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On a k-extension of the Nielsen's β-function

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Abstract

Motivated by the k–digamma function, we introduce a k–extension of the Nielsen's β–function, and further study some properties and inequalities of the new function.

Keywords: Nielsen's β–function, k–extension, k–digamma function, Inequality.

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1. Introduction and Preliminaries

The Nielsen's β–function may be defined by any of the following equivalent forms (see [3], [5], [8], [12]).

$$\begin{aligned}
\beta(x) &= \int_0^1 \frac{t^{x-1}}{1+t}\,dt \quad (x>0) \\
&= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt \quad (x>0) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+x} \quad (x>0) \\
&= \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\} \quad (x>0),
\end{aligned}$$

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where $\psi(u) = \frac{d}{du}\ln\Gamma(u)$ is the digamma or psi function and $\Gamma(u)$ is the Euler's Gamma function. It satisfies the properties:

$$\beta(x+1) = \frac{1}{x} - \beta(x),$$
$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}.$$

Additional properties of this function can also be found in [9]. As shown in [2] and [6], the Nielsen's β–function is very useful in evaluating and estimating certain integrals as well as some mathematical constants. Recently in [10], the authors introduced and studied some properties of a q-analogue of the function. In this paper, we continue the investigation by establishing a k–extension of the function. The paper is motivated by the k–digamma function introduced by Díaz and Pariguan [4]. In the meantime, we state the following definitions which are well–known in the literature.

Definition 1.1. A function $h : I \to \mathbb{R}$ is said to be convex on I if

$$h(ax + (1-a)y) \le ah(x) + (1-a)h(y)$$

holds for all $x, y \in I$ and $a \in [0, 1]$. If h is twice differentiable, then it is said to be convex if and only if $h''(x) \ge 0$ for every $x \in I$.

Definition 1.2. A function $h : I \to \mathbb{R}^+$ is said to be logarithmically convex or in short log–convex if $\ln h$ is convex on I. That is if

$$\ln h(ax + (1-a)y) \le a \ln h(x) + (1-a) \ln h(y),$$

or equivalently

$$h(ax + (1-a)y) \le [h(x)]^a [h(y)]^{1-a}$$

for all $x, y \in I$ and $a \in [0, 1]$.

Definition 1.3. A function $h : I \to \mathbb{R}$ is said to be completely monotonic on I if h has derivatives of all order on I and

$$(-1)^s h^{(s)}(x) \ge 0$$

for all $x \in I$ and $s \in \mathbb{N}$ [13].

We now present our findings in the following sections.

2. k-Extension of Nielsen's β-function

In this section, we introduce a k–extension (also called k–analogue) of the Nielsen's β–function and further study some properties and inequalities involving the new function. We begin by recalling the following definitions concerning the k–Gamma function.

The k–Gamma function (also known as the k–analogue or k–extension of the classical Gamma function) is defined by Díaz and Pariguan [4] for $k > 0$ and $x \in \mathbb{C}\backslash k\mathbb{Z}$ as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt.$$

where $(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k)$ is the Pochhammer k–symbol. Also, the k–Gamma function satisfies the relations (see also [14])

$$\begin{aligned} \Gamma_k(x+k) &= x\Gamma_k(x), \\ \Gamma_k(k) &= 1, \\ \Gamma_k(x)\Gamma_k(k-x) &= \frac{\pi}{k\sin\left(\frac{\pi x}{k}\right)}. \end{aligned} \tag{2.1}$$

Furthermore, the k–analogue of the Euler's beta function is given as

$$\mathbf{B}_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\,dt \quad (x>0, y>0). \tag{2.2}$$

The logarithmic derivative of the k–Gamma function, which is termed the k–digamma function, is defined as

$$\begin{aligned} \psi_k(x) = \frac{d}{dx}\ln\Gamma_k(x) &= \frac{\ln k-\gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x}{nk(nk+x)} \\ &= \frac{\ln k-\gamma}{k} + \sum_{n=0}^{\infty}\left(\frac{1}{nk+k} - \frac{1}{nk+x}\right) \qquad (2.3) \\ &= \int_0^\infty \left(\frac{2e^{-t}-e^{-kt}}{kt} - \frac{e^{-xt}}{1-e^{-kt}}\right)dt, \qquad (2.4) \end{aligned}$$

where $\gamma = 0.57721\dots$ is the Euler–Mascheroni's constant. It satisfies the properties (see also [7])

$$\begin{aligned} \psi_k(x+k) &= \frac{1}{x} + \psi_k(x), \\ \psi_k(k) &= \frac{\ln k-\gamma}{k}, \\ \psi_k(k-x) - \psi_k(x) &= \frac{\pi}{k}\cot\left(\frac{\pi x}{k}\right). \end{aligned} \tag{2.5}$$

