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Yang-Laplace transform method Volterra and Abel's integro-differential equations of fractional order

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Abstract

This study outlines the local fractional integro–differential equations carried out by the local fractional calculus. The analytical solutions within local fractional Volterra and Abel's integral equations via the Yang–Laplace transform are discussed. Some illustrative examples will be discussed. The obtained results show the simplicity and efficiency of the present technique with application to the problems for the local fractional integral equations.

Keywords: Local fractional calculus, Volterra and Abel's integral equations, Yang–Laplace transform.
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1. Introduction

Fractional derivatives and fractional calculus have a long history and there are a number of applications in applied mathematics and engineering. Finding the solution for the differential and integral equations is one of the hot topics among the mathematicians and engineers. There are several analytical and numerical techniques for solving them [1], such as the spectral Legendre–Gauss–Lobatto collocation method [3], the shifted Jacobi–Gauss–Lobatto collocation method [4], the variation iteration method [6], the heat–balance integral method [7], the Adomian decomposition method [11], the finite element method [22], and the finite difference method [9].

Recently the theory of local fractional calculus as one of the practical tools to handle the fractal and continuously non–differentiable functions, was successfully implemented in real world problems

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which modelled by local fractional calculus. For detailed information about recent developments local fractional equations, please refer to [2], [5], [8, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

The main target of this paper is to take into account to application of Yang–Laplace transform to Volterra and local fractional Abel's integro–differential equations with local fractional derivative and local fractional integral. The remainder of this work is organized as follows: in Section 2, we give a brief description of the local fractional calculus, while, in Section 3, we show that for the local fractional Volterra integral equations the Yang–Laplace transform method can be applied successfully. In Section 4, we show that how to applied the Yang–Laplace transform to local fractional Abel's integral equations. Some examples are given in Section 5, while some conclusions and further directions of research are discussed in Section 6.

2. Preliminaries

The objective of this section is to state the prerequisite definitions and also to summarized the necessary equalities for local fractional calculus.

2.1. Local Fractional Calculus

Recall the set R^α of real line numbers and use the Gao–Yang–Kang's idea to describe the definition of the local fractional calculus, see [13, 19]. Recently, the theory of Yang's fractional sets [13] was introduced as follows. For $0 < \alpha \leq 1$, we have the following α–type sets:

Z^α	The α–type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$.
Q^α	The α–type set of the rational numbers is defined as the set $\{m^\alpha = (p/q)^\alpha : p, q \in Z, \, q \neq 0\}$.
J^α	The α–type set of the irrational numbers is defined as the set $\{m^\alpha \neq (p/q)^\alpha : p, q \in Z, \, q \neq 0\}$.
R^α	The α–type set of the real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

Table 1: α–type sets

If a^α, b^α and c^α belongs the set R^α of real line numbers, then

(i) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α;

(ii) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;

(iii) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$;

(iv) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;

(v) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;

(vi) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;

(vii) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2.1. (Yang, [13]) A non–differentiable function $f : R \to R^\alpha$, $x \to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b), we denote $f(x) \in C_\alpha(a, b)$.

Definition 2.2. (Yang, [13]) The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0}\frac{\Delta^\alpha\left(f(x)-f(x_0)\right)}{(x-x_0)^\alpha},$$

where $\Delta^\alpha\left(f(x)-f(x_0)\right)\cong\Gamma(\alpha+1)\left(f(x)-f(x_0)\right).$

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \ldots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \ldots$

Definition 2.3. (Yang, [13]) Let $f(x) \in C_\alpha[a, b]$. Then the local fractional integral is defined by,

$${}_aI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, \ldots, N-1$ and $a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b$ is partition of interval $[a, b]$.

Here, it follows that ${}_aI_b^\alpha f(x) = 0$ if $a = b$ and ${}_aI_b^\alpha f(x) = -{}_bI_a^\alpha f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_aI_x^\alpha f(x)$, then we denoted by $f(x) \in I_x^\alpha[a, b]$.

