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Characterizations of the set containment with star-shaped constraints

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Abstract

In this paper, we first give a separation theorem for a closed star-shaped set at the origin and a point outside it in terms of separation by an upper semi-continuous and super-linear function, and also, we introduce a ν-star-shaped-conjugation. By using this facts, we present characterizations of the set containment with infinite star-shaped constraints defined by weak inequalities. Next, we give characterizations of the set containment with infinite evenly radiant constraints defined by strict or weak inequalities. Finally, we give a characterization of the set containment with an upper semi-continuous and radiant constraint, in a reverse star-shaped set, defined by a co-star-shaped constraint. These results have many applications in Mathematical Economics, in particular, in Utility Theory.

Keywords: star-shaped function; co-star-shaped function; set containment; ν-star-shaped-conjugation; weak separation.
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1. Introduction

The study of separation properties of closed star-shaped sets received increasing attention in recent years, starting with [10, 11, 16] in Euclidean spaces, and [19, 20] in infinite dimensional spaces. The separation property plays a crucial role in the study of convex optimization problems. The separation property for two convex sets easily follows from a simple fact. If a point does not belong to a closed convex set, then this point can be separated from this set. Generalizations of this assertion were studied in the framework of abstract convexity. Note that in contrast with the classical case, the

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nonlinear separation of a point from a set does not imply the separation property for two sets. A very interesting notion of separability of star-shaped sets by a finite collection of linear functionals in $(\mathbb{R}^n)^*$ has been introduced and studied in [16]. In particular, it was shown that every point x does not belong to a closed star-shaped set U at the origin in a normed linear space X can be separated from U by means of a continuous super-linear function p defined on X such that $p(x) > 1$ and $p(u) \leq 1$ for all $u \in U$. Recently, separability of two disjoint star-shaped sets in terms of separation by a sequence of linear functionals $\{x_i^*\}_{i\in\mathbb{N}} \subset X^*$ defined on a Banach space X was given in [8].

Now, we use the later separability for two star-shaped sets and give various characterizations of the set containment in a Banach space X. The set containment problem consists of characterizing the inclusion $F \subseteq G$, motivated by general non-polyhedral knowledge-based data classification, where

$$F := \{x \in X : f_i(x) < 0,\ \forall\ i \in I,\ f_j(x) \leq 0,\ \forall\ j \in J\},$$

$$G := \{x \in X : g_s(x) \leq 0,\ \forall\ s \in S,\ g_t(x) \geq 0,\ \forall\ t \in T\},$$

and I, J, S, T are index sets, $I \cap J = \emptyset,\ I \cup J \neq \emptyset,\ S \cap T = \emptyset,\ S \cup T \neq \emptyset$, and $f_r,\ g_\ell : X \longrightarrow [-\infty, +\infty]$ are functions.

The set containment characterizations have been studied by many researchers, see [3, 4, 5, 6, 14]. The first characterizations were given by Mangasarian [6] for linear systems and for systems involving differentiable convex functions, with finite index sets I and J. These dual characterizations are provided in terms of Farkas' Lemma and the duality theorems of convex programming problems. Also, Goberna and Rodríguez [4] provided characterizations of the set containment for linear systems containing strict inequalities and weak inequalities as well as equalities. Furthermore, Goberna, Jeyakumar and Dinh [3] characterized set containments with convex inequalities which can be either weak or strict. These dual characterizations are also provided by the Fenchel's conjugate. Recently, dual characterizations of the set containments with strict cone-convex inequalities in Banach spaces were given in [2].

It is well known that the Fenchel's conjugate plays very important roles to consider dual problems of convex minimization problems. Similar researches of conjugates of quasi-convex functions have been studied. But the epigraph of a star-shaped function is no longer convex. This causes a fundamental difference between convex and star-shaped duality. For general star-shaped functions, we have to use extra-parameters to obtain dual representations. Similar to the λ-quasi-conjugate, we have to use the λ-star-shaped-conjugate ($\lambda \in \mathbb{R}$). The λ-quasi-conjugate has been used by Greenberg and Pierskalla [3], which has an extra-parameter, and plays an important role in quasi-convex optimization and in the theory of surrogate duality corresponding to that of the Fenchel's conjugate in convex optimization and Lagrangian duality. Singer [12, 13] introduced the λ-semi-conjugate, which also has an extra-parameter, and studied the level set of the λ-semi-conjugate and quasi-convex optimization. If we want to avoid the extra-parameter, then we often need to restrict the class of quasi-convex functions. Thach [17, 18] established two dualities without the extra-parameter for a general quasi-convex minimization (maximization) problem by using concepts of H-quasi-conjugate and R-quasi-conjugate.

More recently, Suzuki and Kuroiwa [14] established the set containment characterizations with I a finite set and J an arbitrary set, assuming the quasi-convexity of f_i for each $i \in I$, the linearity (or the quasi-concavity) of h_j for each $j \in J$, and inequalities in F are strict and in G are strict (or weak, respectively). These dual characterizations are provided in terms of level sets of H-quasi-conjugate and R-quasi-conjugate functions. Moreover, they established [15] the set containment characterizations, assuming that all f_i are quasi-convex, all h_j are linear, I and J are possibly infinite, and the inequalities in F and G can be either weak or strict. Furthermore, they considered a reverse convex system (i.e., all f_i are quasi-convex and all h_j are quasi-concave), containing both weak and strict inequalities. These dual characterizations are provided in terms of level sets of λ-quasi-conjugate and λ-semi-conjugate functions.

In this paper, we establish the set containment characterizations, assuming that all f_i are star-shaped functions, all h_j are upper semi-continuous and super-linear functions, I and J are arbitrary index sets, and the inequalities in F and G can be either weak or strict. Furthermore, we consider a reverse star-shaped system (i.e., all f_i are upper semi-continuous and radiant functions and all h_j are co-star-shaped functions), containing both weak or strict inequalities. These dual characterizations are provided in terms of level sets of ν-star-shaped-conjugation functions.

The structure of the paper is as follows. In Section 2, we provide definitions, notations and preliminary results related to star-shaped functions and their ν-conjugate functions. In Section 3, we first give a separation theorem for a closed star-shaped set at the origin and a point outside it, and by using this fact, we present characterizations of the set containment with infinite star-shaped constraints. A separation theorem for a closed star-shaped set at the origin and a point outside it in terms of separation by an upper semi-continuous and super-linear function is given in Section 4, and moreover, by using this fact, the characterizations of the set containment with infinite evenly radiant constraints (strict or weak inequalities) are presented in Section 4. In Section 5, we give a characterization of the set containment with an upper semi-continuous and radiant constraint, in a reverse star-shaped set, defined by a co-star-shaped constraint.

2. Preliminaries

We start this section by fixing notations and preliminaries that will be used later. Let X be a real Banach space with a Schauder basis $\{x_n\}_{n\geq 1} \subset X$, and let X^* be its dual space with $\langle \cdot, \cdot \rangle : X \times X^* \longrightarrow \mathbb{R}$ is the duality pairing between X and X^*. For any subset A of X, we denote by $\operatorname{int} A$ the interior of A and by $\operatorname{cl} A$ the closure of A.

We recall that a function $f : X \longrightarrow \mathbb{R}$ is said to be super-linear if
(1) $f(x + y) \geq f(x) + f(y)$ for all $x, y \in X$.
(2) $f(\lambda x) = \lambda f(x)$ for all $x \in X$ and all $\lambda > 0$.

Definition 2.1. *Let L be the set of all upper semi-continuous (u.s.c) and super-linear functions $f : X \longrightarrow \mathbb{R}$.*

Remark 2.1. *It should be noted that $0 \in L$ and L is a convex cone that contains X^*, where X^* is the dual space of the Banach space X.*

Definition 2.2. *[10] A subset U of X is called a radiant set if $\lambda U \subseteq U$ for all $\lambda \in (0, 1]$.*

Definition 2.3. *[10] A function $f : X \longrightarrow [-\infty, +\infty]$ is called radiant if*

$$f(\lambda x) \leq \lambda f(x),\ \forall\ x \in X,\ \forall\ \lambda \in (0, 1].$$

Remark 2.2. *It is easy to see that if the function $f : X \longrightarrow [-\infty, +\infty]$ is radiant, then every lower level set of f,*

$$[f \leq \alpha] := \{x \in X : f(x) \leq \alpha\}$$

is a radiant set for each $\alpha \geq 0$. *Moreover, every strict lower level set of* f,

$$[f < \alpha] := \{x \in X : f(x) < \alpha\}$$

is a radiant set for each $\alpha \geq 0$.

