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Optimally Local Dense Conditions for the Existence of Solutions for Vector Equilibrium Problems

Masoomeh Shokouhnia[a], Sirous Moradi[a,*], Somaye Jafari[a]

[a]*Department of Mathematics, Faculty of Science, Arak University, Arak 38156-8-8349, Iran*

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Abstract

In this paper, by using C-sequentially sign property for bifunctions, we provide sufficient conditions that ensure the existence of solutions of some vector equilibrium problems in Hausdorff topological vector spaces which ordered by a cone. The conditions which we consider are not imposed on the whole domain of the operators involved, but just on a locally segment-dense subset of the domain.

Keywords: locally segment-dense set, vector equilibrium problem, C-sequentially sign property
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1. Introduction

Let X be a real Hausdorff, locally convex topological vector space and K be a nonempty subset of X. An equilibrium problem associated to f and K, or briefly $EP(f,K)$ in the sense of Blum and Oettli [6], is stated as follows:

$$find\ x^* \in K \quad such\ that \quad f(x^*, x) \geq 0 \quad for\ all\ x \in K,$$

that $f : K \times K \longrightarrow \mathbb{R}$ is a bifunction. We denote the set of solutions $EP(f,K)$, by $S(f,K)$. This problem is also called Ky Fan inequality due to his contribution to this field [10]. It is well known that some important problems such as convex programs, variational inequalities, fixed point, Nash equilibrium models and minimax problems can be formulated as an equilibrium problem (see e.g. [6, 9, 25, 27]).

In 2000, Giannessi [14], formulated the equilibrium problem for the case of vector bifunction and several extensions of the scalar equilibrium problem to the vector case have been considered. These

*Corresponding author
Email addresses: shokoohniam63@yahoo.com (Masoomeh Shokouhnia), s-moradi@araku.ac.ir (Sirous Moradi), s.jafari.math@gmail.com (Somaye Jafari)

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vector equilibrium problems, much like their scalar counterpart, offer a unified framework for treating vector optimization, vector variational inequalities and cone saddle point problems, to name just a few [15, 16, 1, 3, 4, 5].

In 2015 László and Viorel [23] introduced a notion of a self-segment-dense set in order to establish some existence results for set-valued equilibrium problems, where the conditions are imposed on a self-segment-dense subset of the domain of the involved bifunction. Jafari et al. in [19], presented a new concept "locally segment-dense set" and study existence results for equilibrium problems where the conditions are imposed only on a locally segment-dense subset in the domain of the involved bifunction. The natural question that comes to mind is that whether this results can be also true for vector equilibrium problems or not.

Recently, László [22] has shown that the results obtained in [23] are exactly true on vector cases. In this paper, we consider the results obtained in [19] where the bifunction is vector-valued. The paper is organized as follows. In Sect.2, we recall and introduce some definitions and auxiliary results needed for the definitions and proofs of results in the next sections. Afterwards, we provide some definitions of C-convexity, C-quasimonotonity and C-sequentially sign property. Section 3 is devoted to the main result of the paper, where we obtained existence results for vector equilibrium problems.

2. Preliminaries

Let X be a real Hausdorff, locally convex topological vector space. For a nonempty set $D \subseteq X$, we denote by $intD$ its interior, by clD its closure, by $convD$ its convex hull. We say that $P \subseteq D$ is dense in D iff $D \subseteq clP$. Recall that a set $C \subseteq X$ is a cone iff $tc \in C$ for all $c \in C$ and $t > 0$. The cone C is called a convex cone iff $C+C=C$. The cone C is called a pointed cone iff $C \bigcap (-C) = \{0\}$. Note that a closed, convex and pointed cone C induces a partial ordering on X, that is, $z_1 \leqslant z_2 \Leftrightarrow z_2 - z_1 \in C$ and $z_1 < z_2 \Leftrightarrow z_2 - z_1 \in intC$. In the sequel, when we use $intC$, we assume implicitly that the cone C has nonempty interior. It is obvious that $C + C\backslash\{0\} = C\backslash\{0\}$ and $intC + C = intC$. A well-known example of a closed, convex and pointed cone, with nonempty interior, is the nonnegative orthant of $\mathbb{R}^n$, that is,

$$\mathbb{R}^n_+ := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, i \in \{1, 2, \ldots, n\}\}.$$

