Int. J. Nonlinear Anal. Appl. 9 (2018) No. 2, 179-190
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2018.12908.1665

Fixed point theorems for weak contractions in dualistic partial metric spaces

Muhammad Nazam[a,b,*], Muhammad Arshad[b]

[a]Department of Mathematics, Allama Iqbal Open University Islamabad, Pakistan
[b]Department of Mathematics and Statistics, International Islamic University, Islamabad Pakistan

(Communicated by S. Saeidi)

Abstract

In this paper, we describe some topological properties of dualistic partial metric spaces and establish some fixed point theorems for weak contraction mappings of rational type defined on dualistic partial metric spaces. These results are generalizations of some existing results in the literature. Moreover, we present examples to illustrate our result.

Keywords: Fixed point, Dualistic partial metric, Weak contractions.
2010 MSC: Primary 47H10; Secondary 54H25.

1. Introduction

Let M be a metric space. A map $T : M \to M$ is a contraction if for each $x, y \in M$, there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y).$$

A map $T : M \to M$ is a φ-weak contraction if for each $x, y \in M$, there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$, and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)).$$

The concept of the weak contraction was defined by Alber and Guerre–Delabriere [3] in 1997. Actually in [3], the authors defined such mappings for single–valued maps on Hilbert spaces and proved the existence of fixed points. Rhoades [12], in 2001 showed that most results of [3] are still true for any

*Corresponding author
Email addresses: nazim254.butt@gmail.com (Muhammad Nazam), marshadzia@iiu.edu.pk (Muhammad Arshad)

Received: October 2017 *Revised:* July 2018

Banach space. Also Rhoades [12] proved the following very interesting fixed point theorem which is one of generalizations of the Banach contraction principle because it contains contractions as special cases ($\varphi(t) = (1-k)t$).

Theorem 1.1. (Rhoades, [12, Theorem 2]) Let (E, d) be a complete metric space and A be a φ–weak contraction on E. If $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function with $\varphi(t) > 0$ for all $t \in (0,\infty)$ and $\varphi(0) = 0$, then A has a unique fixed point.

Karapinar *et al.* [6] in 2013, proved the following result.

Theorem 1.2. Let (X, p) be a complete metric space and $T : X \to X$ be a mapping satisfying

$$\varphi(p(T(x), T(y))) \leq \varphi(M(x, y)) - \psi(M(x, y)) \text{ for all } x, y \in X,$$

where

$$M(x, y) = \max\left\{p(x, y), \frac{p(y, T(y))(1 + p(x, T(x)))}{1 + p(x, y)}\right\},$$

$\varphi : [0,\infty) \to [0,\infty)$ is a continuous and monotone non–decreasing function with $\varphi(t) = 0$ if and only if $t = 0$ and $\psi : [0,\infty) \to [0,\infty)$ is a lower semi–continuous function with $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

In this paper, we shall establish aforementioned theorem in the context of dualistic partial metric spaces. We shall show, with the help of examples, that the new result allows us to find fixed points of the self–mappings in some cases in which the results in partial metric spaces cannot be applied.

2. Dualistic Partial Metric

Throughout this paper the letters $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, set of real numbers and set of natural numbers, respectively.

Definition 2.1. (Consistent Semilattice) Let $(X, \preceq)$ be a poset such that

(1) for all $x, y \in X$ $x \wedge y \in X$,

(2) if $\{x, y\} \subseteq X$ is consistent, then $x \vee y \in X$,

then $(X, \preceq)$ with (1) and (2) is called consistent semilattice.

Definition 2.2. (Valuation Space) A valuation space is a consistent semilattice $(X, \preceq)$ and a function $\mu : X \to \mathbb{R}$, called valuation, such that

(1) if $x \preceq y$ and $x \neq y$, $\mu(x) < \mu(y)$ and

(2) if $\{x, y\} \subseteq X$ is consistent, then

$$\mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y).$$

Matthews generalized the notion of metric, as follows:

Definition 2.3. (Matthews, [7]) Let X be a non-empty set. If the function $p : X \times X \to \mathbb{R}_0^+$ satisfies following properties, for all $x, y, z \in X$

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(p_2) $p(x,x) \le p(x,y)$;
(p_3) $p(x,y) = p(y,x)$;
(p_4) $p(x,z) + p(y,y) \le p(x,y) + p(y,z)$.

Then p is a partial metric.

Definition 2.4. (O'Neill, [9]) Let X be a non-empty set. The function $p^* : X \times X \to \mathbb{R}$ complying with following conditions

(p_1^*) $x = y \Leftrightarrow p^*(x,x) = p^*(x,y) = p^*(y,y)$;
(p_2^*) $p^*(x,x) \le p^*(x,y)$;
(p_3^*) $p^*(x,y) = p^*(y,x)$;
(p_4^*) $p^*(x,z) + p^*(y,y) \le p^*(x,y) + p^*(y,z)$;

is called dualistic partial metric on X and the pair (X, p^*) is known as dualistic partial metric space.