Lemma 2.4. (Yang, [13]) (Local fractional integration is anti–differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a, b]$, then we have

$${}_aI_b^\alpha f(x) = g(b) - g(a).$$

Lemma 2.5. (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha[a, b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha[a, b]$, then we have

$${}_aI_b^\alpha f(x)g^{(\alpha)}(x) = \left. f(x)g(x)\right|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x)g(x).$$

Lemma 2.6. (Yang, [13]) We have the following properties of local fractional calculus

i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha};$

ii) $\dfrac{1}{\Gamma(\alpha+1)}\displaystyle\int_a^b x^{k\alpha}(dx)^\alpha = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right),\ k \in R.$

3. Yang–Laplace transform method for local fractional Volterra integro–differential equations

In local fractional Volterra integral equations, at least one of the limits of local fractional integration must be a variable. There are two kinds of local fractional Volterra integral equations based on the place of unknown function. For the first kind local fractional Volterra integral equations, the unknown function $u(x)$ appears only inside local fractional integral sign in the form:

$$m(x) = \frac{1}{\Gamma(1+\alpha)}\int_0^x K(x,t)u(t)(dt)^\alpha,$$

such that $m(x) : \mathbb{R} \to \mathbb{R}^\alpha$ and $K(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^\alpha$. On the other hand, local fractional Volterra integral equations of the second kind, the unknown function $u(x)$ appears both inside and outside the local fractional integral sign. The second kind local fractional Volterra integral equation is represented by the form:

$$u(x) = m(x) + \frac{\lambda}{\Gamma(1+\alpha)} \int_0^x K(x,t)u(t)(dt)^\alpha.$$

In addition to these the local fractional Volterra integro-differential equations of the second and first kinds arise along the same line as local fractional Volterra integral equations with one or more of local derivatives in addition to the local fractional integral operators. The local fractional Volterra integro–differential equations of second and first kinds are represented by respectively

$$u^{(n\alpha)}(x) = m(x) + \frac{\lambda}{\Gamma(1+\alpha)} \int_0^x K(x,t)u(t)(dt)^\alpha$$

and

$$m(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x K_1(x,t)u(t)(dt)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_0^x K_2(x,t)u^{(n\alpha)}(t)(dt)^\alpha.$$

The Yang–Laplace transform method is a spectacular technique that can be utilized for solving local fractional Volterra integro–differential equations.

Theorem 3.1. (Yang–Laplace transform) Let $f : \mathbb{R} \to \mathbb{R}^\alpha$. The Yang–Laplace transform of $f(x)$ is given by

$$\mathcal{L}_\alpha\{f(x)\} = f_s^\alpha(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha \quad 0 < \alpha \leq 1,$$

where $E_\alpha(x^\alpha) = \sum_{k=0}^\infty \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ and $x^\alpha, s^\alpha \in \mathbb{R}^\alpha$.

The Yang–Laplace transform $f_s^\alpha(s)$ may fail to exist. If $f(x)$ has infinite discontinuities or if it grows up swiftly, then $f_s^\alpha(s)$ does not exist. Furthermore, a significant necessary condition for the existence of the Yang–Laplace transform $f_s^\alpha(s)$ is that $f_s^\alpha(s)$ have to vanish as s approaches at infinity.

Theorem 3.2. (Inverse Yang–Laplace transform) The inverse Yang–Laplace transform of $f_s^\alpha(s)$ is given by

$$\mathcal{L}_\alpha^{-1}\{f_s^\alpha(s)\} = f(x) = \frac{1}{(2\pi)^\alpha} \int_{\beta - i\infty}^{\beta + i\infty} E_\alpha(s^\alpha x^\alpha) f_s^\alpha(s)(ds)^\alpha \quad 0 < \alpha \leq 1,$$

where $E_\alpha(x^\alpha) = \sum_{k=0}^\infty \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$, $\quad s^\alpha \in \beta^\alpha + i^\alpha \infty^\alpha$ and $\quad Re(s^\alpha) = \beta^\alpha > 0^\alpha$.

Another significant theorem that will be used in solving local fractional Volterra integro–differential equations is the convolution theorem.

Definition 3.3 (Convolution). The convolution of two functions is the function $f * g$ defined by

$$(f * g)(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x f(t)g(x-t)(dt)^\alpha.$$

As further results, the properties of the convolution of the non–differentiable functions for convenience lead as

1. $(f * g)(x) = (g * f)(x)$;
2. $(f * (g + h))(x) = ((f + g) * h)(x)$.