Remark 2.1. The integral representation (2.4) which is appearing for the first time, is derived as follows. In the work [11], a (p,k)–analogue of the digamma function was given as

$$\psi_{p,k}(x) = \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt \quad (p\in\mathbb{N}, k>0),$$

where $\lim_{p\to\infty}\psi_{p,k}(x) = \psi_k(x)$ and $\lim_{k\to1}\psi_{p,k}(x) = \psi_p(x)$. By using the relation $\ln x = \int_0^\infty \frac{e^{-t}-e^{-xt}}{t}\,dt$ (see [1, p. 230]), we obtain

$$\psi_{p,k}(x) = \frac{1}{k}\int_0^\infty \frac{e^{-t}-e^{-pt}}{t}\,dt + \frac{1}{k}\int_0^\infty \frac{e^{-t}-e^{-kt}}{t}\,dt - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt.$$

Then

$$\begin{aligned} \psi_k(x) = \lim_{p\to\infty}\psi_{p,k}(x) &= \frac{1}{k}\int_0^\infty \frac{2e^{-t}-e^{-kt}}{t}\,dt - \int_0^\infty \frac{e^{-xt}}{1-e^{-kt}}\,dt \\ &= \int_0^\infty \left(\frac{2e^{-t}-e^{-kt}}{kt} - \frac{e^{-xt}}{1-e^{-kt}}\right)dt. \end{aligned}$$

Also, it is worth noting from (2.4) that,

$$\lim_{k\to1}\psi_k(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right)dt = \psi(x), \text{ (see [1, p. 259])}.$$

Motivated by these definitions, we introduce the k–extension of the Nielsen's β–function in the following definition.

Definition 2.2. The k-extension of the Nielsen's β-function is defined for $k > 0$ by the following equivalent forms:

$$\beta_k(x) = \frac{k}{2}\left\{\psi_k\left(\frac{x+k}{2}\right) - \psi_k\left(\frac{x}{2}\right)\right\} \quad (x > 0) \tag{2.6}$$

$$= \sum_{n=0}^{\infty}\left(\frac{k}{2nk+x} - \frac{k}{2nk+x+k}\right) \quad (x > 0) \tag{2.7}$$

$$= \int_0^{\infty} \frac{e^{-\frac{xt}{k}}}{1+e^{-t}}\,dt \quad (x > 0) \tag{2.8}$$

$$= \int_0^{1} \frac{t^{\frac{x}{k}-1}}{1+t}\,dt \quad (x > 0), \tag{2.9}$$

where $\beta_k(x) = \beta(x)$ if $k = 1$.

Remark 2.3. Representations (2.7) and (2.8) are respectively derived from (2.3) and (2.4), and by a change of variable, (2.9) is obtained from (2.8).

Proposition 2.4. *The function $\beta_k(x)$ satisfies the functional equation*

$$\beta_k(x+k) = \frac{k}{x} - \beta_k(x) \tag{2.10}$$

and the reflection formula

$$\beta_k(x) + \beta_k(k-x) = \frac{\pi}{\sin\left(\frac{\pi x}{k}\right)}. \tag{2.11}$$

Proof . By using representation (2.9), we obtain

$$\beta_k(x+k) + \beta_k(x) = \int_0^1 \frac{t^{\frac{x}{k}} + t^{\frac{x}{k}-1}}{1+t}\,dt = \int_0^1 t^{\frac{x}{k}-1}\,dt = \frac{k}{x}.$$