Example 2.5. Let p be a partial metric defined on a non empty set X. The function $p^* : X \times X \to \mathbb{R}$ defined by

$$p^*(x,y) = p(x,y) - p(x,x) - p(y,y) \text{ for all } x, y \in X.$$

It is easy to see that p^* satisfies conditions $(p_1^*) - (p_4^*)$ and hence defines a dualistic partial metric on X. We note that $p^*(x,y)$ may have negative value.

Remark 2.6. We observe that, unlike partial metric case, if p^* is a dualistic partial metric then $p^*(x,y) = 0$ does not imply $x = y$. The self distance $p^*(x,x)$ referred to as the size or weight of x, is a feature used to describe the amount of information contained in x. The smaller $p^*(x,x)$ the more defined x is, x being totally defined if $p^*(x,x) = 0$. The restriction of p^* to $\mathbb{R}_0^+$, is a partial metric.

Example 2.7. *Suppose that* $(\mathbb{R}, \preceq, \mu)$ *is a valuation space, then* $p^*(x,y) = \mu(x \vee y)$ *defines a dualistic partial metric on* $\mathbb{R}$.

Proof . Axioms (p_2^*) and (p_3^*) are immediate. For (p_1^*) we proceed as

$$\text{if } p^*(x,x) = p^*(x,y) = p^*(y,y), \text{ then } \mu(x \vee y) = \mu(x) = \mu(y) \text{ implies } x = y.$$

Converse is obvious. We prove (p_4^*) in detail

$$\begin{aligned} p^*(x,z) + p^*(y,y) &= \mu(x \vee z) + \mu(y) \\ &\le \mu(x \vee y \vee z) + \mu[(x \vee y) \wedge (y \vee z)] \\ &= \mu(x \vee y \vee z) + \mu(x \vee y) + \mu(y \vee z) - \mu(x \vee y \vee z) \\ &= \mu(x \vee y) + \mu(y \vee z) = p^*(x,y) + p^*(y,z). \end{aligned}$$

□

For a dualistic partial metric space (X, p^*), we immediately have a natural definition (although slightly different from the one given in [7]) for the open balls:

$$B_\epsilon^*(x; p^*) = \{y \in X | p^*(x,y) < p^*(x,x) + \epsilon\} \text{ for all } x \in X, \epsilon > 0.$$

Unlike their metric counterpart, some dualistic partial metric open balls may be empty. For example if $p^*(x,x) \neq 0$, then

$$\begin{aligned} B_{p^*(x,x)}^*(x; p^*) &= \{y \in X | p^*(x,y) < 2p^*(x,x)\} \\ &= \{y \in X | p(x,y) - p(x,x) - p(y,y) < -2p(x,x)\} \\ &= \{y \in X | p(x,y) + p(x,x) < p(y,y)\} = \Phi. \end{aligned}$$

Proposition 2.8. *The set $\{B^*_\epsilon(x;p^*);$ for all $x \in X$, $\epsilon > 0\}$ of open balls forms the basis for dualistic partial metric topology denoted by $\mathcal{T}[p^*]$.*

Proof . It is obvious that

$$X = \cup_{x\in X} B^*_\epsilon(x;p^*)$$

and for any two open balls $B^*_\epsilon(x;p^*)$, $B^*_\delta(y;p^*)$ we note that

$$B^*_\epsilon(x;p^*) \cap B^*_\delta(y;p^*) = \cup\left\{B^*_\kappa(c;p^*)|\, c \in B^*_\epsilon(x;p^*) \cap B^*_\delta(y;p^*)\right\}$$

where

$$\kappa = p^*(c,c) + \min\left\{\epsilon - p^*(x,c), \delta - p^*(y,c)\right\}.$$

□

Proposition 2.9. *Each dualistic partial metric topology is T_0 topology.*

Proof . Suppose $p^* : X \times X \to \mathbb{R}$ is a dualistic partial metric and $x \neq y$. Then with out loss of generality, we have $p^*(x,x) < p^*(x,y)$ for all $x, y \in X$. Choose $\epsilon = p^*(x,y) - p^*(x,x)$, since

$$p^*(x,x) < p^*(x,x) + \epsilon = p^*(x,y),$$

so $x \in B^*_\epsilon(x;p^*)$ and $y \notin B^*_\epsilon(x;p^*)$ because otherwise we obtain an absurdity $(p^*(x,y) < p^*(x,y))$. □

Proposition 2.10. *Every open ball in a dualistic partial metric space is an open set.*