In the convolution theorem for the Yang–Laplace transform, it was remarked that if the kernel $K(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^\alpha$ of the local fractional Volterra integral equation of second kind:

$$u(x) = m(x) + \frac{\lambda}{\Gamma(1+\alpha)} \int_0^x K(x,t)u(t)(dt)^\alpha,$$

depends on the difference $x-t$, then it is called a *difference kernel*. Then the local fractional Volterra integral equation of second kind owing difference kernel can be expressed as

$$u(x) = m(x) + \frac{\lambda}{\Gamma(1+\alpha)} \int_0^x K(x-t)u(t)(dt)^\alpha.$$

3.1. Yang-Laplace transform method for local fractional Volterra integral equations of the first kind and second kind

Let consider two mappings $f(x)$ and $g(x)$ that possess the necessary conditions of the existence of Yang–Laplace transform for each. Let the Yang–Laplace transforms for the mappings $f(x)$ and $g(x)$ be given by $\mathcal{L}_\alpha\{f(x)\} = f_s^\alpha(s)$ and $\mathcal{L}_\alpha\{g(x)\} = g_s^\alpha(s)$. The Yang–Laplace convolution product of these two mappings is described by

$$\mathcal{L}_\alpha\{f(x) * g(x)\} = \mathcal{L}_\alpha \left\{ \frac{1}{\Gamma(1+\alpha)} \int_0^x f(t)g(x-t)(dt)^\alpha \right\} = f_s^\alpha(s)g_s^\alpha(s).$$

Starting from this point of view, we will focus on special Volterra integral equations where the kernel is a difference kernel. Note that we will perform the Yang–Laplace transform method and the inverse Yang–Laplace transform method using the elementary transforms given in [13]. By taking Yang–Laplace transform of both sides of local fractional Volterra integral equations we deduce that

$$\mathcal{U}(s) = \mathcal{M}(s) + \lambda\mathcal{K}(s)\mathcal{U}(s) \tag{3.1}$$

where $\mathcal{U}(s) = \mathcal{L}_\alpha\{u(x)\}$, $\mathcal{M}(s) = \mathcal{L}_\alpha\{m(x)\}$ and $\mathcal{K}(s) = \mathcal{L}_\alpha\{K(x,t)\}$. Solving (3.1) for $\mathcal{U}(s)$ gives

$$\mathcal{U}(s) = \frac{\mathcal{M}(s)}{1 - \lambda\mathcal{K}(s)} \quad 1 - \lambda\mathcal{K}(s) \neq 0. \tag{3.2}$$

The solution $u(x)$ is obtained by taking the inverse Yang–Laplace transform of both sides of (3.2) where we find

$$u(x) = \mathcal{L}_\alpha^{-1} \left\{ \frac{\mathcal{M}(s)}{1 - \lambda\mathcal{K}(s)} \right\}. \tag{3.3}$$

Note that the right side of (3.3) can be solved by using the Yang–Laplace transform of ordinary functions given in [13].

Remark 3.4. If the unknown function $u(x)$ appears only under the integral sign of Volterra equation, the local fractional integral equation is named a first kind local fractional Volterra integral equation. Under this circumstances the Yang–Laplace transform technique solution of first kind local fractional Volterra integral equation given as

$$u(x) = \mathcal{L}_\alpha^{-1} \left\{ \frac{\mathcal{M}(s)}{\mathcal{K}(s)} \right\}.$$

where $\mathcal{K}(s) \neq 0$.

3.2. Yang–Laplace transform method for local fractional Volterra integro–differential equations of the second kind

The local fractional Volterra integro–differential equation depend on the difference kernel can be expressed as

$$u^{(n\alpha)}(x) = m(x) + \frac{\lambda}{\Gamma(1+\alpha)} \int_0^x K(x-t)u(t)(dt)^\alpha. \tag{3.4}$$