Let Z be a locally convex Hausdorff topological vector spaces, $K \subseteq X$ be a nonempty subset and $C \subseteq Z$ be a convex and pointed cone with nonempty interior. For the bifunction $f : K \times K \longrightarrow Z$, the vector equilibrium problem (VEP) introduced in [5], consists in finding $x_0 \in K$, such that

$$f(x_0, y) \notin -intC, \forall y \in K. \tag{2.1}$$

Also, the strong vector equilibrium problem (SVEP) consists in finding $x_0 \in K$, such that

$$f(x_0, y) \notin -C\backslash\{0\}, \forall y \in K. \tag{2.2}$$

It can easily be observed that for $Z = \mathbb{R}$ and $C = \mathbb{R}^+ = [0, +\infty)$, the previous problems reduce to the classical scalar equilibrium problem. Note that, if $intC \neq \emptyset$ and $x_0 \in K$ is a solution of strong vector equilibrium problem, then x_0 is also a solution of vector equilibrium problem.

In this paper we mainly focus on vector equilibrium problems (VEP) which have attracted by many authors in recent years within the fields of vector optimization and vector variational inequalities (see, for instance, [7, 12, 17, 28] and the references therein).

The set of solutions of vector equilibrium problems (VEP) is denoted by $S(f, K, C)$. We say that an element $\bar{x} \in K$ is a local Minty solution, if there exists a neighbourhood U of $\bar{x}$ such that

$$f(y, \bar{x}) \notin intC, \forall y \in K \cap U.$$

The set of all local Minty solutions is denoted by $M_L(f,K,C)$. Notice that if $A \subseteq B$, then $M_L(f,B,C)\cap A \subseteq M_L(f,A,C)$ and $S(f,B,C)\cap A \subseteq S(f,A,C)$.

Definition 2.1. *[29] A map $f : K \longrightarrow Z$ is said to be C-lower semicontinuous (C-upper semicontinuous) at $x\in K$, iff for any neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(u)\in V+C$ $\Big(\ f(u)\in V-C\ \Big)$ for all $u\in U\cap K$.*

Obviously, if f is continuous at $x\in K$, then it is also C-lower semicontinuous at $x\in K$. Assume that C has nonempty interior. According to[31], f is C-lower semicontinuous at $x\in K$ iff for any $k\in intC$, there exists a neighbourhood U of x, such that $f(u)\in f(x)+k+intC$ for all $u\in U\cap K$. Even if $intC=\emptyset$ the strongly C-lower semicontinuity of f can be introduced as follows: f is strongly C-lower semicontinuous at $x\in K$ iff for any $k\in C\backslash\{0\}$, there exists a neighbourhood U of x such that $f(u)\in f(x)+k+C\backslash\{0\}$ for all $u\in U\cap K$.

Remark 2.2. *The map $f{:}K \longrightarrow Z$ is C-upper semicontinuous, (strongly C-upper semicontinuous) at $x\in K$ iff the map $-f$ is C-lower semicontinuous, (strongly C-lower semicontinuous) at $x\in K$.*

We say that f is C-lower semicontinuous, (strongly C-lower semicontinuous, C-upper semicontinuous, strongly C-upper semicontinuous) on K, if f is C-lower semicontinuous, (strongly C-lower semicontinuous, C-upper semicontinuous, strongly C-upper semicontinuous) at every $x\in K$. Obviously, if f is C-lower (resp. upper) semicontinuous on a subset A of X, then the restriction $f|A : A \longrightarrow Z$ of f on A is C-lower (resp. upper) semicontinuous on A. The function f is said to be C-continuous on D, if it is C-lower semicontinuous and C-upper semicontinuous on D.

Lemma 2.3. *[29] If $f : K \longrightarrow Z$ is a C-lower semicontinuous function, then the set $\{x\in K : f(x)\notin intC\}$ is closed in K.*

The following definition will be used in the sequel.