Next, by using (2.5), (2.6) and some trigonometric identities, we obtain

$$\begin{aligned}
&\beta_k(x) + \beta_k(k-x) \\
&= \frac{k}{2}\left\{\psi_k\left(\frac{x}{2}+\frac{k}{2}\right) - \psi_k\left(\frac{x}{2}\right) + \psi_k\left(k-\frac{x}{2}\right) - \psi_k\left(\frac{k}{2}-\frac{x}{2}\right)\right\} \\
&= \frac{k}{2}\left\{\psi_k\left(k-\left(\frac{k}{2}-\frac{x}{2}\right)\right) - \psi_k\left(\frac{k}{2}-\frac{x}{2}\right) + \psi_k\left(k-\frac{x}{2}\right) - \psi_k\left(\frac{x}{2}\right)\right\} \\
&= \frac{k}{2}\left\{\frac{\pi}{k}\cot\left(\frac{\pi}{2}-\frac{\pi x}{2k}\right) + \frac{\pi}{k}\cot\left(\frac{\pi x}{2k}\right)\right\} \\
&= \frac{\pi}{2}\left\{\cot\left(\frac{\pi}{2}-\frac{\pi x}{2k}\right) + \cot\left(\frac{\pi x}{2k}\right)\right\} \\
&= \frac{\pi}{2}\left\{\tan\left(\frac{\pi x}{2k}\right) + \cot\left(\frac{\pi x}{2k}\right)\right\} = \frac{\pi}{2\cos\left(\frac{\pi x}{2k}\right)\sin\left(\frac{\pi x}{2k}\right)} = \frac{\pi}{\sin\left(\frac{\pi x}{k}\right)}.
\end{aligned}$$

This completes the proof. □

Remark 2.5. It can be deduced from (2.1), (2.2) and (2.11) that the function $\beta_k(x)$ is related to the k-analogue of the Euler's beta function, $\mathbf{B}_k(x,y)$ in the following ways

$$\beta_k(x)+\beta_k(k-x)=k\mathbf{B}_k(x,k-x),$$

$$\beta_k(x)=-k\frac{d}{dx}\left[\ln \mathbf{B}_k\left(\frac{x}{2},\frac{k}{2}\right)\right].$$

By successive applications of (2.10), we obtain the generalized form

$$\beta_k(x+nk)=\sum_{s=0}^{n-1}\frac{(-1)^{n+1+s}k}{x+sk}+(-1)^n\beta_k(x),$$

where $n\in\mathbb{N}$. Also, successive differentiation of (2.6), (2.8), (2.9) and (2.10) yields respectively

$$\begin{aligned}\beta_k^{(n)}(x)&=\frac{k}{2^{n+1}}\left\{\psi_k^{(n)}\left(\frac{x+k}{2}\right)-\psi_k^{(n)}\left(\frac{x}{2}\right)\right\}\quad (x>0)\\ &=\frac{(-1)^n}{k^n}\int_0^\infty\frac{t^ne^{-\frac{xt}{k}}}{1+e^{-t}}\,dt\quad (x>0) \qquad (2.12)\\ &=\frac{1}{k^n}\int_0^1\frac{(\ln t)^nt^{\frac{x}{k}-1}}{1+t}\,dt\quad (x>0),\end{aligned}$$

$$\beta_k^{(n)}(x+k)=(-1)^n\frac{n!k}{x^{n+1}}-\beta_k^{(n)}(x)\quad (x>0) \qquad (2.13)$$

for $n\in\mathbb{N}_0$.

Remark 2.6. It follows readily from representation (2.12) that:

(i) $\beta_k(x)$ is positive and decreasing;
(ii) $\beta_k^{(n)}(x)$ is positive and decreasing if $n\in\mathbb{N}_0$ is even;
(iii) $\beta_k^{(n)}(x)$ is negative and increasing if $n\in\mathbb{N}_0$ is odd.

Theorem 2.7. *The function* $\beta_k(x)$ *is*

(a) *logarithmically convex on* $(0,\infty)$*;*
(b) *completely monotonic on* $(0,\infty)$*.*

Proof . (a) Let $r>1$, $s>1$ and $\frac{1}{r}+\frac{1}{s}=1$ and $x,y\in(0,\infty)$. Then by (2.9) and the Hölder's inequality, we obtain

$$\begin{aligned}\beta_k\left(\frac{x}{r}+\frac{y}{s}\right)&=\int_0^1\frac{t^{\frac{x}{kr}+\frac{y}{ks}-1}}{1+t}\,dt\\ &=\int_0^1\frac{t^{\frac{x-k}{kr}}}{(1+t)^{\frac{1}{r}}}\frac{t^{\frac{y-k}{ks}}}{(1+t)^{\frac{1}{s}}}\,dt\\ &\le\left(\int_0^1\frac{t^{\frac{x}{k}-1}}{1+t}\,dt\right)^{\frac{1}{r}}\left(\int_0^1\frac{t^{\frac{y}{k}-1}}{1+t}\,dt\right)^{\frac{1}{s}}\\ &=\left[\beta_k(x)\right]^{\frac{1}{r}}\left[\beta_k(y)\right]^{\frac{1}{s}}.\end{aligned}$$

Hence $\beta_k(x)$ is logarithmically convex on $(0,\infty)$.