In order to solve local fractional Volterra integro–differential equations by using the Yang–Laplace transform method, it is key feature to use the Yang–Laplace transforms of the local fractional derivatives of $u(x)$. We can easily deduce that

$$\mathcal{L}_\alpha\{u^{(n\alpha)}(x)\} = s^{n\alpha}\mathcal{L}_\alpha\{u(x)\} - s^{(n-1)\alpha}u(0) - s^{(n-2)\alpha}u^{(\alpha)}(0) - \cdots - u^{((k-1)\alpha)}(0).$$

The Yang–Laplace transform method can be performed likewise by following the same steps used before in previous subsection. Namely we first perform the Yang–Laplace transform to both sides of (3.4), use the proper Yang–Laplace transform for the local fractional derivative of $u(x)$, and then solve for $\mathcal{U}(s)$. After then we use the inverse Yang–Laplace transform of both sides of the resultant equation to get the solution $u(x)$ of the equation.

3.3. Yang–Laplace transform method for local fractional Volterra integro–differential equations of the first kind

The standard format of the local fractional Volterra integro–differential equation of the first kind depend on difference kernel is given by

$$\frac{1}{\Gamma(1+\alpha)} \int_0^x K_1(x-t)u(t)(dt)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_0^x K_2(x-t)u^{(n\alpha)}(t)(dt)^\alpha = m(x) \quad K_2(x-t) \neq 0, \tag{3.5}$$

where initial conditions are prescribed. Taking the Yang–Laplace transform of both sides of (3.5) gives

$$\mathcal{L}_\alpha\{K_1(x-t) * u(t)\} + \mathcal{L}_\alpha\{K_2(x-t) * u^{(n\alpha)}(t)\} = \mathcal{L}_\alpha\{m(x)\},$$

so that

$$\mathcal{K}_1(s)\mathcal{U}(s) + \mathcal{K}_2(s)(s^{n\alpha}\mathcal{L}_\alpha\{u(x)\} - s^{(n-1)\alpha}u(0) - s^{(n-2)\alpha}u^{(\alpha)}(0) - \cdots - u^{((k-1)\alpha)}(0)) = \mathcal{M}(s),$$

where $\mathcal{U}(s) = \mathcal{L}_\alpha\{u(x)\}$, $\mathcal{M}(s) = \mathcal{L}_\alpha\{m(x)\}$, $\mathcal{K}_1(s) = \mathcal{L}_\alpha\{K(x,t)\}$ and $\mathcal{K}_2(s) = \mathcal{L}_\alpha\{K(x,t)\}$. Using the provided initial conditions and solving for $\mathcal{U}(s)$ we deduce that

$$\mathcal{U}(s) = \frac{\mathcal{M}(s) + \mathcal{K}_2(s)(s^{(n-1)\alpha}u(0) - s^{(n-2)\alpha}u^{(\alpha)}(0) - \cdots - u^{((k-1)\alpha)}(0))}{\mathcal{K}_1(s) + s^{n\alpha}\mathcal{K}_2(s)}, \tag{3.6}$$

on the condition

$$\mathcal{K}_1(s) + s^{n\alpha}\mathcal{K}_2(s) \neq 0.$$

By taking the inverse Laplace transform of both sides of (3.6), the exact solution is easily obtained.

4. Yang–Laplace transform method for local fractional Abel's integral equations

Abel in 1823 examined the motion of a small particle that slides down along a smooth unknown curve, in a vertical plane, under the affect of the gravity. The particle takes the time $m(x)$ to move from the highest point of vertical height x to the lowest point 0 on the curve. The Abel's problem is derived to find the equation of that curve.

4.1. Yang–Laplace transform method for local fractional Abel's integral equations

Local fractional version of Abel's integral equations can be represented as

$$m(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{1}{\sqrt{x^\alpha - t^\alpha}} u(t)(dt)^\alpha, \tag{4.1}$$

where $m(x)$ is a prespecified data function, and $u(x)$ is the solution that will be determined. It is to be noted that Abel's local fractional integral equation (4.1) is also named local fractional Volterra integral equation of the first kind. Moreover the kernel $K(x,t)$ in Abel's local fractional integral equation (4.1) is

$$K(x,t) = \frac{1}{\sqrt{x^\alpha - t^\alpha}},$$

where

$$K(x,t) \to \infty \quad \text{as} \quad t \to x.$$

It is interesting to point out that although Abel's local fractional integral equation is a local fractional Volterra integral equation of the first kind, the solution technique of local fractional Volterra integral equation of the first kind except Yang–Laplace transform method are not practicable here. For instance the series solution cannot be performed in this case particularly if $u(x)$ is not analytic. Furthermore, converting Abel's local fractional integral equation to a second kind local fractional Volterra equation is not existing since we cannot use Leibniz rule due to the singularity behaviour of the kernel in (4.1).