Definition 2.4. *[10] Let* S *be a non-empty subset of* X.
(1) *The kernel of* S *is the set of all points* $s \in S$ *such that* $s + \lambda(x - s) \in S$ *for all* $x \in S$ *and all* $\lambda \in [0, 1]$, *i.e.,*

$$kernS := \{s \in S : s + \lambda(x - s) \in S,\ \forall\ x \in S,\ \forall\ \lambda \in [0, 1]\}.$$

(2) *A non-empty subset* S *of* X *is called star-shaped if* $kernS \neq \emptyset$.

Definition 2.5. *[10] A subset* U *of* X *is called a star-shaped set at the origin* (*at* $0 \in X$), *if* $0 \in kernU$, *or equivalently,* $\lambda U \subseteq U$ *for all* $\lambda \in [0, 1]$.

Definition 2.6. *A function* $f : X \longrightarrow [-\infty, +\infty]$ *is called star-shaped if every lower level set of* f,

$$[f \leq \alpha] := \{x \in X : f(x) \leq \alpha\},$$

is a star-shaped set at the origin for each $\alpha \geq f(0)$.

The proof of the following proposition is similar to that of Proposition 2.4 in [19], and therefore, we omit its proof.

Proposition 2.1. *A function* $f : X \longrightarrow [-\infty, +\infty]$ *is star-shaped if and only if every strict lower level set of* f,

$$[f < \alpha] := \{x \in X : f(x) < \alpha\},$$

is a star-shaped set at the origin for each $\alpha > f(0)$.

It is worth noting that if $f : X \longrightarrow [-\infty, +\infty]$ is a star-shaped function, then, $f(0) = -\infty$ or $f(0) = \min_{x \in X} f(x)$ or $f \equiv +\infty$. The usefulness of star-shaped functions in Mathematical Economics (particularly in Utility Theory) has been shown in [21].

Definition 2.7. *A subset* U *of* X *is called reverse star-shaped if its complement* $U^c := X \setminus U$ *is a star-shaped set at the origin, i.e., if either* $U = X$ *or* $0 \notin U$ *and* $x \in U$, $\lambda \geq 1$ *imply that* $\lambda x \in U$.

Remark 2.3. *It should be noted that the empty set* $\emptyset$ *and the space* X *are both star-shaped and reverse star-shaped. In the sequel, we will at times refer to star-shaped sets at the origin and to reverse star-shaped sets to mean proper star-shaped and proper reverse star-shaped sets, i.e., sets which are different from both* $\emptyset$ *and* X.

Definition 2.8. *A function* $f : X \longrightarrow [-\infty, +\infty]$ *is called co-star-shaped if every upper level set of* f,

$$[f \geq \alpha] := \{x \in X : f(x) \geq \alpha\},$$

is a reverse star-shaped set for each $\alpha > f(0)$.

Definition 2.9. *Let A be a non-empty subset of X, and let $\nu \in \mathbb{R}$. We define the following different ν-polarities for the set A.*

$$\begin{aligned} A_\nu^{\vee} &:= \{p \in L : p(a) \leq \nu, \ \forall\ a \in A\}, \\ A_\nu^{\wedge} &:= \{p \in L : p(a) < \nu, \ \forall\ a \in A\}, \\ A_\nu^{\triangledown} &:= \{p \in L : p(a) \geq \nu, \ \forall\ a \in A\}, \\ A_\nu^{\vartriangle} &:= \{p \in L : p(a) > \nu, \ \forall\ a \in A\}. \end{aligned}$$

Remark 2.4. *It is easy to check that $A_\nu^{\vee}$ and $A_\nu^{\wedge}$ are star-shaped sets at the origin for each $\nu \in (0, +\infty)$ (also, see Remark 2.1). Moreover, $A_\nu^{\vee}$ is closed in L under the point-wise convergence of functions for each $\nu \in \mathbb{R}$. Consequently, $A_\nu^{\vee}$ is a closed star-shaped set at the origin in L for each $\nu \in (0, +\infty)$.*

Remark 2.5. *Let $A \subseteq B \subseteq X$, and let $\star \in \{\vee, \wedge, \triangledown, \vartriangle\}$ and $\nu \in \mathbb{R}$. Then, $B_\nu^{\star} \subseteq A_\nu^{\star}$.*

Proof: This is an immediate consequence of Definition 2.9. ■

In the following, we define the ν-star-shaped-conjugation (ν-star-shaped-duality) of functions based upon the ν-polarities, which given in Definition 2.9 (also, for similar definitions, see [9, 20]).

Definition 2.10. *Let $f : X \longrightarrow [-\infty, +\infty]$ be a function, and let $\nu \in \mathbb{R}$. We define the ν-star-shaped-conjugates of the function f as follows.*

$$\begin{aligned} f_\nu^{\vee} : L \longrightarrow [-\infty, +\infty], \ \text{by} \ \ f_\nu^{\vee}(p) &:= \nu + \sup\{-f(x) : x \in X, \ p(x) > \nu\}, \\ f_\nu^{\wedge} : L \longrightarrow [-\infty, +\infty], \ \text{by} \ \ f_\nu^{\wedge}(p) &:= \nu + \sup\{-f(x) : x \in X, \ p(x) \geq \nu\}, \\ f_\nu^{\triangledown} : L \longrightarrow [-\infty, +\infty], \ \text{by} \ \ f_\nu^{\triangledown}(p) &:= \nu + \sup\{-f(x) : x \in X, \ p(x) < -\nu\}, \\ f_\nu^{\vartriangle} : L \longrightarrow [-\infty, +\infty], \ \text{by} \ \ f_\nu^{\vartriangle}(p) &:= \nu + \sup\{-f(x) : x \in X, \ p(x) < \nu\}. \end{aligned}$$

Proposition 2.2. *Let $f : X \longrightarrow [-\infty, +\infty]$ be a function, and let $\star \in \{\vee, \wedge, \triangledown, \vartriangle\}$ and $\nu \in \mathbb{R}$. Then,*

$$\{x \in X : f(x) < \alpha\}_\nu^{\star} = \{p \in L : f_\nu^{\star}(p) \leq \nu - \alpha\},$$

for each $\alpha \in \mathbb{R}$.

Proof: This is an immediate consequence of Definition 2.10. ■

The proof of the following result is similar to that one of [9], and therefore, we omit its proof.

Proposition 2.3. *Let I be an arbitrary index set, and let $\nu \in \mathbb{R}$. Let $\{A_i\}_{i\in I}$ be a collection of subsets of X and $\star \in \{\vee, \wedge, \triangledown, \vartriangle\}$. Then,*

$$\left(\bigcup_{i\in I} A_i\right)_\nu^{\star} = \bigcap_{i\in I} A_{i,\nu}^{\star} \ .$$

3. Characterizations of the Set Containment with Infinite Star-Shaped Constraints

In this section, we first define the closed star-shaped hull of a subset A of X. Next, we give a separation theorem for a closed star-shaped set at the origin and a point outside it. Finally, by using this fact, we present characterizations of the set containment with infinite star-shaped constraints defined by weak inequalities. We start with the following definition.

Definition 3.1. *Let $\nu \in \mathbb{R}$. The ν-bipolar of a subset A of X, denoted by $A_\nu^{\vee\vee}$, and defined as follows:*

$$A_\nu^{\vee\vee} := \{x \in X : p(x) \leq \nu, \ \forall \ p \in A_\nu^{\vee}\}.$$

Remark 3.1. *In view of Remark 2.4, $A_\nu^{\vee\vee}$ is a closed star-shaped set at the origin in X for each $\nu \in (0, +\infty)$. Moreover, it is clear that $A \subseteq A_\nu^{\vee\vee}$ for each $\nu \in \mathbb{R}$. Note that $0 \in A_\nu^{\vee\vee}$ for each $\nu \in (0, +\infty)$ (because $p(0) = 0$ for each $p \in L$). Also, in view of Definition 2.2, it follows that $A_\nu^{\vee\vee}$ is a closed radiant set for each $\nu \in (0, +\infty)$.*

Definition 3.2. *The closed star-shaped hull of a subset A of X, denoted by $\operatorname{cls} A$, and defined as follows:*

$$\operatorname{cls} A := \bigcap \{B \subseteq X : B \textit{ is a closed star-shaped set at the origin and } B \supseteq A\},$$

i.e., $\operatorname{cls} A$ is the smallest closed star-shaped subset of X at the origin that contains A.