(b) It follows easily from (2.12) that

$$(-1)^n\beta_k^{(n)}(x) = \frac{(-1)^{2n}}{k^n}\int_0^\infty \frac{t^n e^{-\frac{xt}{k}}}{1+e^{-t}}\,dt \geq 0.$$

Thus, $\beta_k(x)$ is completely monotonic on $(0,\infty)$. □

Remark 2.8. The log–convexity of $\beta_k(x)$ implies that:

(a) the Turan–type inequality $\beta_k(x)\beta_k''(x) - (\beta_k'(x))^2 \geq 0$ holds for $x>0$;

(b) the function $\frac{\beta_k'(x)}{\beta_k(x)}$ is increasing on $(0,\infty)$.

Theorem 2.9. *The inequality*

$$\beta_k(x+k)\beta_k(y+k) \leq (\ln 2)\beta_k(x+y+k)$$

holds for $x,y\in[0,\infty)$.

Proof . Let F and λ be defined for $x,y\in[0,\infty)$ as

$$F(x,y) = \frac{\beta_k(x+k)\beta_k(y+k)}{\beta_k(x+y+k)}$$

and

$$\lambda(x,y) = \ln F(x,y) = \ln\beta_k(x+k) + \ln\beta_k(y+k) - \ln\beta_k(x+y+k).$$

Then by fixing y, we obtain

$$\lambda'(x,y) = \frac{\beta_k'(x+k)}{\beta_k(x+k)} - \frac{\beta_k'(x+y+k)}{\beta_k(x+y+k)} \leq 0,$$

since $\frac{\beta_k'(x)}{\beta_k(x)}$ is increasing for $x>0$. Thus, $\lambda(x,y)$ is nonincreasing. Consequently, $F(x,y)$ is also nonincreasing. Then for $x\geq 0$, we have $F(x,y)\leq F(0,y)$ which gives

$$\frac{\beta_k(x+k)\beta_k(y+k)}{\beta_k(x+y+k)} \leq \beta_k(k) = \ln 2.$$

□

Theorem 2.10. *The inequality*

$$\beta_k(x)\beta_k(x+y+z) - \beta_k(x+y)\beta_k(x+z) > 0$$

holds for positive real numbers x, y *and* z.

Proof . Let h be defined for positive real numbers x and z as

$$h(x) = \frac{\beta_k(x+z)}{\beta_k(x)}.$$

Then, it suffices to show that h is increasing. Let $\eta(x) = \ln h(x)$. Then

$$\eta'(x) = \frac{\beta_k'(x+z)}{\beta_k(x+z)} - \frac{\beta_k'(x)}{\beta_k(x)} > 0.$$

Thus, $\eta(x)$ and consequently $h(x)$ are increasing. Hence for $y>0$, we have $h(x+y)>h(x)$ which gives the desired result. □

Theorem 2.11. *The inequality*

$$\frac{[\beta_k(1+k)]^a}{\beta_k(a+k)} \leq \frac{[\beta_k(x+k)]^a}{\beta_k(ax+k)} \leq (\ln 2)^{a-1} \tag{2.14}$$

holds for $a \geq 1$ and $x \in [0,1]$. It reverses if $0 < a \leq 1$.