Proof . Let (X,p^*) be a dualistic partial metric space and $B^*_\epsilon(v;p^*)$ be an open ball, centered at v, of radius $\epsilon > 0$. We show that for $x \neq v$,

$$x \in B^*_\delta(x;p^*) \subseteq B^*_\epsilon(v;p^*).$$

Suppose that $x \in B^*_\epsilon(v;p^*)$ then using (p^*_1) and (p^*_2), we have

$$p^*(x,x) < p^*(x,v) < p^*(v,v) + \epsilon. \tag{2.1}$$

Take $\delta = \epsilon + p^*(v,v) - p^*(x,x)$, (2.1) implies $p^*(x,x) < p^*(x,x) + \delta$. Thus, $x \in B^*_\delta(x;p^*)$. Next we show that

$$B^*_\delta(x;p^*) \subseteq B^*_\epsilon(v;p^*).$$

Suppose that $y \in B^*_\delta(x;p^*)$, then

$$p^*(x,y) < p^*(x,x) + \delta;$$
$$p^*(x,y) < p^*(x,x) + \epsilon + p^*(v,v) - p^*(x,x) = \epsilon + p^*(v,v),$$

which implies that $y \in B^*_\epsilon(v;p^*)$. □

If (X,p^*) is a dualistic partial metric space, then the function $d_{p^*} : X \times X \to \mathbb{R}^+_0$ defined by

$$d_{p^*}(x,y) = p^*(x,y) - p^*(x,x),$$

is a quasi metric on X such that $\mathcal{T}[p^*] = \mathcal{T}[d_{p^*}]$ where $B_\epsilon(x;d_{p^*}) = \{y \in X | d_{p^*}(x,y) < \epsilon\}$. In this case, $d^s_{p^*}(x,y) = \max\{d_{p^*}(x,y), d_{p^*}(y,x)\}$ defines a metric on X, known as induced metric.

The following definition and Lemma describe the convergence criteria established by Oltra and Valero in [8].

Definition 2.11. (Oltra and Valero, [8]) Let (X, p^*) be a dualistic partial metric space.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, p^*) is called a Cauchy sequence if $\lim_{n,m\to\infty} p^*(x_n, x_m)$ exists and is finite.

(2) A dualistic partial metric space (X, p^*) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}[p^*]$, to a point $\upsilon \in X$ such that

$$p^*(x, x) = \lim_{n,m\to\infty} p^*(x_n, x_m).$$

Lemma 2.12. (Oltra and Valero, [8])

(1) Every Cauchy sequence in $(X, d^s_{p^*})$ is also a Cauchy sequence in (X, p^*).

(2) A dualistic partial metric (X, p^*) is complete if and only if the metric space $(X, d^s_{p^*})$ is complete.

(3) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $\upsilon \in X$ with respect to $\mathcal{T}[(d^s_{p^*})]$ if and only if

$$\lim_{n\to\infty} p^*(\upsilon, x_n) = p^*(\upsilon, \upsilon) = \lim_{n\to\infty} p^*(x_n, x_m).$$

3. Fixed Point Theorem

Let

$$\mathcal{M}(x, y) = \max\left\{|p^*(x, y)|, \left|\frac{p^*(y, T(y))(1 + p^*(x, T(x))}{1 + p^*(x, y)}\right|\right\}.$$

Theorem 3.1. *Let (X, p^*) be a complete dualistic partial metric space and $T : X \to X$ be a self–mapping satisfying*

$$\varphi(|p^*(T(x), T(y))|) \leq \varphi(\mathcal{M}(x, y)) - \psi(\mathcal{M}(x, y)) \text{ for all } x, y \in X. \tag{3.1}$$

If $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and monotone non–decreasing function with $\varphi(t) = 0$ if and only if $t = 0$ and $\psi : [0, \infty) \to [0, \infty)$ is a lower semi–continuous function with $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof . Let x_0 be an initial point of X and let us define Picard iterative sequence $\{x_n\}$ by

$$x_n = T(x_{n-1}) \text{ for all } n \in \mathbb{N}.$$

If there exists a positive integer i such that $x_i = x_{i+1}$, then $x_i = x_{i+1} = T(x_i)$, so x_i is a fixed point of T. In this case proof is complete. On the other hand if $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then by contractive condition (3.1), we have for $x_n, x_{n+1} \in X$

$$\varphi(|p^*(x_n, x_{n+1})|) \leq \varphi(\mathcal{M}(x_{n-1}, x_n)) - \psi(\mathcal{M}(x_{n-1}, x_n)), \tag{3.2}$$

where

$$\begin{aligned}\mathcal{M}(x_{n-1}, x_n) &= \max\left\{|p^*(x_{n-1}, x_n)|, \left|\frac{p^*(x_n, x_{n+1})(1 + p^*(x_{n-1}, x_n)}{1 + p^*(x_{n-1}, x_n)}\right|\right\} \\ &= \max\{|p^*(x_{n-1}, x_n)|, |p^*(x_n, x_{n+1})|\}.\end{aligned}$$