In the following, we give a separation theorem that separates strictly a closed star-shaped set at the origin and a point outside it. The proof is similar to that of Theorem 3.1 in [20], and therefore, we omit its proof.

Theorem 3.1. *Let A be a closed star-shaped subset of X at the origin, and let $x \in X$ be a point such that $x \notin A$. Then there exists a continuous super-linear function $p_1 : X \longrightarrow \mathbb{R}$ such that $p_1(x) > 1$ and $p_1(a) \leq 1$ for all $a \in A$. It should be noted that $p_1 \in L$, where L defined in Definition 2.1.*

Corollary 3.1. *Let A be a closed star-shaped subset of X at the origin, and let $x \in X$ be a point such that $x \notin A$. Let $\nu \in (0, +\infty)$. Then there exists $p \in L$ such that $p(x) > \nu$ and $p(a) \leq \nu$ for all $a \in A$.*

Proof: By Theorem 3.1, there exists a continuous super-linear function $p_1 : X \longrightarrow \mathbb{R}$ such that $p_1(x) > 1$ and $p_1(a) \leq 1$ for all $a \in A$. Now, put $p := \nu p_1$. Then, by using Remark 2.1, $p \in L$. Moreover, $p(x) > \nu$ and $p(a) \leq \nu$ for all $a \in A$. ∎

Theorem 3.2. *Let A be a subset of X, and let $\nu \in (0, +\infty)$. Then, $\operatorname{cls} A = A_\nu^{\vee\vee}$, and hence, in particular, $A = A_\nu^{\vee\vee}$ if and only if A is closed and star-shaped set at the origin. (It is worth noting that $\operatorname{cls} A = A$ if and only if A is closed and star-shaped set at the origin.)*

Proof: Since, in view of Remark 3.1, $A_\nu^{\vee\vee}$ is a closed and star-shaped set at the origin and $A \subseteq A_\nu^{\vee\vee}$, it follows from Definition 3.2 that $\operatorname{cls} A \subseteq A_\nu^{\vee\vee}$. Conversely, assume that $x \notin \operatorname{cls} A$. Then, by Corollary 3.1, there exists $p \in L$ such that $p(x) > \nu$ and $p(a) \leq \nu$ for all $a \in A$. So, by using Definition 2.9, there exists $p \in A_\nu^{\vee}$ such that $p(x) > \nu$. Therefore, we conclude from Definition 3.1 that $x \notin A_\nu^{\vee\vee}$. Hence, $A_\nu^{\vee\vee} \subseteq \operatorname{cls} A$, which implies that $\operatorname{cls} A = A_\nu^{\vee\vee}$. ∎

Proposition 3.1. *Let I be an arbitrary index set, and let A_i be a closed star-shaped subset of X at the origin $(i \in I)$ and $\nu \in (0, +\infty)$. Then,*

$$\left(\bigcap_{i\in I} A_i\right)_\nu^\vee = \operatorname{cls} \bigcup_{i\in I} A_{i,\nu}^\vee.$$

Proof: Due to Proposition 2.3, we have

$$\left(\bigcup_{i\in I} A_{i,\nu}^\vee\right)_\nu^\vee = \bigcap_{i\in I} A_{i,\nu}^{\vee\vee}. \tag{3.1}$$

Since A_i is a closed and star-shaped set at the origin, it follows from Theorem 3.2 that $A_{i,\nu}^{\vee\vee} = A_i$ for each $i \in I$. Therefore, by using (3.1), we obtain

$$\left(\bigcup_{i\in I} A_{i,\nu}^\vee\right)_\nu^\vee = \bigcap_{i\in I} A_i. \tag{3.2}$$

Now, put $B := \bigcup_{i\in I} A_{i,\nu}^\vee$. So, it follows from (3.2) that

$$B_\nu^\vee = \bigcap_{i\in I} A_i. \tag{3.3}$$

On the other hand, in view of Theorem 3.2, one has

$$\operatorname{cls} B = B_\nu^{\vee\vee}.$$

This together with (3.3) implies that

$$\operatorname{cls} B = \left(\bigcap_{i\in I} A_i\right)_\nu^\vee,$$

which completes the proof. ∎

Proposition 3.2. *Let A be a subset of X, and let $0 \neq p \in L$ be arbitrary and $\nu \in \mathbb{R}$. Then the following assertions are equivalent.*
(i) $A \subseteq \{x \in X : p(x) \leq \nu\}$.
(ii) $p \in A_\nu^\vee$.

Proof: $[(i) \Longleftrightarrow (ii)]$. We have, by Definition 2.9,

$$\begin{aligned} A \subseteq \{x \in X : p(x) \leq \nu\} &\Longleftrightarrow (x \in A \Longrightarrow p(x) \leq \nu) \\ &\Longleftrightarrow (p(x) \leq \nu, \ \forall\, x \in A) \\ &\Longleftrightarrow p \in A_\nu^\vee. \end{aligned}$$

∎

Theorem 3.3. *Let $f : X \longrightarrow (-\infty, +\infty]$ be a function, and let $0 \neq p \in L$ and $\nu, \alpha \in \mathbb{R}$ be arbitrary. Then the following assertions are equivalent.*
(i) $\{x \in X : f(x) < \alpha\} \subseteq \{x \in X : p(x) \leq \nu\}$.
(ii) $p \in \{x \in X : f(x) < \alpha\}_\nu^\vee$.
(iii) $-p \in \{p \in L : f_\nu^\triangledown(p) \leq \nu - \alpha\}$.

Proof: The implication $[(i) \Longleftrightarrow (ii)]$ follows from Proposition 3.2.
$[(i) \Longrightarrow (iii)]$. Suppose that (i) holds. Then the implication $(f(x) < \alpha \Longrightarrow p(x) \leq \nu)$ holds, or equivalently, the implication $(p(x) > \nu \Longrightarrow f(x) \geq \alpha)$ holds. This together with Definition 2.10 implies that

$$\begin{aligned} f_\nu^\nabla(-p) &= \nu + \sup\{-f(x) : x \in X,\ -p(x) < -\nu\} \\ &= \nu + \sup\{-f(x) : x \in X,\ p(x) > \nu\} \\ &\leq \nu - \alpha, \end{aligned}$$

and hence, (iii) holds.
$[(iii) \Longrightarrow (i)]$. Assume that (iii) holds. Then, by Definition 2.10, one has

$$\begin{aligned} \nu - \alpha &\geq f_\nu^\nabla(-p) \\ &= \nu + \sup\{-f(x) : x \in X,\ -p(x) < -\nu\} \\ &= \nu + \sup\{-f(x) : x \in X,\ p(x) > \nu\}. \end{aligned}$$

This implies that the implication $(p(x) > \nu \Longrightarrow f(x) \geq \alpha)$ holds, or equivalently, the implication $(f(x) < \alpha \Longrightarrow p(x) \leq \nu)$ holds, which implies (i) holds. ■

Lemma 3.1. *Let $f : X \longrightarrow (-\infty, +\infty]$ be a function, and let $\nu \in \mathbb{R}$. Then,*

$$\bigcup_{n=1}^{\infty}\{x \in X : f(x) \leq \alpha + \frac{1}{n}\}_\nu^\vee = \bigcup_{n=1}^{\infty}\{x \in X : f(x) < \alpha + \frac{1}{n}\}_\nu^\vee,\ (\alpha \in \mathbb{R}).$$

Proof: It is obvious. ■

Theorem 3.4. *Let I and J be arbitrary index sets. Let $f_i : X \longrightarrow (-\infty, +\infty]$ be a lower semi-continuous (l.s.c) and star-shaped function $(i \in I)$, and let $0 \neq p_j \in L$ $(j \in J)$ and $\nu \in (0, +\infty)$. Then the following assertions are equivalent.*
(i) We have,

$$\bigcap_{i \in I}\{x \in X : f_i(x) \leq \alpha\} \subseteq \bigcap_{j \in J}\{x \in X : p_j(x) \leq \nu\},\ (\alpha \geq f_i(0),\ \textit{for each } i \in I).$$

