*, y^*) \in X \times X$ is a coupled fixed point of f. Let $\varepsilon > 0$. Using the continuity of f at the point (x^*, y^*), given $\frac{\varepsilon}{2} > 0$, there exists a $\delta > 0$ such that $d(x^*, u) + d(y^*, v) < \delta$ implies that $d(f(x^*, y^*), f(u, v)) < \frac{\varepsilon}{2}$. Since $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$, for $\eta = \min(\frac{\varepsilon}{2}, \frac{\delta}{2}) > 0$, there exist $N \in \mathbb{N}$ such that, for $n \geq N$,

$$d(x_n, x^*) < \eta \quad \textit{and} \quad d(y_n, y^*) < \eta$$

Now, for $n \in \mathbb{N}, n \geq N$ we have

$$\begin{aligned} d(f(x^*, y^*), x^*) &\leq d\big(f(x^*, y^*), x_{n+1}\big) + d\big(x_{n+1}, x^*\big) \\ &= d\big(f(x^*, y^*), f(x_n, y_n)\big) + d\big(x_{n+1}, x^*\big) \\ &< \frac{\varepsilon}{2} + \eta \leq \varepsilon \end{aligned}$$

This implies that $f(x^*, y^*) = x^*$. Similarly, we can show that $f(y^*, x^*) = y^*$. $\square$

In the next theorem, the continuity hypothesis of f has been omitted.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and d be a complete metric on X. Suppose that X has following property:*

(1) *If a nondecreasing sequence $x_n \to x$, then $x_n \leq x$, $\forall n \in \mathbb{N}$.*

(2) *If a nonincreasing sequence $y_n \to y$, then $y_n \geq y$, $\forall n \in \mathbb{N}$.*

Assume that $f : X \times X \to X$ is a mapping such that:

(*i*) *$f : X \times X \to X$ has the mixed monotone property on X.*

(*ii*) *(x_0, y_0) is an $\leq$-coupled fixed point of f.*

(*iii*) *There exist $\alpha, \beta \in (0, \frac{1}{2})$ with*

$$d(f(x, y), f(u, v)) \leq \alpha d(x, f(x, y)) + \beta d(u, f(u, v)) \quad \forall\, x \geq u,\ y \leq v.$$

Then, there exists $(x^*, y^*) \in X \times X$ such that $f(x^*, y^*) = x^*$ and $f(y^*, x^*) = y^*$.

Proof . Define $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X, by $x_n = f(x_{n-1}, y_{n-1})$ and $y_n = f(y_{n-1}, x_{n-1})$. Similar to the proof of Theorem 2.1, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are nondecreasing and nonincreasing sequences respectively and $x_n \to x^*$ and $y_n \to y^*$ for some $x^*, y^* \in X$. Therefore, $x_n \leq x^*$ and $y_n \geq y^*$ for all $n \in \mathbb{N}$. We now show that (x^*, y^*) is a coupled fixed point of f. By (iii) we have

$$\begin{aligned} d(f(x^*,y^*),x^*) &\leq d(f(x^*,y^*),x_{n+1}) + d(x_{n+1},x^*) \\ &= d(f(x^*,y^*),f(x_n,y_n)) + d(x_{n+1},x^*) \\ &\leq \alpha d(x^*,f(x^*,y^*)) + \beta d(x_n,f(x_n,y_n)) + d(x_{n+1},x^*) \\ &\leq \alpha d(x^*,f(x^*,y^*)) + \beta \times (\frac{\beta}{1-\alpha})^n d(x_1,x_0) + d(x_{n+1},x^*). \end{aligned}$$

Therefore

$$d(f(x^*,y^*),x^*) \leq (\frac{\beta}{1-\alpha})^{n+1} d(x_1,x_0) + \frac{1}{1-\alpha} d(x_{n+1},x^*).$$

Letting $n \to \infty$, then $d(f(x^*,y^*),x^*) = 0$, which implies that $f(x^*,y^*) = x^*$. Similarly, we can show that $f(y^*,x^*) = y^*$. $\square$

Let $(X, \leq)$ be a partially ordered set. In the rest of the paper, we endow the product space $X \times X$ with the following partial order:

$$\forall (x,y),(u,v) \in X \times X, \quad (x,y) \leq (u,v) \Leftrightarrow x \leq u, \quad y \geq v.$$

We can prove that the components of the coupled fixed points are equal, if the product space $X \times X$ endowed with the above partial order has the property which every pair of elements has either a lower bound or an upper bound. It is know that this condition is equivalent with for every $(x,y),(u,v) \in X \times X$, there exists a $(z,w) \in X \times X$ that is comparable to (x,y) and (u,v) ([9]).