If $\mathcal{M}(x_{n-1}, x_n) = |p^*(x_n, x_{n+1})|$ then inequality (3.2) implies,

$$\begin{aligned}\varphi(|p^*(x_n, x_{n+1})|) &\leq \varphi(|p^*(x_n, x_{n+1})|) - \psi(|p^*(x_n, x_{n+1})|)\\ &< \varphi(|p^*(x_n, x_{n+1})|),\end{aligned}$$

which is a contradiction to $|p^*(x_n, x_{n+1})| > 0$. Hence $\mathcal{M}(x_{n-1}, x_n) = |p^*(x_{n-1}, x_n)|$. Thus,

$$\varphi(|p^*(x_n, x_{n+1})|) \quad < \quad \varphi(|p^*(x_{n-1}, x_n)|) \text{ entails } |p^*(x_n, x_{n+1})| \leq |p^*(x_{n-1}, x_n)|.$$

This shows that $\{|p^*(x_n, x_{n+1})|\}_{n\in\mathbb{N}}$ is a non increasing sequence of positive real numbers. There exists a number $\mathcal{L} \geq 0$ such that $\lim_{n\to\infty} |p^*(x_n, x_{n+1})| = \mathcal{L}$. We claim that $\mathcal{L} = 0$. On contrary suppose that $\mathcal{L} > 0$ and taking the upper limit of

$$\varphi(|p^*(x_n, x_{n+1})|) \leq \varphi(|p^*(x_{n-1}, x_n)|) - \psi(|p^*(x_{n-1}, x_n)|),$$

we get

$$\begin{aligned}\varphi(\mathcal{L}) &\leq \varphi(\mathcal{L}) - \lim_{n\to\infty} \inf \psi(|p^*(x_{n-1}, x_n)|);\\ \varphi(\mathcal{L}) &\leq \varphi(\mathcal{L}) - \psi(\mathcal{L}) < \varphi(\mathcal{L}),\end{aligned}$$

which is a contradiction, so $\mathcal{L} = 0$. Hence

$$\lim_{n\to\infty} |p^*(x_n, x_{n+1})| = 0 \text{ and then } \lim_{n\to\infty} p^*(x_n, x_{n+1}) = 0. \tag{3.3}$$

We use (3.1) to find the self distance $p^*(x_n, x_n)$, as follows:

$$\varphi(|p^*(x_n, x_n)|) \leq \varphi(\mathcal{M}(x_{n-1}, x_{n-1})) - \psi(\mathcal{M}(x_{n-1}, x_{n-1})), \tag{3.4}$$

where

$$\mathcal{M}(x_{n-1}, x_{n-1})) = \max\left\{|p^*(x_{n-1}, x_{n-1})|, \left|\frac{p^*(x_n, x_{n-1})(1 + p^*(x_{n-1}, x_n)}{1 + p^*(x_{n-1}, x_{n-1})}\right|\right\}.$$

If

$$\mathcal{M}(x_{n-1}, x_{n-1}) = \left|\frac{p^*(x_n, x_{n-1})(1 + p^*(x_{n-1}, x_n)}{1 + p^*(x_{n-1}, x_{n-1})}\right|,$$

then taking the upper limit $\lim_{n\to\infty}$ on (3.4) and using (3.3), we obtain

$$\lim_{n\to\infty} \varphi(|p^*(x_n, x_n)|) \leq 0 \Rightarrow \lim_{n\to\infty} \varphi(|p^*(x_n, x_n)|) = 0.$$

The continuity of φ implies $\lim_{n\to\infty} |p^*(x_n, x_n)| = 0$. Similarly if $\mathcal{M}(x_{n-1}, x_{n-1}) = |p^*(x_{n-1}, x_{n-1})|$ then

$$\varphi(|p^*(x_n, x_n)|) \leq \varphi(|p^*(x_{n-1}, x_{n-1})|) - \psi(|p^*(x_{n-1}, x_{n-1})|);$$

$$\varphi(|p^*(x_n, x_n)|) \quad < \quad \varphi(|p^*(x_{n-1}, x_{n-1})|) \text{ implies } |p^*(x_n, x_n)| \leq |p^*(x_{n-1}, x_{n-1})|.$$

Thus, $\{|p^*(x_n, x_n)|\}_{n\in\mathbb{N}}$ is a non increasing sequence of positive real numbers and continuing as in case of (3.3) we get,

$$\lim_{n\to\infty} p^*(x_n, x_n) = 0. \tag{3.5}$$

Since $d_{p^*}(x_n, x_{n+1}) = p^*(x_n, x_{n+1}) - p^*(x_n, x_n)$, so using (3.5) we get

$$\lim_{n\to\infty} d_{p^*}(x_n, x_{n+1}) = 0. \tag{3.6}$$

Now we show that $\{x_n\}$ is a Cauchy sequence in $(X, d^s_{p^*})$. For this we have to show that

$$\lim_{n,m\to\infty} d^s_{p^*}(x_n, x_m) = 0.$$

That is

$$\lim_{n,m\to\infty} d_{p^*}(x_n, x_m) = 0 = \lim_{n,m\to\infty} d_{p^*}(x_m, x_n).$$