Definition 2.4. *[8] Let X be a real Hilbert space, and let S be a nonempty subset of X. Suppose that x is a point not lying in S. Suppose further that there exists a point $s\in S$ whose distance to x is minimal. Then s is called a closest point or a projection of x onto S. The vector $x-s$ is called a proximal normal direction to S at s. Any nonnegative multiple of such a vector is called a proximal normal to S at s, and the set of all proximal normals to S at s is denoted by $N_S^P(s)$. It is clear that $N_S^P(s)$ is in fact a cone.*

2.1. Locally Segment-Dense Sets

Let X be a real Hausdorff locally convex topological vector space and $x,y\in X$. Suppose that $[x,y]{:=}\{(1-t)x+ty : t\in[0,1]\}$ is the closed segment joining x and y. The semiopen segments [x, y[,]x, y] and the open segment]x, y[are defined analogously. The well-known segment-dense sets have been introduced by Luc [24]. Let $K \subseteq X$ be a convex set. We say that $U \subseteq K$ is segment-dense in K iff for each $x\in K$, there exists $y\in U$ such that x is a cluster point of the set $[x,y]\cap U$.

Recently, Lászlá and Viorel [23] introduced a notion of a self-segment-dense set, which is slightly different from the notion of the segment-dense set introduced by Luc [24]. Let K be a convex subset of X and $U \subseteq K \subseteq X$. The set U is called self segment-dense in K iff U is dense in K and for every $x,y\in U, cl([x,y]\cap U)=[x,y]$.

Obviously, a segment-dense set is always dense, but the converse is not always true, except in one-dimensional setting. It is easy to see that in one dimension, the notions of a segment-dense set and a self-segment-dense set reduce to the notion of a dense set. The difference between the notions of a segment-dense set and a self-segment-dense set in example 2.1 was demonstrated by László and Viorel [23], while an example of a subset that is dense but not self-segment-dense was also provided in example 2.2 in [23]. Jafari et al. in [19], presented a concept of "locally segment-dense sets". Let K be a convex subset of X and $D \subseteq K \subseteq X$. The set D is called locally segmentdense in K, iff for every $x, y \in D, cl([x, y] \cap D) = [x, y]$; and for every $x \in D$ and $y \in K$, the set $]x, y] \cap D$ is nonempty. Notice that it can be concluded from condition (i), $cl([x, z] \cap D) = [x, z]$ for every $z \in]x, y] \cap D$. As the next example shows, we can find locally segment-dense sets in K, which is neither segment-dense in K nor self-segment-dense in it.

Example 2.5. *Let $X = K := \mathbb{R}^2$, and let $D := \{(x, y) : x \in \mathbb{Q} \cap]-1, 1[, y \in]-1, 1[\}$, where $\mathbb{Q}$ denotes the set of all rational numbers. It is clear that D is locally segment-dense in K, but not dense in K.*

It must be noted that even in one dimension, the concept of a locally segment-dense is different from the concepts of a segment-dense set and a self-segment-dense set. For example, let $X = K := \mathbb{R}$ and $D :=]-1, 1[\cap \mathbb{Q}$. Then D is a locally segment-dense set in K, while it is neither segment-dense in K nor self-segment-dense in K.

Remark 2.6. *[19] It is worth mentioning that if U is a convex open neighbourhood of an element $x \in X$, then U is locally segment-dense in X. Indeed, every convex algebraically open subset $U \subseteq X$ is locally segment-dense in X. We recall that U is algebraically open (due to [20]) if $U = core(U)$, where*

$$core(U) := \{\bar{x} \in U : \ \forall x \in X \ \exists \bar{t} > 0 \ \ such\ that\ \bar{x} + tx \in U, \ \forall t \in [0, \bar{t}]\}.$$

Remark 2.7. *Suppose D be a locally segment-dense set in K. If $x \in D$ and $y \in K$, then there can be found $\{z_n\} \subset]x, y] \cap D$ such that $z_n \to x$ as $n \to +\infty$. This is due to the definition of locally segment-dense set D in K, which allows us to find $z \in]x, y] \cap D$ such that $cl([x, z] \cap D) = [x, z]$.*

The next result obtained in [23] is also valid for locally segment-dense sets.