(ii) We have,

$$p_j \in \operatorname{cls} \bigcup_{i \in I} A_{i,\nu}^{\vee\vee},\ \forall\ j \in J,$$

where

$$A_i := \{p \in L : f_{i,\nu}^\vee(p) < \nu - \alpha\},\ (i \in I).$$

Proof: First, note that since f_i is a lower semi-continuous function, it follows that

$$B_i := \{x \in X : f_i(x) \leq \alpha\},\ (i \in I)$$

is a closed set for each $i \in I$. Moreover, since f_i is a star-shaped function, it follows from Definition 2.6 that

$$B_i := \{x \in X : f_i(x) \leq \alpha\},\ (i \in I)$$

is a star-shaped set at the origin, and hence, B_i is a closed star-shaped set at the origin for each $i \in I$. Now, let

$$D_\alpha := \bigcap_{i\in I}\{x \in X : f_i(x) \leq \alpha\}.$$

Thus, due to Proposition 3.2, we have (i) is equivalent to $[p_j \in D^{\vee}_{\alpha,\nu}$ for each $j \in J]$.
On the other hand, since B_i is a closed star-shaped set at the origin for each $i \in I$, so one has

$$\begin{aligned}
&D^{\vee}_{\alpha,\nu}\\
&= \operatorname{cls}\bigcup_{i\in I}\{x \in X : f_i(x) \leq \alpha\}^{\vee}_{\nu} && \text{(by Proposition 3.1)}\\
&= \operatorname{cls}\bigcup_{i\in I}\Big(\bigcap_{n=1}^{\infty}\{x \in X : f_i(x) \leq \alpha + \frac{1}{n}\}\Big)^{\vee}_{\nu}\\
&= \operatorname{cls}\bigcup_{i\in I}\Big(\operatorname{cls}\bigcup_{n=1}^{\infty}\{x \in X : f_i(x) \leq \alpha + \frac{1}{n}\}^{\vee}_{\nu}\Big) && \text{(by (3.4) and Proposition 3.1)}\\
&= \operatorname{cls}\bigcup_{i\in I}\Big(\operatorname{cls}\bigcup_{n=1}^{\infty}\{x \in X : f_i(x) < \alpha + \frac{1}{n}\}^{\vee}_{\nu}\Big) && \text{(by Lemma 3.1)}\\
&= \operatorname{cls}\bigcup_{i\in I}\Big(\operatorname{cls}\bigcup_{n=1}^{\infty}\{p \in L : f^{\vee}_{i,\nu}(p) \leq \nu - \alpha - \frac{1}{n}\}\Big) && \text{(by Proposition 2.2)}\\
&= \operatorname{cls}\bigcup_{i\in I}\Big(\operatorname{cls}\{p \in L : f^{\vee}_{i,\nu}(p) < \nu - \alpha\}\Big)\\
&= \operatorname{cls}\bigcup_{i\in I}\operatorname{cls} A_i\\
&= \operatorname{cls}\bigcup_{i\in I}A^{\vee\vee}_{i,\nu}. && \text{(by Theorem 3.2)}
\end{aligned}$$

Hence, (i) is equivalent to (ii).
Note that since f_i is a lower semi-continuous and star-shaped function and $\alpha \geq f_i(0)$ for each $i \in I$ (and hence, $\alpha + \frac{1}{n} \geq f_i(0)$ for each $i \in I$), it follows from Definition 2.6 that

$$\{x \in X : f_i(x) \leq \alpha + \frac{1}{n}\} \tag{3.4}$$

is a closed star-shaped set at the origin for each $i \in I$. ■

4. Characterizations of the Set Containment with Infinite Evenly Radiant Constraints

In this section, we first give a separation theorem for a closed star-shaped set at the origin and a point outside it in terms of separation by an upper semi-continuous and super-linear function. Finally, by using this fact, we present characterizations of the set containment with infinite evenly radiant constraints defined by strict or weak inequalities.

Throughout this section, we assume that X is a real Banach space with a Schauder basis $\{x_n\}_{n\in\mathbb{N}}$. We recall (for more details, see [1, 7]) that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Banach space X is called a

Schauder basis for X if for each $x \in X$ there exists a unique sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ of scalars such that

$$x = \sum_{n=1}^{\infty} \alpha_n x_n.$$

It is convenient to present the separability in terms of a collection of linear functionals. Consider a sequence of linear functionals $\mathcal{F} := \{x_n^*\}_{n\in\mathbb{N}}$ in X^* (the dual space of X), and let

$$T_{\mathcal{F}} := \bigcap_{n\in\mathbb{N}} \{x \in X : \langle x_n^*, x\rangle < 0\}, \tag{4.1}$$

and

$$T^{\mathcal{F}} := \bigcap_{n\in\mathbb{N}} \{x \in X : \langle x_n^*, x\rangle > 0\}. \tag{4.2}$$

It should be noted that if the elements x_n^* $(n \in \mathbb{N})$ are linearly independent, then both $T_{\mathcal{F}}$ and $T^{\mathcal{F}}$ are non-empty sets.

Definition 4.1. *[8] Let $A \subseteq X$ and $x \notin A$. We say that a sequence $\{x_i^*\}_{i\in\mathbb{N}}$ of linearly independent elements in X^* strictly separates A and x if there exists $\varepsilon \in (0,1)$ with the following property: for each $a \in A$, there exists $i_a \in \mathbb{N}$ such that $\langle x_{i_a}^*, a\rangle < 1-\varepsilon$ and $\langle x_i^*, x\rangle = 1$ for all $i \in \mathbb{N}$.*

Definition 4.2. *[8] Let U and V be subsets of X and $\{x_i^*\}_{i\in\mathbb{N}}$ be a sequence of linearly independent elements in X^*. The sets U and V are said to be weakly separated by $\{x_i^*\}_{i\in\mathbb{N}}$ if for each pair $(u,v) \in U \times V$ there exists $j \in \mathbb{N}$ such that $\langle x_j^*, u\rangle \le \langle x_j^*, v\rangle$.*

The following theorem shows that each point, which does not belong to a closed star-shaped set at the origin, can be strictly separated from this set. The proof is similar to that of Theorem 4.1 in [8], and therefore, we omit its proof.

Theorem 4.1. *Let $U \subset X$ be a closed star-shaped set at the origin, and let $x \notin U$. Then, U and x are strictly separated by a sequence $\{x_i^*\}_{i\in\mathbb{N}}$ of linearly independent elements in X^*.*

Proposition 4.1. *Let $A \subset X$ be a closed star-shaped set at the origin, and let $x \in X$ be such that $x \notin A$. Then there exists $p_1 \in L$ such that $p_1(a) < 1$ for all $a \in A$ and $p_1(x) = 1$.*

Proof: Since A is a closed and star-shaped set at the origin and $x \notin A$, it follows from Theorem 4.1 that there exists a sequence $\{x_i^*\}_{i\in\mathbb{N}}$ of linearly independent elements in X^* such that strictly separates A and x. Therefore, in view of Definition 4.1, there exists $\varepsilon \in (0,1)$ with the following property: for each $a \in A$, there exists $i_a \in \mathbb{N}$ such that $\langle x_{i_a}^*, a\rangle < 1-\varepsilon$ and $\langle x_i^*, x\rangle = 1$ for all $i \in \mathbb{N}$. Now, we define the function $p_1 : X \longrightarrow \mathbb{R}$ by

$$p_1(z) := \inf_{i\in\mathbb{N}} \langle x_i^*, z\rangle, \ \forall\ z \in X.$$

Since $\mathcal{F} := \{x_i^*\}_{i\in\mathbb{N}}$ is a sequence of linearly independent elements in X^*, we conclude from (4.1) and (4.2) that $T_{\mathcal{F}} \neq \emptyset$ and $T^{\mathcal{F}} \neq \emptyset$, and hence, p_1 is a real valued function. Clearly, p_1 is an upper semi-continuous and super-linear function, and so, $p_1 \in L$. It is easy to check that $p_1(a) < 1$ for all $a \in A$, and moreover, $p_1(x) = 1$, which completes the proof. ■