Proof . Let $a \geq 1$, $Q(x) = \frac{[\beta_k(x+k)]^a}{\beta_k(ax+k)}$ and $g(x) = \ln Q(x)$ for $x \geq 0$. Then

$$g'(x) = a\left[\frac{\beta_k'(x+k)}{\beta_k(x+k)} - \frac{\beta_k'(ax+k)}{\beta_k(ax+k)}\right] \leq 0.$$

Thus, $Q(x)$ is nonincreasing. Then for $x \in [0,1]$, we have $Q(1) \leq Q(x) \leq Q(0)$ which yields the result (2.14). If $0 < a \leq 1$, then we obtain $g'(x) \geq 0$ which implies that $Q(x)$ is nondecreasing. Then for $x \in [0,1]$, we obtain $Q(0) \leq Q(x) \leq Q(1)$ which gives the reverse of (2.14). □

3. Some results involving $\left|\beta_k^{(n)}(x)\right|$

In this section we study some properties and inequalities of the function $\left|\beta_k^{(n)}(x)\right|$ where $n \in \mathbb{N}_0$. To start with, we note that $\left|\beta_k^{(n)}(x)\right| = (-1)^n \beta_k^{(n)}(x)$ for all $n \in \mathbb{N}_0$. This together with relation (2.13) yields

$$\left|\beta_k^{(n)}(x+k)\right| = \frac{n!k}{x^{n+1}} - \left|\beta_k^{(n)}(x)\right|. \tag{3.1}$$

We also note that, if $f(x) = \left|\beta_k^{(n)}(x)\right|$, then $f'(x) = -\left|\beta_k^{(n+1)}(x)\right|$. This implies that the $f(x)$ is decreasing for all $n \in \mathbb{N}$.

Proposition 3.1. *Let Δ_n be defined for $x > 0$ and $n \in \mathbb{N}$ as*

$$\Delta_n(x) = \frac{x^{n+1}}{n!}\left|\beta_k^{(n)}(x)\right|.$$

Then,

$$\lim_{x\to 0} \Delta_n(x) = k \quad \text{and} \quad \lim_{x\to 0} \Delta_n'(x) = 0.$$

Proof . It follows from (3.1) that

$$\lim_{x\to 0} \Delta_n(x) = \lim_{x\to 0}\left\{k - \frac{x^{n+1}}{n!}\left|\beta_k^{(n)}(x+k)\right|\right\} = k.$$

Also,

$$\lim_{x\to 0} \Delta_n'(x) = \lim_{x\to 0}\left\{\frac{x^{n+1}}{n!}\left|\beta_k^{(n+1)}(x+k)\right| - \frac{(n+1)x^n}{n!}\left|\beta_k^{(n)}(x+k)\right|\right\} = 0.$$

□

Theorem 3.2. *Let $n \in \mathbb{N}_0$, $r > 1$, $s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$. Then, the inequality*

$$\left|\beta_k^{(n)}\left(\frac{x}{r} + \frac{y}{s}\right)\right| \leq \left|\beta_k^{(n)}(x)\right|^{\frac{1}{r}}\left|\beta_k^{(n)}(y)\right|^{\frac{1}{s}}, \tag{3.2}$$

holds for $x, y > 0$.

Proof . Similarly, by the relation (2.12) and the Hölder's inequality, we obtain

$$\begin{aligned}\left|\beta_k^{(n)}\left(\frac{x}{r}+\frac{y}{s}\right)\right| &= \frac{1}{k^n}\int_0^\infty \frac{t^n e^{-(\frac{x}{kr}+\frac{y}{ks})t}}{1+e^{-t}}\,dt \\ &= \frac{1}{k^n}\int_0^\infty \frac{t^{\frac{n}{r}} e^{-\frac{xt}{kr}}}{(1+e^{-t})^{\frac{1}{r}}}\frac{t^{\frac{n}{s}} e^{-\frac{yt}{ks}}}{(1+e^{-t})^{\frac{1}{s}}}\,dt \\ &\leq \left(\frac{1}{k^n}\int_0^\infty \frac{t^n e^{-\frac{xt}{k}}}{1+e^{-t}}\,dt\right)^{\frac{1}{r}}\left(\frac{1}{k^n}\int_0^\infty \frac{t^n e^{-\frac{yt}{k}}}{1+e^{-t}}\,dt\right)^{\frac{1}{s}} \\ &= \left|\beta_k^{(n)}(x)\right|^{\frac{1}{r}}\left|\beta_k^{(n)}(y)\right|^{\frac{1}{s}},\end{aligned}$$

which completes the proof. □

Remark 3.3. Inequality (3.2) implies that the function $\left|\beta_k^{(n)}(x)\right|$ is log–convex for all $n \in \mathbb{N}_0$. This further implies that:

(a) the inequality $\left|\beta_k^{(n+2)}(x)\right| . \left|\beta_k^{(n)}(x)\right| - \left|\beta_k^{(n+1)}(x)\right|^2 \geq 0$ holds;

(b) the function $\left|\beta_k^{(n+1)}(x)\right| / \left|\beta_k^{(n)}(x)\right|$ is decreasing.