Lemma 2.8. *Let X be a real Hausdorff locally convex topological vector space, K be a convex subset of X, and let $U \subseteq K$ be such that for every $x, y \in U$, it holds that $cl([x, y] \cap U) = [x, y]$. Then for all finite subsets $\{u_1, u_2, \ldots, u_n\} \subseteq U$, one has*

$$cl(conv\{u_1, u_2, \ldots, u_n\} \cap U) = conv\{u_1, u_2, \ldots, u_n\}.$$

2.2. C-convexity, C-quasimonotone and C-sequentially sign property

In the sequal, we suppose X and Z are real Hausdorff locally convex topological vector spaces, D is a locally segment-dense set in K (a nonempty subset of X) and $f : X \longrightarrow Z$ is a function. Assume also that $C \subseteq Z$ is a convex and pointed cone with $intC \neq \emptyset$ that C induces a partial ordering on Z.

Definition 2.9. *[26, 21] The function f is C-convex on D, iff for all $x, y \in D$ and $t \in [0, 1]$ such that $tx + (1 - t)y \in D$, then $tf(x) + (1 - t)f(y) - f(tx + (1 - t)y) \in C, \forall t \in [0, 1]$. f is said to be C-concave iff -f is C-convex.*

Definition 2.10. *[13, 2] The function f is C-quasimonotone on D, iff for $x, y \in D$,*

$$f(x, y) \in intC \Rightarrow f(y, x) \notin intC.$$

Definition 2.11. *[2] The function f is properly C-quasimonotone on D, iff for every subset of finite elements $\{x_1, x_2, \ldots, x_n\} \subseteq D$ and every $\bar{x} \in conv\{x_1, x_2, \ldots, x_n\} \cap D$, there exists $i \in \{1, 2, \ldots, n\}$ such that $f(x_i, \bar{x}) \notin intC$.*

Motivated by the notion of the strong upper sign property recently introduced in [18], we define C-strong upper sign property and a useful notion of C-sequentially sign property for vector bifunctions. The following definitions are vector versions of the definitions introduced in [18].

Definition 2.12. *We say that f has the C-strong upper sign property with respect to the first variable at $x \in K \subseteq X$, iff for every $y \in K$ the following implication holds:*

$$\exists \delta \in]0,1] : f(z_t, x) \notin intC, \ \forall t \in]0, \delta[\Rightarrow f(x, y) \notin -intC$$

where $z_t = tx + (1-t)y$.

Definition 2.13. *Let K a convex subset of X and D be a locally segment-dense set in K. We say that f has the C-sequentially sign property with respect to the first variable at $x \in K \subseteq X$, iff for every $y \in K$ the following implication holds:*

$$if\ \{z_n\} \subset]x, y] \cap D : z_n \to x \ and\ f(z_n, x) \notin intC, \ for\ all\ n \in \mathbb{N}\ then\ f(x, y) \notin -intC.$$

We say that f has the C-sequentially sign property on D, iff f has this property at every $x \in D$. The following result highlights a large class of bifunctions, which have the C-sequentially sign property and is a vector version of proposition 2.2 in [19].

Proposition 2.14. *Let K a convex subset of X and D be a locally segment-dense set in K and $f : K \times K \longrightarrow Z$ a bifunction, which satisfies the following conditions:*

(1) for every $x \in D, f(x, x) \in C$;

(2) for every $y \in K, f(\cdot, y)$ is C-upper semicontinuous on D;

(3) for every $x, y_1 \in D$ and $y_2 \in K$ the following implication holds:

$$f(x, y_1) \notin intC, \ f(x, y_2) \notin C \Rightarrow f(x, z_t) \notin C, \ \forall t \in]0, 1[, \tag{2.3}$$

where $z_t = ty_1 + (1-t)y_2$.

Then f has the C-sequentially sign property on D.