Corollary 4.1. *Let $\nu \in (0, +\infty)$. Let $A \subset X$ be a closed and star-shaped set at the origin, and let $x \in X$ be such that $x \notin A$. Then there exists $p \in L$ such that $p \in A^{\wedge}_{\nu}$ and $p(x) = \nu$.*

Proof: By Proposition 4.1, there exists $p_1 \in L$ such that $p_1(a) < 1$ for all $a \in A$ and $p_1(x) = 1$. Let $p := \nu p_1$. Then, in view of Remark 2.1, $p \in L$ and $p(a) < \nu$ for all $a \in A$ and $p(x) = \nu$. Therefore, in view of Definition 2.9, one has $p \in A^{\wedge}_{\nu}$, and moreover, $p(x) = \nu$. ∎

Definition 4.3. *Let A be a subset of X, and let $\nu \in \mathbb{R}$. We say that A is an ν-evenly radiant set if, for each $x \in A^c := X \setminus A$, there exists $p \in L$ such that $p \in A^{\wedge}_{\nu}$ and $p(x) \geq \nu$.*

Remark 4.1. *In view of Corollary 4.1, every closed star-shaped set at the origin is ν-evenly radiant for each $\nu \in (0, +\infty)$.*

By the following lemma, we present examples of ν-evenly radiant sets.

Lemma 4.1. *Let $f : X \longrightarrow [-\infty, 0]$ be a function, and let $\nu \in (0, +\infty)$. Let*

$$B := \{p \in L : f^{\wedge}_{\nu}(p) \leq \nu - \alpha\}, \ (\alpha \leq 0).$$

Then, B is an ν-evenly radiant set.

Proof: If we show that B is a closed star-shaped set at the origin, then the result follows from Remark 4.1. First, we show that B is a closed set. To this end, let $\{p_n\}_{n\geq 1} \subset B$ and $p \in L$ be such that $p_n \longrightarrow p$ point-wise, as $n \longrightarrow +\infty$, i.e., $p_n(t) \longrightarrow p(t)$ for each $t \in X$, as $n \longrightarrow +\infty$. Since $p_n \in B$ for all $n \geq 1$, it follows that $f^{\wedge}_{\nu}(p_n) \leq \nu - \alpha$ for all $n \geq 1$. This together with Definition 2.10 implies that

$$\nu - \alpha \geq f^{\wedge}_{\nu}(p_n) = \nu + \sup\{[-f(x)] : x \in X, \ p_n(x) \geq \nu\}, \ \forall \ n \geq 1.$$

This implies that

$$\nu - \alpha \geq \nu - f(x), \ \forall \ x \in X \text{ with } \ p_n(x) \geq \nu, \ n = 1, 2, \cdots .$$

Since $p_n(t) \longrightarrow p(t)$ for each $t \in X$, as $n \longrightarrow +\infty$, we conclude that

$$\nu - \alpha \geq \nu - f(x), \ \forall \ x \in X \text{ with } \ p(x) \geq \nu.$$

So, by using Definition 2.10, we have

$$\nu - \alpha \geq \nu + \sup\{[-f(x)] : x \in X, \ p(x) \geq \nu\} = f^{\wedge}_{\nu}(p).$$

This implies that $p \in B$, and so, B is closed.

Now, we show that B is a radiant set. Let $p \in B$ and $\lambda \in (0, 1]$ be arbitrary. In view of Definition 2.10 and the fact that $0 < \lambda \leq 1$, one has

$$\begin{aligned} f^{\wedge}_{\nu}(\lambda p) &= \nu + \sup\{[-f(x)] : x \in X, \ \lambda p(x) \geq \nu\} \\ &= \nu + \sup\{[-f(x)] : x \in X, \ p(x) \geq \frac{\nu}{\lambda}\} \\ &\leq \nu + \sup\{[-f(x)] : x \in X, \ p(x) \geq \nu\} \\ &= f^{\wedge}_{\nu}(p) \\ &\leq \nu - \alpha. \end{aligned}$$

This implies that $f^{\wedge}_{\nu}(\lambda p) \leq \nu - \alpha$, and hence, $\lambda p \in B$, i.e., B is a radiant set. Also, $0 \in B$ because $0 \in L$, $\nu > 0$ and $\alpha \leq 0$, and so, by Definition 2.10, we have

$$\begin{aligned} f^{\wedge}_{\nu}(0) &= \nu + \sup\{-f(x) : x \in X,\ 0(x) \geq \nu\} \\ &= \nu + 0 \\ &= \nu \\ &\leq \nu - \alpha. \end{aligned}$$

That is, $0 \in B$. Therefore, in view of Definition 2.5, B is a closed star-shaped set at the origin, which completes the proof. ■

Definition 4.4. *Let A be a subset of X, and let $\nu \in \mathbb{R}$. We define the wedge ν-bipolar of A by*

$$A^{\wedge\wedge}_{\nu} := \{x \in X : p(x) < \nu,\ \forall\ p \in A^{\wedge}_{\nu}\}.$$

Theorem 4.2. *Let A be a subset of X, and let $\nu \in \mathbb{R}$. Then, A is ν-evenly radiant if and only if $A = A^{\wedge\wedge}_{\nu}$.*

Proof: Assume that A is an ν-evenly radiant set. Let $x \in A^c$ be arbitrary. Then, by Definition 4.3, there exists $p \in L$ such that $p \in A^{\wedge}_{\nu}$ and $p(x) \geq \nu$. So, in view of Definition 4.4, $x \notin A^{\wedge\wedge}_{\nu}$, and hence, $A^{\wedge\wedge}_{\nu} \subseteq A$. Clearly, $A \subseteq A^{\wedge\wedge}_{\nu}$. Thus, $A = A^{\wedge\wedge}_{\nu}$.
Conversely, suppose that $A = A^{\wedge\wedge}_{\nu}$. But, it follows from Definition 4.4 that

$$A^{\wedge\wedge}_{\nu} = \bigcap_{p \in A^{\wedge}_{\nu}} \{x \in X : p(x) < \nu\}.$$

So, this together with the fact that $A = A^{\wedge\wedge}_{\nu}$ implies that, for each $x \in A^c$ (and so, $x \notin A^{\wedge\wedge}_{\nu}$), there exists $p \in A^{\wedge}_{\nu}$ such that $p(x) \geq \nu$. Therefore, by using Definition 4.3, A is an ν-evenly radiant set. ■

Definition 4.5. *Let $\nu \in \mathbb{R}$. The ν-evenly radiant hull of a subset A of X is defined by*

$$\operatorname{erad} A := \bigcap_{p \in A^{\wedge}_{\nu}} \{x \in X : p(x) < \nu\}.$$

In view of Definition 4.4, we have $\operatorname{erad} A = A^{\wedge\wedge}_{\nu}$. *Clearly,* $A \subseteq \operatorname{erad} A$.

It is worth noting that, in view of Theorem 4.2, A is ν-evenly radiant if and only if $A = \operatorname{erad} A$.

Proposition 4.2. *Let I be an arbitrary index set, and let $\nu \in \mathbb{R}$. Let A_i be an ν-evenly radiant subset of X $(i \in I)$. Then,*

$$\left(\bigcap_{i \in I} A_i\right)^{\wedge}_{\nu} = \operatorname{erad} \bigcup_{i \in I} A^{\wedge}_{i,\nu}.$$

Proof: Due to Proposition 2.3, we have

$$\left(\bigcup_{i \in I} A^{\wedge}_{i,\nu}\right)^{\wedge}_{\nu} = \bigcap_{i \in I} A^{\wedge\wedge}_{i,\nu}. \tag{4.3}$$

Since A_i is an ν-evenly radiant set, it follows from Theorem 4.2 that $A_{i,\nu}^{\wedge\wedge} = A_i$ for each $i \in I$. Therefore, by using (4.3), we obtain

$$\left(\bigcup_{i\in I} A_{i,\nu}^{\wedge}\right)_{\nu}^{\wedge} = \bigcap_{i\in I} A_i. \tag{4.4}$$

Now, put $C := \bigcup_{i\in I} A_{i,\nu}^{\wedge}$. So, it follows from (4.4) that

$$C_{\nu}^{\wedge} = \bigcap_{i\in I} A_i. \tag{4.5}$$

On the other hand, in view of Definition 4.5, one has

$$\operatorname{erad} C = C_{\nu}^{\wedge\wedge}.$$

This together with (4.5) implies that

$$\operatorname{erad} C = \left(\bigcap_{i\in I} A_i\right)_{\nu}^{\wedge},$$

which completes the proof. ■

Proposition 4.3. *Let A be a subset of X, and let $0 \neq p \in L$ be arbitrary and $\nu \in \mathbb{R}$. Then the following assertions are equivalent.*
(1) $A \subseteq \{x \in X : p(x) < \nu\}$.
(2) $p \in A_{\nu}^{\wedge}$.