Theorem 2.3. *In addition to the hypothesis of Theorem 2.1, assume that every pair of elements of $X \times X$ has an upper bound or a lower bound in $X \times X$. Then, there exists an $x \in X$ such that $f(x,x) = x$.*

Proof . The existence of coupled fixed point (x,y) proved in Theorem 2.1. We first show that if $(x^*,y^*) \in X \times X$ is another coupled fixed point of f, then $x = x^*, y = y^*$. To see this, if (x,y) comparable to (x^*,y^*) with respect to the order in $X \times X$, it is obvious, because in this case $x \geq x^*,\ y \leq y^*$. Hence with (iii) we see that

$$\begin{aligned} d(x,x^*) + d(y,y^*) &= d(f(x,y),f(x^*,y^*)) + d(f(y,x),f(y^*,x^*)) \\ &\leq \alpha d\big(x,f(x,y)\big) + \beta d\big(x^*,f(x^*,y^*)\big) \\ &\quad + \alpha d\big(y^*,f(y^*,x^*)\big) + \beta d\big(y,f(y,x)\big) \\ &= \alpha d(x,x) + \beta d(x^*,x^*) + \alpha d(y^*,y^*) + \beta d(y,y) = 0. \end{aligned}$$

So $x = x^*, y = y^*$. Finally, assume that (x,y) is not comparable to (x^*,y^*), then, there exists a $(z_1,z_2) \in X \times X$ that is comparable to (x,y) and (x^*,y^*). Thus, by the the mixed monotone property of f for any $n \in \mathbb{N}$, $(f^n(z_1,z_2), f^n(z_2,z_1))$ is comparable to $(f^n(x,y), f^n(y,x))$ and $(f^n(x^*,y^*), f^n(y^*,x^*))$. This follows that

$$\begin{aligned} d(x,x^*) + d(y,y^*) &= d\big(f^{n+1}(x,y),f^{n+1}(x^*,y^*)\big) + d\big(f^{n+1}(y,x),f^{n+1}(y^*,x^*)\big) \\ &\leq d(f^n(x,y),f^n(z_1,z_2)) + d(f^n(y,x),f^n(z_2,z_1)) \\ &\quad + d(f^n(z_1,z_2),f^n(x^*,y^*)) + d(f^n(z_2,z_1),f^n(y^*,x^*)) \\ &\leq \alpha(\frac{\beta}{1-\alpha})^n d(x,f(x,y)) + \beta(\frac{\beta}{1-\alpha})^n d(z_1,f(z_1,z_2)) \\ &\quad + \alpha(\frac{\beta}{1-\alpha})^n d(z_1,f(z_1,z_2)) + \beta(\frac{\beta}{1-\alpha})^n d(x^*,f(x^*,y^*)) \end{aligned}$$

letting $n \to \infty$ so that $d\big((x,y),(x^*,y^*)\big) = 0$, it follows that $x = x^*, y = y^*$. Now, we note that if (x,y) is a coupled fixed point of f, then (y,x) is also another coupled fixed point of f. Therefore $x = y$. □

In the next theorem, we give another condition that prove the components of the coupled fixed points in Theorem 2.1 are equal.

Theorem 2.4. *In addition to the hypothesis of Theorem 2.1, if the $\leq$-coupled fixed point of f are comparable. Then the components of the coupled fixed point are equal.*

Proof . Let (x_0,y_0) be the $\leq$-coupled fixed point of f such that $x_0 \leq y_0$. The mixed monotonicity of f implies that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Suppose that (x,y) is the coupled fixed point of f. Now, using the triangular inequality, (2.1) and (2.2), we get

$$\begin{aligned} d(x,y) &\leq d(x,x_{n+1}) + d(x_{n+1},y) \leq d(x,x_{n+1}) + d(x_{n+1},y_{n+1}) + d(y_{n+1},y) \\ &\leq d(x,x_{n+1}) + d(f(y_n,x_n),f(x_n,y_n)) + d(y_{n+1},y) \\ &\leq d(x,x_{n+1}) + \alpha d(y_n,f(y_n,x_n)) + \beta d(x_n,f(x_n,y_n)) + d(y_{n+1},y) \\ &= d(x,x_{n+1}) + \alpha d(y_{n+1},y_n) + \beta d(x_{n+1},x_n) + d(y_{n+1},y) \\ &\leq d(x,x_{n+1}) + \alpha(\frac{\alpha}{1-\beta})^n d(y_1,y_0) + \beta(\frac{\beta}{1-\alpha})^n d(x_1,x_0) + d(y_{n+1},y). \end{aligned}$$

Letting $n \to \infty$, since $x_n \to x$ and $y_n \to y$, we obtain $d(x,y) = 0$, that is $x = y$. Similarly, if $x_0 \geq y_0$, then it is possible to show $x_n \geq y_n$ for all n, and that $d(x,y) = 0$. □