Theorem 3.4. *Let $n \in \mathbb{N}_0$. Then the inequality*

$$\left|\beta_k^{(n)}(x+y)\right| < \left|\beta_k^{(n)}(x)\right| + \left|\beta_k^{(n)}(y)\right| \tag{3.3}$$

holds for $x, y > 0$.

Proof . Let $F_k(x,y) = \left|\beta_k^{(n)}(x+y)\right| - \left|\beta_k^{(n)}(x)\right| - \left|\beta_k^{(n)}(y)\right|$ for $n \in \mathbb{N}_0$. Without loss of generality, let y be fixed. Then,

$$\begin{aligned}F_k'(x,y) &= \left|\beta_k^{(n+1)}(x)\right| - \left|\beta_k^{(n+1)}(x+y)\right| \\ &> 0,\end{aligned}$$

since $\left|\beta_k^{(n)}(x)\right|$ is decreasing for all $n \in \mathbb{N}_0$. Thus, $F_k(x,y)$ is increasing. Moreover,

$$\begin{aligned}\lim_{x\to\infty} F_k(x,y) &= \lim_{x\to\infty}\left\{\left|\beta_k^{(n)}(x+y)\right| - \left|\beta_k^{(n)}(x)\right| - \left|\beta_k^{(n)}(y)\right|\right\} \\ &= -\left|\beta_k^{(n)}(y)\right| \\ &< 0.\end{aligned}$$

Therefore, $F_k(x,y) \leq \lim_{x\to\infty} F_k(x,y) < 0$ which gives the result (3.3). □

Theorem 3.5. *Let $n \in \mathbb{N}_0$, $a > 0$, and $x > 0$. Then the inequalities*

$$\left|\beta_k^{(n)}(ax)\right| \leq a\left|\beta_k^{(n)}(x)\right| \quad \textit{if} \quad a \geq 1, \tag{3.4}$$

and

$$\left|\beta_k^{(n)}(ax)\right| \geq a\left|\beta_k^{(n)}(x)\right| \quad \textit{if} \quad a \leq 1, \tag{3.5}$$

are satisfied.

Proof . Let $a \geq 1$ and $H_k(x) = \left|\beta_k^{(n)}(ax)\right| - a\left|\beta_k^{(n)}(x)\right|$. Then,

$$\begin{aligned} H_k'(x) &= a\left\{\left|\beta_k^{(n+1)}(x)\right| - \left|\beta_k^{(n+1)}(ax)\right|\right\} \\ &\geq 0. \end{aligned}$$

Hence, $H_k(x)$ is nondecreasing. Moreover, $\lim_{x\to\infty} H_k(x) = 0$. Therefore, $H_k(x) \leq \lim_{x\to\infty} H_k(x) = 0$ which gives the result (3.4). Similarly, if $a \leq 1$, we obtain $H_k'(x) \leq 0$ and $H_k(x) \geq \lim_{x\to\infty} H_k(x) = 0$ yielding the result (3.5). □

Remark 3.6. It is interesting to note that the result (3.3) coincides with (3.4) if $x = y$ in (3.3) and $a = 2$ in (3.4).

The following lemma is known in the literature as the the convolution theorem for Laplace transforms:

Lemma 3.7. Let $f(t)$ and $g(t)$ be any two functions with convolution $f * g = \int_0^t f(s)g(t-s)\,ds$. Then the Laplace transform of the convolution is given as

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\}.$$

That is

$$\int_0^\infty \left[\int_0^t f(s)g(t-s)\,ds\right] e^{-xt}\,dt = \int_0^\infty f(t)e^{-xt}\,dt \int_0^\infty g(t)e^{-xt}\,dt. \tag{3.6}$$

Theorem 3.8. *Let G_k be defined for $k > 0$, $n \in \mathbb{N}_0$ and $x > 0$ as*

$$G_k(x) = kx\left|\beta_k^{(n)}(x)\right|.$$

Then, $G_k(x)$ is decreasing.