Proof: The proof is similar to that of Proposition 3.2. ■

Theorem 4.3. *Let $g : X \longrightarrow (-\infty, +\infty]$ be a function, and let $0 \neq p \in L$ and $\nu, \beta \in \mathbb{R}$ be arbitrary. Then the following assertions are equivalent.*
(i) $\{x \in X : g(x) < \beta\} \subseteq \{x \in X : p(x) \geq \nu\}$.
(ii) $p \in \{x \in X : g(x) < \beta\}_{\nu}^{\triangledown}$.
(iii) $p \in \{p \in L : g_{\nu}^{\triangle}(p) \leq \nu - \beta\}$.

Proof: $[(i) \Longleftrightarrow (ii)]$. We have, by Definition 2.9,

$$\begin{aligned} & \{x \in X : g(x) < \beta\} \subseteq \{x \in X : p(x) \geq \nu\} \\ \Longleftrightarrow\ & \Big(x \in \{x \in X : g(x) < \beta\} \Longrightarrow p(x) \geq \nu\Big) \\ \Longleftrightarrow\ & \Big(p(x) \geq \nu,\ \forall\, x \in \{x \in X : g(x) < \beta\}\Big) \\ \Longleftrightarrow\ & p \in \{x \in X : g(x) < \beta\}_{\nu}^{\triangledown}. \end{aligned}$$

$[(i) \Longrightarrow (iii)]$. Suppose that (i) holds. Then the implication $(g(x) < \beta \Longrightarrow p(x) \geq \nu)$ holds, or equivalently, the implication $(p(x) < \nu \Longrightarrow g(x) \geq \beta)$ holds. This together with Definition 2.10 implies that

$$\begin{aligned} g_{\nu}^{\triangle}(p) &= \nu + \sup\{-g(x) : x \in X,\ p(x) < \nu\} \\ &\leq \nu - \beta, \end{aligned}$$

and hence, (iii) holds.
$[(iii) \Longrightarrow (i)]$. Assume that (iii) holds. Then, by Definition 2.10, one has

$$\begin{aligned} \nu - \beta &\geq g_{\nu}^{\triangle}(p) \\ &= \nu + \sup\{-g(x) : x \in X,\ p(x) < \nu\}. \end{aligned}$$

This implies that the implication $(p(x) < \nu \Longrightarrow g(x) \geq \beta)$ holds, or equivalently, the implication $(g(x) < \beta \Longrightarrow p(x) \geq \nu)$ holds. Hence, (i) follows. ■

Definition 4.6. *Let $\nu \in \mathbb{R}$. A function $f : X \longrightarrow [-\infty, +\infty]$ is called ν-evenly radiant if every lower level set of f,*

$$[f \leq \alpha] := \{x \in X : f(x) \leq \alpha\},$$

is an ν-evenly radiant set for each $\alpha \in \mathbb{R}$. Also, a function $f : X \longrightarrow [-\infty, +\infty]$ is called strictly ν-evenly radiant if every strict lower level set of f,

$$[f < \alpha] := \{x \in X : f(x) < \alpha\},$$

is an ν-evenly radiant set for each $\alpha \in \mathbb{R}$.

Theorem 4.4. *Let I and J be arbitrary index sets, and let $\nu \in \mathbb{R}$. Let $f_i : X \longrightarrow (-\infty, +\infty]$ be a strictly ν-evenly radiant function $(i \in I)$, and let $0 \neq p_j \in L$ $(j \in J)$. Then the following assertions are equivalent.*
(i) We have,

$$\bigcap_{i\in I}\{x \in X : f_i(x) < \alpha\} \subseteq \bigcap_{j \in J}\{x \in X : p_j(x) < \nu\},\ (\alpha \in \mathbb{R}).$$

(ii) We have,

$$p_j \in \operatorname{erad} \bigcup_{i \in I}\{p \in L : f_{i,\nu}^{\wedge}(p) \leq \nu - \alpha\},\ \forall\ j \in J.$$

Proof: Since f_i is a strictly ν-evenly radiant function, it follows from Definition 4.6 that

$$C_i := \{x \in X : f_i(x) < \alpha\}$$

is an ν-evenly radiant set for each $i \in I$. Now, let

$$G_{\alpha} := \bigcap_{i \in I}\{x \in X : f_i(x) < \alpha\}.$$

Thus, due to Proposition 4.3, we have (i) is equivalent to $[p_j \in G_{\alpha,\nu}^{\wedge}$ for each $j \in J]$.
On the other hand, since C_i is an ν-evenly radiant set for each $i \in I$, we obtain

$$\begin{aligned} G_{\alpha,\nu}^{\wedge} &= \operatorname{erad} \bigcup_{i \in I}\{x \in X : f_i(x) < \alpha\}_{\nu}^{\wedge} && \text{(by Proposition 4.2)} \\ &= \operatorname{erad} \bigcup_{i \in I}\{p \in L : f_{i,\nu}^{\wedge}(p) \leq \nu - \alpha\}. && \text{(by Proposition 2.2)} \end{aligned}$$

Hence, (i) is equivalent to (ii). ■

Lemma 4.2. *Let $f : X \longrightarrow (-\infty, +\infty]$ be a function, and let $\nu \in \mathbb{R}$. Then,*

$$\bigcup_{n=1}^{\infty}\{x \in X : f(x) \leq \alpha + \frac{1}{n}\}_{\nu}^{\wedge} = \bigcup_{n=1}^{\infty}\{x \in X : f(x) < \alpha + \frac{1}{n}\}_{\nu}^{\wedge},\ (\alpha \in \mathbb{R}).$$

Proof: It is obvious. ■

Theorem 4.5. *Let I and J be arbitrary index sets, and let $\nu \in \mathbb{R}$. Let $f_i : X \longrightarrow (-\infty, +\infty]$ be an ν-evenly radiant function $(i \in I)$, and let $0 \neq p_j \in L$ $(j \in J)$. Then the following assertions are equivalent.*
(i) *We have,*

$$\bigcap_{i \in I}\{x \in X : f_i(x) \leq \alpha\} \subseteq \bigcap_{j \in J}\{x \in X : p_j(x) < \nu\},\ (\alpha \in \mathbb{R}).$$

(ii) *We have,*

$$p_j \in \operatorname{erad} \bigcup_{i \in I} B_{i,\nu}^{\wedge\wedge},\ \forall\ j \in J,$$

where

$$B_i := \{p \in L : f_{i,\nu}^{\wedge}(p) < \nu - \alpha\},\ (i \in I,\ \alpha \in \mathbb{R}).$$

Proof: Since f_i is an ν-evenly radiant function, it follows from Definition 4.6 that

$$D_i := \{x \in X : f_i(x) \leq \alpha\}$$

is an ν-evenly radiant set for each $i \in I$. Now, let

$$H_\alpha := \bigcap_{i \in I}\{x \in X : f_i(x) \leq \alpha\}.$$

Thus, due to Proposition 4.3, we have (i) is equivalent to $[p_j \in H_{\alpha,\nu}^{\wedge}$ for each $j \in J]$.