3. An application to periodic boundary value problems

In this section, on the basis of the coupled fixed point theorems in Section 2, we study the existence of a unique solution of a periodic boundary value problem.

$$\begin{cases} u' = f(t,u) + g(t,u) \\ u(0) = u(T), \end{cases} \qquad \forall t \in I = [0,T],$$

where $T > 0$ and $f,g \in C(I \times \mathbb{R}, \mathbb{R})$. As a first step we obtain a solution of the following boundary value equations system:

$$\begin{cases} u' = f(t,u) + g(t,v) \\ v' = f(t,v) + g(t,u) \\ u(0) = u(T), \quad v(0) = v(T). \end{cases} \qquad \forall t \in I = [0,T], \tag{3.1}$$

This problem is equivalent to the following integral equations system:

$$\begin{cases} u(t) = u(T) + \int_0^t \big(f(s,u(s)) + g(s,v(s))\big)ds \\ v(t) = v(T) + \int_0^t \big(f(s,v(s)) + g(s,u(s))\big)ds \end{cases} \qquad \forall t \in I = [0,T].$$

Let $X = C(I,\mathbb{R})$ be the metric space of all continuous functions $u : I \to \mathbb{R}$, endowed with the metric

$$d(u,v) = \sup_{t \in I} |u(t) - v(t)| \qquad \forall u,v \in X.$$

Also, if we define the order relation in X as follows, then X is a partially ordered set:

$$u \leq v \Leftrightarrow u(t) \leq v(t) \qquad \forall t \in I.$$

Suppose that we consider a monotone nondecreasing sequence $\{u_n\}_{n=1}^{\infty}$ in X converging to $u \in X$. Then, for every $t \in I$, the sequence of real numbers

$$u_1(t) \leq u_2(t) \leq \cdots \leq u_n(t) \leq \cdots,$$

converges to $u(t)$. Therefore, for all $t \in I$, $n \in \mathbb{N}$, $u_n(t) \leq u(t)$. Hence $u_n \leq u$, for all $n \in \mathbb{N}$. Similarly, we can verify that the limit $v(t)$ of a monotone nonincreasing sequence $\{v_n\}_{n=1}^{\infty}$ in X is a lower bound for all the elements in the sequence. Moreover, for any $f, g \in X$, $M(t) = \max\{f(t), g(t)\}$ and $m(t) = \min\{f(t), g(t)\}$, for all $t \in I$, are in X and are the upper and lower bounds of f, g, respectively. Also, $(X \times X, d)$ is a complete metric space if we define

$$d((u_1, v_1), (u_2, v_2)) = \sup_{t \in I} |u_1(t) - u_2(t)| + \sup_{t \in I} |v_1(t) - v_2(t)|.$$

Also, $X \times X = C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a partially ordered set if we define the order relation in X as follows:

$$(u_1, v_1) \leq (u_2, v_2) \Longleftrightarrow u_1(t) \leq u_2(t) \quad and \quad v_1(t) \geq v_2(t) \quad \forall t \in I.$$

Define, for $t \in I$,

$$F(u, v)(t) = u(T) + \int_0^t \big(f(s, u(s)) + g(s, v(s))\big) ds.$$

Note that if $(u, v) \in X \times X$ is a coupled fixed point of F, we have

$$u(t) = F(u, v)(t) \;\; and \;\; v(t) = F(v, u)(t) \quad \forall t \in I$$

and it is a solution of (3.1) satisfying the boundary conditions $u(0) = u(T)$ and $v(0) = v(T)$.

Theorem 3.1. *Suppose $f, g \in C(I \times \mathbb{R}, \mathbb{R})$ such that f is nondecreasing in its second argument and g is nonincreasing in its second argument. Suppose that there exist mixed $\leq$-solution (u_0, v_0) of (1.1). Then the boundary value equations system (3.1) having a unique solution in $C(I, \mathbb{R})$.*

Proof . We show that the operator $F : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to C(I, \mathbb{R})$ has a unique coupled fixed point in $C(I, \mathbb{R}) \times C(I, \mathbb{R})$. To achieve this, we verify that F satisfies the hypotheses of Theorem 2.2. For any $u_1 \geq u_2$ we have

$$\begin{aligned} F(u_1, v)(t) &= u_1(T) + \int_0^t \big(f(s, u_1(s)) + g(s, v(s))\big) ds \\ &\geq u_2(T) + \int_0^t \big(f(s, u_2(s)) + g(s, v(s))\big) ds = F(u_2, v)(t). \end{aligned}$$

Also, if $v_1 \leq v_2$, then

$$\begin{aligned} F(u, v_1)(t) &= u(T) + \int_0^t \big(f(s, u(s)) + g(s, v_1(s))\big) ds \\ &\geq u(T) + \int_0^t \big(f(s, u(s)) + g(s, v_2(s))\big) ds = F(u, v_2)(t). \end{aligned}$$