Proof . By using the relation $\frac{n!}{x^{n+1}} = \int_0^\infty t^n e^{-xt}\,dt$ for $x > 0$ and $n \in \mathbb{N}_0$, which is derived from the Gamma function, and by the convolution theorem for Laplace transforms (3.6), we obtain the following.

$$\begin{aligned} G_k'(x) &= k\left|\beta_k^{(n)}(x)\right| - kx\left|\beta_k^{(n+1)}(x)\right| \\ &= x\left[\frac{k}{x}\left|\beta_k^{(n)}(x)\right| - k\left|\beta_k^{(n+1)}(x)\right|\right], \\ \frac{G_k'(x)}{x} &= \frac{k}{x}\left|\beta_k^{(n)}(x)\right| - k\left|\beta_k^{(n+1)}(x)\right| \\ &= \int_0^\infty e^{-\frac{xt}{k}}\,dt \cdot \frac{1}{k^n}\int_0^\infty \frac{t^n e^{-\frac{xt}{k}}}{1+e^{-t}}\,dt - \frac{k}{k^{n+1}}\int_0^\infty \frac{t^{n+1}e^{-\frac{xt}{k}}}{1+e^{-t}}\,dt \\ &= \frac{1}{k^n}\int_0^\infty\left[\int_0^t \frac{s^n}{1+e^{-s}}\,ds\right]e^{-\frac{xt}{k}}\,dt - \frac{1}{k^n}\int_0^\infty \frac{t^{n+1}e^{-\frac{xt}{k}}}{1+e^{-t}}\,dt \\ &= \frac{1}{k^n}\int_0^\infty A_n(t)e^{-\frac{xt}{k}}\,dt, \end{aligned}$$

where

$$A_n(t) = \int_0^t \frac{s^n}{1+e^{-s}}\,ds - \frac{t^{n+1}}{1+e^{-t}}.$$

Then $A_n(0) = \lim_{t\to 0^+} A_n(t) = 0$. Furthermore,

$$\begin{aligned} A_n'(t) &= \frac{t^n}{1+e^{-t}} - \frac{(n+1)t^n}{1+e^{-t}} - \frac{t^{n+1}e^{-t}}{(1+e^{-t})^2} \\ &= -\frac{t^n}{1+e^{-t}}\left[n + \frac{te^{-t}}{1+e^{-t}}\right] < 0, \end{aligned}$$

which implies that $A_n(t)$ is decreasing. Then for $t > 0$, we obtain $A_n(t) < A_n(0) = 0$. Thus, $G_k'(x) < 0$ which completes the proof. □

Theorem 3.9. *Let $k > 0$ and $n \in \mathbb{N}_0$. Then the inequality*

$$\left|\beta_k^{(n)}(xy)\right| < \left|\beta_k^{(n)}(x)\right| + \left|\beta_k^{(n)}(y)\right|, \tag{3.7}$$

holds for $x > 0$ and $y \geq 1$.

Proof . Let $T_k(x,y) = k\left|\beta_k^{(n)}(xy)\right| - k\left|\beta_k^{(n)}(x)\right| - k\left|\beta_k^{(n)}(y)\right|$ for $k > 0$, $n \in \mathbb{N}_0$, $x > 0$ and $y \geq 1$. Let y be fixed. Then

$$\begin{aligned} T_k'(x,y) &= -ky\left|\beta_k^{(n+1)}(xy)\right| + k\left|\beta_k^{(n+1)}(x)\right| \\ &= \frac{1}{x}\left\{kx\left|\beta_k^{(n+1)}(x)\right| - kxy\left|\beta_k^{(n+1)}(xy)\right|\right\} \\ &\geq 0, \end{aligned}$$

since $kx\left|\beta_k^{(n)}(x)\right|$ is decreasing. Hence, $T_k(x,y)$ is nondecreasing. Then for $0 < x < \infty$, we obtain

$$T_k(x,y) \leq \lim_{x\to\infty} T_k(x,y) = -k\left|\beta_k^{(n)}(y)\right| < 0,$$

which gives the result (3.7). □

4. Concluding Remarks

Motivated by the k–digamma function, we have introduced a k–extension of the Nielsen's β–function, and further studied some properties and inequalities concerning the new function.

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