On the other hand, since D_i is an ν-evenly radiant set for each $i \in I$, it follows that

$$
\begin{aligned}
& H^{\wedge}_{\alpha,\nu} \\
&= \operatorname{erad} \bigcup_{i\in I} \{x \in X : f_i(x) \le \alpha\}^{\wedge}_{\nu} && \text{(by Proposition 4.2)} \\
&= \operatorname{erad} \bigcup_{i\in I} \left(\bigcap_{n=1}^{\infty} \{x \in X : f_i(x) \le \alpha + \frac{1}{n}\} \right)^{\wedge}_{\nu} \\
&= \operatorname{erad} \bigcup_{i\in I} \left(\operatorname{erad} \bigcup_{n=1}^{\infty} \{x \in X : f_i(x) \le \alpha + \frac{1}{n}\}^{\wedge}_{\nu} \right) && \text{(by (4.6) and Proposition 4.2)} \\
&= \operatorname{erad} \bigcup_{i\in I} \left(\operatorname{erad} \bigcup_{n=1}^{\infty} \{x \in X : f_i(x) < \alpha + \frac{1}{n}\}^{\wedge}_{\nu} \right) && \text{(by Lemma 4.2)} \\
&= \operatorname{erad} \bigcup_{i\in I} \left(\operatorname{erad} \bigcup_{n=1}^{\infty} \{p \in L : f^{\wedge}_{i,\nu}(p) \le \nu - \alpha - \frac{1}{n}\} \right) && \text{(by Proposition 2.2)} \\
&= \operatorname{erad} \bigcup_{i\in I} \left(\operatorname{erad} \{p \in L : f^{\wedge}_{i,\nu}(p) < \nu - \alpha\} \right) \\
&= \operatorname{erad} \bigcup_{i\in I} \operatorname{erad} B_i \\
&= \operatorname{erad} \bigcup_{i\in I} B^{\wedge\wedge}_{i,\nu}. && \text{(by Definition 4.5)}
\end{aligned}
$$

Hence, (i) is equivalent to (ii).
Note that since f_i is an ν-evenly radiant function for each $i \in I$, then by Definition 4.6, the set

$$\{x \in X : f_i(x) \le \alpha + \frac{1}{n}\} \tag{4.6}$$

is an ν-evenly radiant set for each $i \in I$. ■

5. Characterization of the Set Containment in a Reverse Star-Shaped Set

In this section, we give a characterization of the set containment with an upper semi-continuous and radiant constraint, in a reverse star-shaped set, defined by a co-star-shaped constraint. Throughout this section, we assume that X is a real Banach space with a Schauder basis $\{x_n\}_{n\in\mathbb{N}}$. We start with the following crucial result, which has been proved in [8, Theorem 4.3], and therefore, we omit its proof.

Theorem 5.1. *Let U and V be star-shaped sets in X such that $\operatorname{int} kernU \neq \emptyset$ and $\operatorname{int} U \cap V = \emptyset$. Then, U and V are weakly separated by a sequence $\{x^*_i\}_{i\in\mathbb{N}}$ of linearly independent elements in X^*.*

Lemma 5.1. *Let $f : X \longrightarrow (-\infty, +\infty]$ be a radiant function, and let*

$$U_0 := \{x \in X : f(x) < \alpha\}, \ (\alpha \ge 0).$$

Assume that $0 \notin U_0$, and $kernU_0 \neq \emptyset$. Let $U := U_0 \cup \{0\}$. Then, U is a star-shaped set.

Proof: Since $kernU_0 \neq \emptyset$, so there exists $u_0 \in kernU_0$. Therefore, it follows from Definition 2.4 that U_0 is a star-shaped set at u_0. Now, we show that U is also a star-shaped set at u_0. To this end, it is enough to show that

$$u_0 + \lambda(u - u_0) \in U, \ \forall \ u \in U, \ \forall \ \lambda \in [0, 1].$$

Since $u_0 \in kernU_0$, it follows from definition 2.4 (1) that

$$u_0 + \lambda(u - u_0) \in U_0, \ \forall \ u \in U_0, \ \forall \ \lambda \in [0, 1],$$

and hence,

$$u_0 + \lambda(u - u_0) \in U, \ \forall \ u \in U_0, \ \forall \ \lambda \in [0, 1]. \tag{5.1}$$

On the other hand, since f is a radiant function, we conclude from Remark 2.2 that U_0 is a radiant set. Thus, in view of Definition 2.2, one has $\gamma x \in U_0$ for all $x \in U_0$ and all $\gamma \in (0, 1]$. This together with the fact that $u_0 \in U_0$ implies that $(1 - \lambda)u_0 \in U_0$ for all $\lambda \in [0, 1)$, and so,

$$(1 - \lambda)u_0 \in U, \ \forall \ \lambda \in [0, 1).$$

Since $0 \in U$, it follows that

$$u_0 + \lambda(0 - u_0) = (1 - \lambda)u_0 \in U, \ \forall \ \lambda \in [0, 1]. \tag{5.2}$$

Hence, we conclude from (5.1) and (5.2) that

$$u_0 + \lambda(u - u_0) \in U, \ \forall \ u \in U, \ \forall \ \lambda \in [0, 1].$$

This together with Definition 2.4 (1) implies that $u_0 \in kernU$, and so, by using Definition 2.4 (2), U is a star-shaped set at u_0. ■

Theorem 5.2. *Let $f : X \longrightarrow (-\infty, +\infty]$ be an upper semi-continuous and radiant function with $f(0) \geq 0$, and let $g : X \longrightarrow (-\infty, +\infty]$ be a co-star-shaped function. Let $\beta \in \mathbb{R}$ be such that $\beta > g(0)$, and let*

$$U_0 := \{x \in X : f(x) < 0\}.$$

Suppose that

$$\operatorname{int} kernU_0 \neq \emptyset.$$

Consider the following assertions.
(i) We have,

$$\{x \in X : f(x) < 0\} \subseteq \{x \in X : g(x) \geq \beta\}.$$

(ii) We have,

$$\{x \in X : f(x) < 0\} \cap \{x \in X : g(x) < \beta\} = \emptyset.$$

(iii) There exist $\nu \in \mathbb{R}$ and

$$-p \in \{p \in L : f^{\triangledown}_{\nu}(p) \leq \nu\},$$

and

$$q \in \{p \in L : g^{\triangle}_{\nu}(p) \leq \nu - \beta\}$$

such that $q(x) \leq p(x)$ for all $x \in X$.
Then, $(i) \Longleftrightarrow (ii)$ and $(ii) \Longrightarrow (iii)$. Hence, in the later implication, we have

$$q - p \in \{p \in L : g_{\nu}^{\triangle}(p) \leq \nu - \beta\} + \{p \in L : f_{\nu}^{\triangledown}(p) \leq \nu\}.$$

Moreover, if (iii) holds with $q < p$ on U_0, then, (i) holds.

Note that since g is a co-star-shaped function, it follows from Definition 2.8 that the set $\{x \in X : g(x) \geq \beta\}$ is a reverse star-shaped set.

Proof: Clearly, the assertions (i) and (ii) are equivalent. Now, suppose that the assertion (ii) holds. We show that the assertion (iii) holds. To this end, let

$$V := \{x \in X : g(x) < \beta\},$$

and

$$U := U_0 \cup \{0\}.$$

Since $0 \notin U_0$ (because $f(0) \geq 0$), and also, by the hypothesis, $kern U_0 \neq \emptyset$ and f is a radiant function, it follows from Lemma 5.1 (with $\alpha = 0$) that U is a star-shaped set. Since f is an upper semi-continuous function, it follows that U_0 is an open subset of X, and so, $\operatorname{int} U_0 = U_0$. Moreover, since by the hypothesis, g is a co-star-shaped function, it follows from Definition 2.7 and Definition 2.8 that V is a star-shaped set at the origin because $\beta > g(0)$. Furthermore, by the hypothesis (ii), we have, $U_0 \cap V = \emptyset$, and hence,

$$\operatorname{int} U \cap V = \operatorname{int} U_0 \cap V = U_0 \cap V = \emptyset,$$

and also, by the hypothesis, $\operatorname{int} kern U = \operatorname{int} kern U_0 \neq \emptyset$. Therefore, we conclude from Theorem 5.1 that U and V are weakly separated by a sequence $\{x_i^*\}_{i \in \mathbb{N}}$ of linearly independent elements in X^*. Thus, in view of Definition 4.2, for each pair $(u, v) \in U \times V$, there exists $j \in \mathbb{N}$ such that

$$\langle x_j^*, u \rangle \leq \langle x_j^*, v \rangle. \tag{5.3}$$

Now, let

$$\begin{aligned}
A_{(u,v)} &:= \{j \in \mathbb{N} : \langle x_j^*, u \rangle \leq \langle x_j^*, v \rangle\}, \text{ for each } (u, v) \in U \times V, \\
\mathcal{A} &:= \bigcap_{(u,v) \in U \times V} A_{(u,v)}, \\
\mathcal{B} &:= \bigcap_{u \in U} A_{(u,0)}, \\
\mathcal{C} &:= \bigcap_{v \in V} A_{(0,v)}.
\end{aligned}$$

Then, in view of (5.3), $A_{(u,v)} \neq \emptyset$ for each $(u, v) \in U \times V$. It is easy to see that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \subseteq \mathcal{C}$ because $0 \in U \cap V$.