Thus, $F(u,v)$ has the mixed monotone property on $C(I,\mathbb{R})$. In addition, for $(x,y) \le (u,v)$, we obtain

$$\begin{aligned}
d(F(u,v),F(x,y)) &= \sup_{t\in I} \big|F(u,v)(t) - F(x,y)(t)\big| \\
&= \sup_{t\in I} \big|u(T) + \int_0^t \big(f(s,u(s)) + g(s,v(s))\big)ds \\
&\quad - x(T) - \int_0^t \big(f(s,x(s)) + g(s,y(s))\big)ds\big| \\
&\le \frac{1}{3}\sup_{t\in I} \big|u(T) - u(t) + \int_0^t \big(f(s,u(s)) + g(s,v(s))\big)ds\big| \\
&\quad + \frac{1}{3}\sup_{t\in I} \big|x(T) - x(t) + \int_0^t \big(f(s,x(s)) + g(s,y(s))\big)ds\big| \\
&= \frac{1}{3}d(u,F(u,v)) + \frac{1}{3}d(x,F(x,y)).
\end{aligned}$$

So F is a generalized Kannan type mapping with $\alpha = \beta = \frac{1}{3}$. Finally, let $(u_0,v_0) \in C^1(I,\mathbb{R})\times C^1(I,\mathbb{R})$ be a mixed $\le$-solution of (1.1). We have

$$\begin{cases} u_0'(t) \le f(t,u_0(t)) + g(t,v_0(t)) & u_0(0) \le u_0(T), \\ v_0'(t) \ge f(t,v_0(t)) + g(t,u_0(t)) & v_0(0) \ge v_0(T). \end{cases}$$

By integration on $[0,t]$, we get

$$\begin{cases} u_0(t) \le u_0(T) + \int_0^t [f(s,u_0(s)) + g(s,v_0(s))]ds, \\ v_0(t) \ge v_0(T) + \int_0^t [f(s,v_0(s)) + g(s,u_0(s))]ds, \end{cases}$$

which implies that

$$u_0 \le F(u_0,v_0), \qquad v_0 \ge F(v_0,u_0).$$

Therefore by Theorem 2.2, F has a coupled fixed point in $C(I,\mathbb{R}) \times C(I,\mathbb{R})$ which is a solution of the boundary value equations system (3.1). □

Remark 3.2. Since the hypothesis of Theorem 2.4 is satisfied, therefore the components of the coupled fixed point (u,v) in Theorem 3.1 are equal. That is, $u(t) = v(t)$, which this implies that $u = F(u,u)$ and thus $u(t)$ is the unique solution of

$$\begin{cases} u'(t) = f(t,u(t)) + g(t,u(t)) \\ u(0) = u(T), \end{cases} \qquad \forall t \in I = [0,T].$$

This establishes that the PBVP (1.1) has an unique solution on I.

4. Conclusion

In this paper, we found several theoretical results of fixed point theorems for the generalized Kanann type contraction mappings in partially ordered metric space. Also, as an application, we proved the existence and uniqueness of solution for a first-order ordinary differential equation with periodic boundary conditions admitting only the existence of a mixed $\le$-solution.

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References

[1] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. 65 (2006) 1379–1393.

[2] K.C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, Cambridge, 1989.

[3] B.L. Buzbee, G.H. Golub and C.W. Nielson, *On direct methods for solving Poisson's equations*, SIAM J. Numer. Anal. 7 (1970) 627–656.

[4] B.S. Choudhury, *A common unique fixed point result in metric spaces involving generalized altering distances*, Math. Commun. 10 (2005) 105–110.

[5] W.L. Green and E.W. Kamen, *Stabilization of linear systems over a commutative normed algebra with applications to spatially distributed parameter dependent systems*, SIAM J. Control Optim. 23 (1985) 1–18.

[6] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Vol. 5. Academic Press, New York, 2014.

[7] S. Heikkila and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Vol. 181. CRC Press, 1994.

[8] G.S. Ladde, V. Lakshmikantham and A. S. Vetsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Vol. 27. Pitman Publishing, 1985.

[9] J.J. Nieto and R. Rodríguez–López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order 22 (2005) 223–239.

[10] J.J. Nieto and R. Rodríguez–López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin. (Engl. Ser.) 23 (2007) 2205–2212.

[11] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. 132 (2004) 1435–1443.

[12] Y. Sun, *A fixed point theorem for mixed monotone operator with applications*, J. Math. Anal. Appl. 156 (1991) 240–252.

[13] X. Zhang, *Common fixed point theorems for some new generalized contractive type mappings*, J. Math. Anal. Appl. 333 (2007) 780–786.

[14] Z. Zhiato, *New fixed point theorems of mixed montone operator and applications*, J. Math. Anal. Appl. 204 (1996) 307–319.