In order to solve local fractional Abel's integral equations we need to take the Yang–Laplace transform of both sides of (4.1) leads to

$$\mathcal{L}_\alpha\{m(x)\} = \mathcal{L}_\alpha\{u(x)\}\mathcal{L}_\alpha\{x^{-\alpha/2}\},$$

or equivalently

$$\mathcal{M}(s) = \mathcal{U}(s)\frac{\Gamma(1-\frac{\alpha}{2})}{s^{\frac{\alpha}{2}}},$$

that gives

$$\mathcal{U}(s) = \mathcal{M}(s)\frac{s^{\frac{\alpha}{2}}}{\Gamma(1-\frac{\alpha}{2})}, \tag{4.2}$$

where Γ is the gamma function. The last equation (4.2) can be rewritten as

$$\mathcal{U}(s) = \frac{s^\alpha}{[\Gamma(1-\frac{\alpha}{2})]^2}\frac{\Gamma(1-\frac{\alpha}{2})}{s^{\frac{\alpha}{2}}}\mathcal{M}(s),$$

which can be rewritten by

$$\mathcal{L}_\alpha\{u(x)\} = \frac{s^\alpha}{[\Gamma(1-\frac{\alpha}{2})]^2}\mathcal{L}_\alpha\{v(x)\}, \tag{4.3}$$

where

$$v(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x (x^\alpha - t^\alpha)^{-1/2} m(t)(dt)^\alpha.$$

Using the fact that

$$\mathcal{L}_\alpha\{v^{(\alpha)}(x)\} = s^\alpha \mathcal{L}_\alpha\{v(x)\} - v(0),$$

into (4.3) we deduce that

$$\mathcal{L}_\alpha\{u(x)\} = \frac{1}{[\Gamma(1-\frac{\alpha}{2})]^2}\mathcal{L}_\alpha\{v^\alpha(x)\}. \tag{4.4}$$

Applying the inverse Yang–Laplace transform to both sides of (4.4) gives the formula

$$u(x) = \frac{1}{[\Gamma(1-\frac{\alpha}{2})]^2}\frac{d^\alpha}{dx^\alpha}\int_0^x \frac{1}{\sqrt{x^\alpha - t^\alpha}}m(t)(dt)^\alpha, \tag{4.5}$$

that will be used for the identification of the solution $u(x)$. Notice that the formula (4.5) will be used for solving Abel's local fractional integral equation, and it is not necessary to use Yang–Laplace transform method for each problem. Abel's problem given by (4.1) can be solved straight–forwardly by using the formula (4.5) where the unknown function $u(x)$ has been replaced by the given function $m(x)$.

4.2. Yang–Laplace transform method for the generalized local fractional Abel's integral equations

The generalized local fractional Abel's integral equations are the singular local fractional integral equation given as

$$m(x) = \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{1}{(x^\alpha - t^\alpha)^\theta}u(t)(dt)^\alpha, \quad 0 < \theta < 1, \tag{4.6}$$

where θ are known constants such that $0 < \theta < 1$, $m(x)$ is a predetermined data function, and $u(x)$ is the solution that will be determined. The Abel's problem discussed previous subsection is a special case of the generalized equation where $\theta = \frac{1}{2}$.