Assumption (N): Assume that $\mathcal{A} \neq \emptyset$.

It is clear that under the Assumption (N), $\mathcal{B}$ and $\mathcal{C}$ are non-empty sets. It should be noted that by using (5.3), we obtain

$$\langle x_j^*, v\rangle \geq 0,\ \forall\ v \in V,\ \forall\ j \in \mathcal{C}, \tag{5.4}$$

and

$$\langle x_j^*, u\rangle \leq 0,\ \forall\ u \in U,\ \forall\ j \in \mathcal{B}. \tag{5.5}$$

Since $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{B}$, it follows from (5.4) and (5.5) that

$$\langle x_j^*, v\rangle \geq 0,\ \forall\ v \in V,\ \forall\ j \in \mathcal{A}, \tag{5.6}$$

and

$$\langle x_j^*, u\rangle \leq 0,\ \forall\ u \in U,\ \forall\ j \in \mathcal{A}.$$

The later inequality implies that

$$\langle x_j^*, u\rangle \leq 0,\ \forall\ u \in U_0,\ \forall\ j \in \mathcal{A}. \tag{5.7}$$

Now, we define the function $p : X \longrightarrow \mathbb{R}$ by

$$p(x) := \sup_{j\in\mathcal{A}}\langle x_j^*, x\rangle,\ \forall\ x \in X,$$

and the function $q : X \longrightarrow \mathbb{R}$ by

$$q(x) = \inf_{j\in\mathcal{A}}\langle x_j^*, x\rangle,\ \forall\ x \in X.$$

Since $\mathcal{F} := \{x_i^*\}_{i\in\mathbb{N}}$ is a sequence of linearly independent elements in X^*, we conclude from (4.1) and (4.2) that $T_{\mathcal{F}} \neq \emptyset$ and $T^{\mathcal{F}} \neq \emptyset$. Thus, p and q are real valued functions, and also, there exist $x_0, y_0 \in X$ such that

$$\langle x_j^*, x_0\rangle < 0,\ \text{and}\ \ \langle x_j^*, y_0\rangle > 0,\ \forall\ j \in \mathbb{N}. \tag{5.8}$$

This implies that $q(x_0) < 0 < p(y_0)$, i.e., p and q are not identically zero, and $p \neq q$. Also, it is easy to check that $-p, q \in L$ and $q(x) \leq p(x)$ for all $x \in X$. Furthermore, it follows from (5.6) and (5.7) that

$$q(v) \geq 0,\ \forall\ v \in V,\ \text{and}\ \ p(u) \leq 0,\ \forall\ u \in U_0. \tag{5.9}$$

So, it follows from (5.9) that there exists $\nu \in \mathbb{R}$ such that

$$p(u) \leq \nu \leq q(v),\ \forall\ u \in U_0,\ \forall\ v \in V. \tag{5.10}$$

Therefore, we conclude from (5.10) that

$$U_0 = \{x \in X : f(x) < 0\} \subseteq \{x \in X : p(x) \leq \nu\}, \tag{5.11}$$

and

$$V = \{x \in X : g(x) < \beta\} \subseteq \{x \in X : q(x) \geq \nu\}. \tag{5.12}$$

Thus, by using (5.11) and Theorem 3.3 (the implication $(i) \Longrightarrow (iii)$), we obtain

$$-p \in \{p \in L : f_\nu^{\triangledown}(p) \leq \nu\}. \tag{5.13}$$

Moreover, by using (5.12) and Theorem 4.3 (the implication $(i) \Longrightarrow (iii)$), we get

$$q \in \{p \in L : g_\nu^{\triangle}(p) \leq \nu - \beta\}. \tag{5.14}$$

Therefore, we conclude from (5.13) and (5.14) that

$$q - p \in \{p \in L : g_\nu^{\triangle}(p) \leq \nu - \beta\} + \{p \in L : f_\nu^{\triangledown}(p) \leq \nu\}.$$

This implies (iii) holds.

$[(iii) \Longrightarrow (i)]$. Assume that there exist $\nu \in \mathbb{R}$ and

$$-p \in \{p \in L : f_\nu^{\triangledown}(p) \leq \nu\}, \tag{5.15}$$

and

$$q \in \{p \in L : g_\nu^{\triangle}(p) \leq \nu - \beta\} \tag{5.16}$$

such that $q < p$ on U_0. So, in view of Definition 2.10 and (5.15), we deduce that

$$\begin{aligned} \nu &\geq f_\nu^{\triangledown}(-p) \\ &= \nu + \sup\{-f(x) : x \in X,\ -p(x) < -\nu\} \\ &= \nu + \sup\{-f(x) : x \in X,\ p(x) > \nu\}. \end{aligned}$$

This implies that the implication $(p(x) > \nu \Longrightarrow f(x) \geq 0)$ holds, or equivalently, the following implication holds.

$$f(x) < 0 \Longrightarrow p(x) \leq \nu. \tag{5.17}$$

Also, we obtain from (5.16) and Definition 2.10 that

$$\begin{aligned} \nu - \beta &\geq g_\nu^{\triangle}(q) \\ &= \nu + \sup\{-g(x) : x \in X,\ q(x) < \nu\}. \end{aligned}$$

This implies that the following implication holds.

$$q(x) < \nu \Longrightarrow g(x) \geq \beta. \tag{5.18}$$

Now, let $x \in U_0 = \{x \in X : f(x) < 0\}$ be arbitrary. Then, $f(x) < 0$. Thus, it follows from (5.17) that $p(x) \leq \nu$. This together with the fact that $q < p$ on U_0 (note that $x \in U_0$) implies that $q(x) < \nu$. Therefore, in view of (5.18), we have $g(x) \geq \beta$, i.e., $x \in \{x \in X : g(x) \geq \beta\}$. Hence, (i) holds. This completes the proof. ■

The following example shows that the set $\mathcal{A}$ in the proof of Theorem 5.2 may be non-empty, and moreover, $q < p$ on U_0.

Example 5.1. *We consider the following two radiant subsets U and V of $\mathbb{R}^2$. Let*

$$U := \mathrm{co}\{(0,0), (\frac{1}{4}, 1), (0,1)\} \cup \mathrm{co}\{(0,0), (\frac{1}{4}, \frac{2}{3}), (\frac{1}{4}, 1)\},$$

and

$$V := \mathrm{co}\{(0,0), (\frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{5})\} \cup \mathrm{co}\{(0,0), (\frac{3}{4}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3})\}.$$

Also, we define the linear functionals $x_1^*, x_2^* : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $\langle x_1^*, (x,y)\rangle := x - y$ and $\langle x_2^*, (x,y)\rangle := x$ for all $(x,y) \in \mathbb{R}^2$. Since U and V are radiant sets and $0 \in U \cap V$, it follows from Definition 2.5 that U and V are star-shaped sets at the origin. It is not difficult to show that x_1^* and x_2^* are *linearly independent, and moreover, U and V are weakly separated by $\{x_1^*, x_2^*\}$, i.e., for each pair $(u,v) \in U \times V$, there exists $j \in \{1,2\}$ such that*

$$\langle x_j^*, u\rangle \leq \langle x_j^*, v\rangle.$$

Also, $\operatorname{int} kernU \neq \emptyset$ *and* $\operatorname{int} U \cap V = \emptyset$. *Finally, it is easy to check that $\mathcal{A} = \{1,2\}$. Furthermore, we have*

$$p(x,y) = \max\{x-y, x\}, \ \forall \ (x,y) \in \mathbb{R}^2,$$

and

$$q(x,y) = \min\{x-y, x\}, \ \forall \ (x,y) \in \mathbb{R}^2.$$

But, we have $U_0 = U \setminus \{(0,0)\}$. It is easy to see that $q < p$ on U_0.

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