Suppose on contrary that

$$\lim_{n,m\to\infty} d_{p^*}(x_n, x_m) \neq 0.$$

Then there exists $\epsilon > 0$ for which we can find two sub sequences $\{x_{n_k}\}, \{x_{m_k}\}$ of $\{x_n\}$ such that n_k is smallest index for which

$$\text{for all } n_k > m_k \ \ d_{p^*}(x_{n_k}, x_{m_k}) \geq \epsilon. \tag{3.7}$$

It follows directly that

$$d_{p^*}(x_{n_k-1}, x_{m_k}) < \epsilon. \tag{3.8}$$

By (3.7) and (3.8) we have

$$\begin{aligned}\epsilon \leq d_{p^*}(x_{n_k}, x_{m_k}) &\leq d_{p^*}(x_{n_k}, x_{n_k-1}) + d_{p^*}(x_{n_k-1}, x_{m_k})\\ &< d_{p^*}(x_{n_k}, x_{n_k-1}) + \epsilon.\end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides in above inequality and from (3.6), we obtain

$$\lim_{k\to\infty} d_{p^*}(x_{n_k}, x_{m_k}) = \epsilon. \tag{3.9}$$

The triangular inequality gives

$$\begin{aligned}d_{p^*}(x_{n_k-1}, x_{m_k-1}) &\leq d_{p^*}(x_{n_k-1}, x_{n_k}) + d_{p^*}(x_{n_k}, x_{m_k-1})\\ &\leq d_{p^*}(x_{n_k-1}, x_{n_k}) + d_{p^*}(x_{n_k}, x_{m_k}) + d_{p^*}(x_{m_k}, x_{m_k-1}).\end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides in above inequality and from (3.6), (3.9), we obtain

$$\lim_{k\to\infty} d_{p^*}(x_{n_k-1}, x_{m_k-1}) = \epsilon. \tag{3.10}$$

By (3.1) for $x_{n_k} \neq x_{m_k}$,

$$\varphi(|p^*(x_{n_k}, x_{m_k})|) \leq \varphi(\mathcal{M}(x_{n_k-1}, x_{m_k-1})) - \psi(\mathcal{M}(x_{n_k-1}, x_{m_k-1})), \tag{3.11}$$

where

$$\mathcal{M}(x_{n_k-1}, x_{m_k-1}) = \max\left\{|p^*(x_{n_k-1}, x_{m_k-1})|, \left|\frac{p^*(x_{m_k-1}, x_{m_k})(1 + p^*(x_{n_k-1}, x_{n_k}))}{1 + p^*(x_{n_k-1}, x_{n_k-1})}\right|\right\}.$$

By (3.5), (3.9) and (3.10) we have

$$\lim_{k\to\infty} \mathcal{M}(x_{n_k-1}, x_{m_k-1}) = \epsilon. \tag{3.12}$$

Now applying $\lim_{k\to\infty}$ in (3.11) along with properties of φ, ψ and (3.12) we get

$$\varphi(\epsilon) \leq \varphi(\epsilon) - \lim_{k\to\infty} \inf \psi(\mathcal{M}(x_{n_k-1}, x_{m_k-1})).$$

That is $\varphi(\epsilon) < \varphi(\epsilon)$, a contradiction. Therefore $\lim_{n,m\to\infty} d_{p^*}(x_n, x_m) = 0$. Similarly we can prove that $\lim_{n,m\to\infty} d_{p^*}(x_m, x_n) = 0$. Hence $\lim_{n,m\to\infty} d^s_{p^*}(x_n, x_m) = 0$ which ensures that $\{x_n\}$ is a Cauchy sequence in $(X, d^s_{p^*})$. Since $(X, d^s_{p^*})$ is a complete metric space, therefore, $\{x_n\}$ converges to a point (say) υ with respect to $\tau(d^s_{p^*})$ in X. By Lemma 2.12 we have

$$\lim_{n\to\infty} d^s_{p^*}(x_n, \upsilon) = 0 \iff \lim_{n\to\infty} p^*(\upsilon, x_n) = p^*(\upsilon, \upsilon) = \lim_{n,m\to\infty} p^*(x_n, x_m). \tag{3.13}$$