To construct a formula that will be used for solving the generalized Abel's local fractional integral equation (4.6), we will perform the Yang-Laplace transform technique in the same way to the approach followed before. By taking the Yang–Laplace transforms of both sides of (4.6) leads to

$$\mathcal{L}_\alpha\{m(x)\} = \mathcal{L}_\alpha\{u(x)\}\mathcal{L}_\alpha\{x^{-\alpha\theta}\},$$

or equivalently

$$\mathcal{M}(s) = \mathcal{U}(s)\frac{\Gamma(1-\alpha\theta)}{s^{(1-\theta)\alpha}},$$

that gives

$$\mathcal{U}(s) = \mathcal{M}(s)\frac{s^{(1-\theta)\alpha}}{\Gamma(1-\alpha\theta)}, \tag{4.7}$$

where Γ is the gamma function. The last equation (4.7) can be rewritten as

$$\mathcal{U}(s) = \frac{s^\alpha}{\Gamma(1-\alpha\theta)\Gamma(1-\alpha+\alpha\theta)}\frac{\Gamma(1-\alpha+\alpha\theta)}{s^{\alpha\theta}}\mathcal{M}(s),$$

which can be rewritten by

$$\mathcal{L}_\alpha\{u(x)\} = \frac{s^\alpha}{\Gamma(1-\alpha\theta)\Gamma(1-\alpha+\alpha\theta)}\mathcal{L}_\alpha\{v(x)\}, \tag{4.8}$$

where

$$v(x) = \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{1}{(x^\alpha - t^\alpha)^{1-\theta}}m(t)(dt)^\alpha.$$

Using the fact that

$$\mathcal{L}_\alpha\{v^{(\alpha)}(x)\} = s^\alpha\mathcal{L}_\alpha\{v(x)\} - v(0),$$

into (4.8) we deduce that

$$\mathcal{L}_\alpha\{u(x)\} = \frac{1}{\Gamma(1-\alpha\theta)\Gamma(1-\alpha+\alpha\theta)}\mathcal{L}_\alpha\{v^\alpha(x)\}. \tag{4.9}$$

Performing the inverse Yang–Laplace transform to both sides of (4.9) gives the formula

$$u(x) = \frac{1}{\Gamma(1-\alpha\theta)\Gamma(1-\alpha+\alpha\theta)}\frac{d^\alpha}{dx^\alpha}\int_0^x \frac{1}{(x^\alpha - t^\alpha)^{1-\theta}} m(t)(dt)^\alpha \quad 0<\theta<1. \tag{4.10}$$

Notice that the exponent of the kernel of the generalized Abel's local fractional integral equation is $-\theta$, but the exponent of the kernel of the formulae (4.10) is $(\theta-1)$. The unknown function in (4.6) has been replaced by $m(t)$ in (4.10).

4.3. Yang–Laplace transform method for the main generalized local fractional Abel's integral equations

It is helpful to present a further generalization to local fractional Abel's integral equation by considering a generalized singular kernel instead of $K(x,t) = \frac{1}{\sqrt{x^\alpha - t^\alpha}}$. The generalized version of kernel will be of the form

$$K(x,t) = \frac{1}{[n(x)-n(t)]^\theta}, \quad 0<\theta<1,$$

where $n(t): \mathbb{R} \to \mathbb{R}^\alpha$. The main generalized local fractional Abel's integral equation is given by

$$m(x) = \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{1}{[n(x)-n(t)]^\theta} u(t)(dt)^\alpha, \quad 0<\theta<1, \tag{4.11}$$

where $n(t): \mathbb{R} \to \mathbb{R}^\alpha$ is strictly monotonically increasing and differentiable function in some interval $0<t<b$, and $\frac{d^\alpha}{dt^\alpha}n(t)$ for every t in the interval. The solution $u(x)$ of (4.11) is given by

$$u(x) = \frac{1}{\Gamma(1+\alpha)B(\theta, 1-\theta)}\frac{d^\alpha}{dx^\alpha}\int_0^x \frac{1}{[n(x)-n(t)]^{(1-\theta)}} m(t)\frac{d^\alpha}{dt^\alpha}n(t)(dt)^\alpha \quad 0<\theta<1,$$

where

$$B(x,y) = \frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{(x-1)\alpha}(1-t)^{(y-1)\alpha}(dt)^\alpha.$$