Since,

$$\lim_{n,m\to\infty} d_{p^*}(x_n, x_m) = 0 \text{ implies } \lim_{n,m\to\infty} p^*(x_n, x_m) = 0,$$

by (3.13)

$$p^*(\upsilon, \upsilon) = 0 = \lim_{n\to\infty} p^*(\upsilon, x_n).$$

Now we prove that υ is fixed point of T. On contrary suppose that $\upsilon \neq T(\upsilon)$, then using (3.1) and Lemma 2.12 we have

$$\varphi(|p^*(x_n, T(\upsilon)|) \leq \varphi(\mathcal{M}(x_{n-1}, \upsilon)) - \psi(\mathcal{M}(x_{n-1}, \upsilon)).$$

Letting $n \to \infty$ and using properties of φ, ψ we get $\varphi(p^*(\upsilon, T(\upsilon))) < \varphi(p^*(\upsilon, T(\upsilon)))$, a contradiction as $p^*(\upsilon, T(\upsilon)) \geq 0$. Hence $\upsilon = T(\upsilon)$ which shows υ is a fixed point of T. Finally, we shall prove the uniqueness. Suppose that ω is another fixed point of T such that $\upsilon \neq \omega$ then from (3.1),

$$\varphi(|p^*(\upsilon, \omega)|) \leq \varphi(\mathcal{M}(\upsilon, \omega)) - \psi(\mathcal{M}(\upsilon, \omega))$$

implies that

$$\varphi(|p^*(\upsilon, \omega)|) < \varphi(|p^*(\upsilon, \omega)|),$$

a contradiction, hence $\upsilon = \omega$ which completes the proof. □

To explain our result, in the following example, we show that if the given mapping has no fixed point then contractive condition (3.1) does not hold.

Example 3.2. Let $X = \mathbb{R}$ and define the mapping $p^*_\vee : X \times X \to X$ by $p^*_\vee(x, y) = \max\{x, y\}$. It is easy to check that $(X, p^*_\vee)$ is a complete dualistic partial metric space. Define the self–mapping $T_0 : \mathbb{R} \to \mathbb{R}$ by

$$T_0(x) = \begin{cases} 0 & \text{if } x \neq 0; \\ -1 & \text{if } x = 0. \end{cases}$$

The mapping T_0 has no fixed point. It is easy to check that T_0 does not satisfy the contractive condition in the statement of Theorem 3.1. Indeed,

$$1 = \varphi(|p^*_\vee(-1, -1)|) = \varphi(|p^*_\vee(T_0(0), T_0(0))|) > \varphi(\mathcal{M}(0, 0)) - \psi(\mathcal{M}(0, 0)),$$

where

$$\mathcal{M}(0, 0) = \max\left\{|p^*_\vee(0, 0)|, \left|\frac{p^*_\vee(0, T_0(0))(1 + p^*_\vee(0, T_0(0)))}{1 + p^*_\vee(0, 0)}\right|\right\} = 0,$$

$\varphi(t) = t$ and $\psi(t) = \frac{t}{1+t}$ for all $t \geq 0$.

Corollary 3.3. *Let (X, p^*) be a complete dualistic partial metric space and $T : X \to X$ be a mapping satisfying*

$$|p^*(T(x), T(y))| \leq h\mathcal{M}(x, y) \text{ for all } x, y \in X.$$

Then T has a unique fixed point.

Proof . Proof follows if we set $\varphi(t) = t$ and $\psi(t) = (1 - h)t$ in theorem 3.1, where $h \in [0, 1[$ and $t \geq 0$. □

Example 3.4. Let $(\mathbb{R}, p^*_\vee)$ be a complete dualistic partial metric space, as defined in Example 3.2 and define the self–mapping $T_1 : \mathbb{R} \to \mathbb{R}$ by

$$T_1(x) = \begin{cases} 0 & \text{if } x = 0; \\ -1 & \text{if } x \geq 2. \end{cases}$$

The mapping T_1 has a unique fixed point $x = 0$. It is easy to check that T_1 satisfies the contractive condition in the statement of Corollary 3.3. Indeed, $\forall\ x \geq y \geq 2$ and $\frac{1}{2} < h < 1$

$$\begin{aligned} &|p^*_\vee(T_1(x), T_1(y))| \leq h \max\left\{|p^*_\vee(x, y)|, \left|\frac{p^*_\vee(y, T_1(y))(1 + p^*_\vee(x, T_1(x))}{1 + p^*(x, y)}\right\}\right|; \\ &1 = |p^*_\vee(-1, -1)| \leq hx, \end{aligned}$$

holds. Also note that for $x = 0 = y$ the contractive condition in the statement of Corollary 3.3 trivially holds.

Following example emphasis the use of absolute value function in contractive condition in the statement of Corollary 3.3.

Example 3.5. Let $X = (-1, 0]$ and $p^*_\vee : X \times X \to \mathbb{R}$ be defined by $p^*_\vee(x, y) = x \vee y$, for all $x, y \in X$. Note that $(X, p^*_\vee)$ is a complete dualistic partial metric space. Let $T_2 : X \to X$ be given by

$$T_2(x) = \frac{x}{2} \text{ for all } x \in X.$$

Since $|x| \leq \left|\frac{y(x+2)}{4(x+1)}\right|$ holds for all $x, y \in X$ with $x \geq y$, thus

$$|p^*_\vee(x, y)| \leq \left|\frac{p^*_\vee(y, T_2(y))(1 + p^*_\vee(x, T_2(x))}{1 + p^*(x, y)}\right|.$$

Consequently, for $h = \frac{4}{5}$, the contractive condition

$$|p^*_\vee(T_2(x), T_2(y))| \leq h \max\left\{|p^*_\vee(x, y)|, \left|\frac{p^*_\vee(y, T_2(y))(1 + p^*_\vee(x, T_2(x))}{1 + p^*(x, y)}\right|\right\},$$

is satisfied and $x = 0$ is a unique fixed point of T. However, we note that for $y = -\frac{3}{4}$, $x = -\frac{1}{2}$, $h = \frac{2}{5}$ the contractive condition

$$p^*_\vee(T_2(x), T_2(y)) \leq h \max\left\{p^*_\vee(x, y), \frac{p^*_\vee(y, T_2(y))(1 + p^*_\vee(x, T_2(x))}{1 + p^*_\vee(x, y)}\right\},$$

does not hold. Hence for $\varphi(t) = t$ and $\psi(t) = (1 - h)t$, Theorem 1.2 is not applicable.

Theorem 3.6, generalizes Theorem 2 in [12], results presented in [3] and Corollary 2.2 in [10].

Theorem 3.6. *Let (X, p^*) be a complete dualistic partial metric space and $T : X \to X$ be a mapping satisfying*

$$|p^*(T(x), T(y))| \leq |\mathcal{M}(x, y)| - \psi(|\mathcal{M}(x, y)|) \text{ for all } x, y \in X, \tag{3.14}$$

where

$$\mathcal{M}(x, y) = \max\left\{p^*(x, y), p^*(y, T(y)), p^*(x, T(x)), \frac{p^*(x, T(y)) + p^*(y, T(x))}{2}\right\},$$

$\psi : [0, \infty) \to [0, \infty)$ is a lower semi–continuous function with $\psi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0$. Then T has a unique fixed point.

Proof . Let x_0 be an initial point of X and let us define Picard iterative sequence $\{x_n\}$ by

$$x_n = T(x_{n-1}) \text{ for all } n \in \mathbb{N}.$$

If there exists a positive integer i such that $x_i = x_{i+1}$, then $x_i = x_{i+1} = T(x_i)$, so x_i is a fixed point of T. In this case proof is complete. On the other hand if $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, then from contractive condition (3.14), we have for $x_n, x_{n+1} \in X$

$$\begin{aligned}
|p^*(x_n, x_{n+1})| &\leq |\mathcal{M}(x_{n-1}, x_n)| - \psi(|\mathcal{M}(x_{n-1}, x_n)|) \\
&\leq \left|\max\left\{\begin{array}{l} p^*(x_{n-1}, x_n), p^*(x_n, x_{n+1}), p^*(x_{n-1}, x_n), \\ \frac{p^*(x_{n-1}, x_{n+1}) + p^*(x_n, x_n)}{2} \end{array}\right\}\right| \\
&= |\max\{p^*(x_{n-1}, x_n), p^*(x_n, x_{n+1})\}| = |p^*(x_{n-1}, x_n)|.
\end{aligned}$$

This shows that $\{|p^*(x_n, x_{n+1})|\}_{n\in\mathbb{N}}$ is a non increasing sequence of positive real numbers. There exists a number $\nabla \geq 0$ such that $\lim_{n\to\infty} |p^*(x_n, x_{n+1})| = \nabla$. We claim that $\nabla = 0$. On contrary suppose that $\nabla > 0$ and taking upper limit of

$$|p^*(x_n, x_{n+1})| \leq |p^*(x_{n-1}, x_n)| - \psi(|p^*(x_{n-1}, x_n)|),$$

we get

$$\begin{aligned}
&\nabla \leq \nabla - \lim_{n\to\infty} \inf \psi(|p^*(x_{n-1}, x_n)|), \\
&\nabla \leq \nabla - \psi(\nabla) < \nabla,
\end{aligned}$$

which is a contradiction, so $\nabla = 0$ and hence

$$\lim_{n\to\infty} |p^*(x_n, x_{n+1})| = 0 \text{ implies } \lim_{n\to\infty} p^*(x_n, x_{n+1}) = 0.$$

Similarly, $\lim_{n\to\infty} p^*(x_n, x_n) = 0$. Continuing as in proof of Theorem 3.1, we get the required result. □

Following corollary appeared in [10].