To prove this formula, we consider the integral

$$\int_0^x \frac{m(y)\frac{d^\alpha}{dt^\alpha}n(y)}{[n(x)-n(y)]^{(1-\theta)}}(dy)^\alpha$$

and substitute for $m(x)$ from (4.11) to obtain

$$\frac{1}{\Gamma(1+\alpha)}\int_0^x \int_0^y \frac{u(t)\frac{d^\alpha}{dt^\alpha}n(y)}{[n(y)-n(t)]^\theta[n(x)-n(y)]^{(1-\theta)}}(dt)^\alpha(dy)^\alpha,$$

where by changing the order of integration we find

$$\frac{1}{\Gamma(1+\alpha)}\int_0^x u(t)\int_t^x \frac{\frac{d^\alpha}{dt^\alpha}n(y)}{[n(y)-n(t)]^\theta[n(x)-n(y)]^{(1-\theta)}}(dy)^\alpha(dt)^\alpha.$$

We can prove that

$$\frac{1}{\Gamma(1+\alpha)}\int_t^x \frac{\frac{d^\alpha}{dt^\alpha}n(y)}{[n(y)-n^(t)]^\theta[n(x)-n(y)]^{(1-\theta)}}(dy)^\alpha = \Gamma(1+\alpha)B(\theta, 1-\theta).$$

This means that

$$\int_0^x \frac{m(y)\frac{d^\alpha}{dt^\alpha}n(y)}{[n(x)-n(y)]^{(1-\theta)}}(dy)^\alpha = B(\theta,1-\theta)\int_0^x u(t)(dt)^\alpha. \tag{4.12}$$

Differentiating both sides of (4.12) gives

$$u(x) = \frac{1}{\Gamma(1+\alpha)B(\theta,1-\theta)}\frac{d^\alpha}{dx^\alpha}\int_0^x \frac{1}{[n(x)-n(t)]^{(1-\theta)}}m(t)\frac{d^\alpha}{dt^\alpha}n(t)(dt)^\alpha \quad 0<\theta<1.$$

5. Examples

The Yang–Laplace transform method for solving local fractional Volterra and Abel's integral equations of the first and second kinds will be exemplified by running the following examples.

Example 1

Considering the following local fractional Volterra integral equation of second kind

$$u(x) = 1 + \frac{1}{\Gamma(1+\alpha)}\int_0^x u(t)(dt)^\alpha.$$

The exact solution of Example 1 is the form

$$u(x) = E_\alpha(x^\alpha).$$

Example 2

Considering the following local fractional Volterra integro-differential equation of second kind

$$u^{(2\alpha)}(x) = -1^\alpha - \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{[\Gamma(1+\alpha)]^2}\int_0^x (x^\alpha - t^\alpha)u(t)(dt)^\alpha,$$

with the initial condition

$$u(0) = 1^\alpha \quad \text{and} \quad u^{(\alpha)}(0) = 1^\alpha.$$

The exact solution of Example 2 is the form

$$u(x) = \sin_\alpha(x^\alpha) + \cos_\alpha(x^\alpha).$$

Example 3

Considering the following local fractional Volterra integro-differential equation of first kind

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^x \cos_\alpha(x^\alpha - t^\alpha)u(t)(dt)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_0^x \sin_\alpha(x^\alpha - t^\alpha)u^{(3\alpha)}(t)(dt)^\alpha \\ &\quad = 1^\alpha + \sin_\alpha(x^\alpha) + \cos_\alpha(x^\alpha),\end{aligned}$$

with the initial condition

$$u(0) = 1^\alpha, \quad u^{(\alpha)}(0) = 1^\alpha \quad \text{and} \quad u^{(2\alpha)}(0) = -1^\alpha.$$

The exact solution of Example 3 is the form

$$u(x) = \frac{x^\alpha}{\Gamma(1+\alpha)} + \cos_\alpha(x^\alpha).$$

Example 4

Considering the following local fractional Abel's integral equation

$$[\Gamma(1-\frac{\alpha}{2})]^2 = \int_0^x \frac{1}{\sqrt{x^\alpha - t^\alpha}} u(t)(dt)^\alpha.$$

The exact solution of Example 4 is the form

$$u(x) = \frac{1}{\sqrt{x^\alpha}}.$$

6. Concluding remarks

The main goal of the current study was to determine the solution of local fractional Volterra and Abel's integro–differential equations using the Yang–Laplace transform. The present study confirms previous findings in case of $\alpha = 1$ and contributes additional evidence. In order to validate our scheme we presented some examples. It is recommended that further research be undertaken in the following areas: system of local fractional integral equations and nonlinear local fractional integral equations.

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