Corollary 3.7. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$d(T(x), T(y)) \leq \mathcal{M}(x, y) - \psi(\mathcal{M}(x, y)) \text{ for all } x, y \in X,$$

where

$$\mathcal{M}(x, y) = \max\left\{d(x, y), d(y, T(y)), d(x, T(x)), \frac{d(x, T(y)) + d(y, T(x))}{2}\right\},$$

$\psi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\psi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0$. Then T has a unique fixed point.

Proof . As the restriction of p^* to $\mathbb{R}_0^+$, is a partial metric p and partial metric with $p(x,x)=0$ is a metric on X. Hence result follows from Theorem 3.6, if we set $p^*(x,y)=p(x,y)$ for all $x,y \in \mathbb{R}_0^+ = X$ along with $p(x,x)=0$ for all $x \in X$. □

The Corollary 3.8 generalizes [8, Theorem 2.3].

Corollary 3.8. *Let (X,p^*) be a complete dualistic partial metric space and $T: X \to X$ be a mapping satisfying*

$$|p^*(T(x),T(y))| \leq k|\mathcal{M}(x,y)| \text{ for all } x,y \in X.$$

Then T has a unique fixed point.

Proof . Set $\psi(t)=(1-k)t$ in Theorem 3.6. □

The Corollary 3.9 shows that the ψ–weak contraction with a function ψ is of Boyd and Wong type [4].

Corollary 3.9. *Let (X,p^*) be a complete dualistic partial metric space and $T: X \to X$ be a mapping satisfying*

$$|p^*(T(x),T(y))| \leq \varphi(|\mathcal{M}(x,y)|) \text{ for all } x,y \in X.$$

Then T has a unique fixed point.

Proof . Set $\varphi(t)=t-\psi(t)$ in Theorem 3.6, $\varphi(t)$) is an upper semi-continuous function from the right. □

The Corollary 3.10 generalizes the main Theorem of Reich [11].

Corollary 3.10. *Let (X,p^*) be a complete dualistic partial metric space and $T: X \to X$ be a mapping satisfying*

$$|p^*(T(x),T(y))| \leq \theta(|\mathcal{M}(x,y)|)|\mathcal{M}(x,y)| \text{ for all } x,y \in X.$$

Then T has a unique fixed point.

Proof . Set $\theta(t)=1-\dfrac{\psi(t)}{t}$ in Theorem 3.6, where $t>0$ and $\theta(0)=0$. □

4. Conclusion

In this paper, we describe some topological properties of dualistic partial metric spaces and establish some fixed point theorems for weak contraction mappings of rational type defined on dual partial metric spaces. We show, with the help of examples, that the new results allow us to find fixed points of mappings in some cases in which the results in partial metric spaces cannot be applied. Our investigations shows that various existing fixed point results can be extended to dualistic partial metric (for example results in [1, 2, 5, 13]). Moreover, many familiar topological properties and principles can be proved in certain dualistic partial metric spaces, although some results might need some advanced assumptions. We are very sure that various properties, including separation axioms, countability, connectedness and compactness can be established for dualistic partial metric topology.

References

[1] M. Abbas, T. Nazir and S. Romaguera, *Fixed point results for generalized cyclic contraction mappings in partial metric spaces,* RACSAM 106 (2012) 287–297.

[2] T. Abdeljawad, E. Karapinar and K. Tas, *Existence and uniqueness of a common fixed point on partial metric spaces,* Appl. Math. Lett. 24 (2011) 1900–1904.

[3] Y.I. Alber and S. Guerre–Delabriere, *Principle of Weakly Contractive Maps in Hilbert Spaces*, New Results in Operator Theory and Its Applications, Birkhäuser, Basel, 1997. 7–22.

[4] D.W. Boyd and J.S. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. 20 (1969) 458–464.

[5] H. Aydi, M. Abbas and C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology Appl. 159 (2012) 3234–3242.

[6] E.karapınar, W.Shatanawi and K.Tas: *Fixed point theorem on partial metric spaces involving rational expressions*, Miskolc Math. Notes,14 (2013) 135–142.

[7] S.G. Matthews, *Partial Metric Topology*, Ann. New York Acad. Sci. 728 (1994) 183–197.

[8] S. Oltra and O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Ist. Mat. Univ. Trieste, 36 (2004) 17–26.

[9] S.J. O'Neill, *Partial metric, valuations and domain theory*, Ann. New York Acad. Sci. 806 (1996) 304–315.

[10] Q. Zhang and Y. Song, *Fixed point theory for generalized φ-weak contractions*, Appl. Math. Lett. 22 (2009) 75–78.

[11] S. Reich, *Some fixed point problems*, Atti. Accad. Naz. Lincei 57 (1974) 194–198.

[12] B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. 4 (2001) 2683–2693.

[13] S. Romaguera, *A Kirk type characterization of completeness for partial metric spaces*, Fixed Point Theory Appl., 2010. (2010), Article ID 493298, 6 pages.