Proof . Suppose by contradiction that f does not have the C- sequentially sign property at some $x_0 \in D$. Hence, there exists $y_0 \in K$, for which there is a sequence $\{z_n\} \subset]x_0, y_0] \cap D$ with $z_n \longrightarrow x_0$ such that $f(z_n, x_0) \notin intC$ for all $n \in \mathbb{N}$ and $f(x_0, y_0) \in -intC$. So there exists a neighbourhood V of $f(x_0, y_0)$ that $V \subseteq -intC$. Since $f(\cdot, y_0)$ is C-upper semicontinuous, there exists a neighbourhood U of x_0 such that,

$$f(u, y_0) \subseteq V - C \subseteq -intC, \ \forall u \in U \cap D.$$

Since $z_n \longrightarrow x_0$, there exists $n_0 \in \mathbb{N}$ that for all $n \geqslant n_0$, $f(z_n, y_0) \in -intC$. Now, for every $n \geqslant n_0$, if we take $x{:=}z_n, y_1{:=}x_0$ and $y_2{:=}y_0$ in (3.2), we deduce that $f(z_n, z_t) \notin C$, for all $t \in]0, 1[$, where $z_t{=}(1-t)x + ty$. The latter contradicts to $f(z_n, z_n) \in C$ for all $n \geqslant n_0$. Thus, $f(x_0, y_0) \notin -intC$ and this completes the proof. □ The following definition is a common generalization of locally segment-dense Minty solution to the vector case.

Definition 2.15. *Let K a convex subset of X and D be a locally segment-dense set in K and $f : K \times K \longrightarrow \mathbb{R}$ a bifunction. We say that $\bar{x}\in D$ is a locally segment-dense Minty solution, iff there exists a neighbourhood U of $\bar{x}$ such that*

$$f(y, \bar{x}) \notin intC, \forall y\in D \cap U.$$

The set of all locally segment-dense Minty solution is denoted by $M_L^D(f, K, C)$. It is worth noting that if K be a subset convex of X, then $M_L(f, K, C) \cap D \subseteq M_L^D(f, K, C)$ and the inclusion may be strict. Hence, there are more possibilities that $M^D(f, K, C)$ can be nonempty. For example, let $X = K := \mathbb{R}$ *and* $D :=] - 1, 1[\cap\mathbb{Q}$. Consider the bifunction $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} -2, & x, y\in D \\ \\ 2, & otherwise \end{cases} \tag{2.4}$$

It is easy to check that $M_L(f, K, C) = \emptyset$ while $M_L^D(f, K, C) \neq \emptyset$.

In the following lemma, we show that the rather large set $M_L^D(f, K, C)$ is a subset of $S(f, K, C)$ under the condition of the C-sequentially sign property of the involved bifunction.

Lemma 2.16. *Let K a convex subset of X and D be a locally segment-dense set in K and $f : K \times K \longrightarrow Z$ a bifunction with the C-sequentially sign property. Then $M^D(f, K, C) \subseteq S(f, K, C)$.*

Proof . Let $\bar{x}$ be an element of $M_L^D(f, K, C)$. Then there exists a neighbourhood U of $\bar{x}$ such that

$$f(y, \bar{x}) \notin intC, \ \forall y\in D \cap U. \tag{2.5}$$

To verify that $\bar{x}\in S(f, K, C)$, take $y_0\in K$. It follows from Remark 2.7, there is a sequence $\{z_n\} \subset]\bar{x}, y] \cap D$ such that $z_n \longrightarrow \bar{x}$. Choose $n_0\in\mathbb{N}$ such that $z_n\in U$ for every $n \geq n_0$. It follows from (2.5) that for all $n \geq n_0$, $f(z_n, \bar{x}) \notin intC$. Now, employing the C-sequentially sign property, we conclude that $f(\bar{x}, y_0) \notin -intC$ and this completes the proof. □

3. Existence Results for Vector Equilibrium Problems

The purpose of this section is to use locally segment-dense sets to obtain some existence results for vector equilibrium problems with unnecessarily compact domains. Hence, we use a kind of a coercivity condition upon the bifunction involved. In the sequel let $F : K \rightrightarrows K$ be a set-valued mapping defined by

$$F(y) := \{x\in K : f(y, x) \notin intC\} \ \ \forall y\in K. \tag{3.1}$$

Theorem 3.1. *Let K be a convex subset of X and D be a locally segment-dense set in K and $f : K \times K \longrightarrow Z$ a bifunction satisfying the following conditions:*

(1) f is C-quasimonotone on D, which is not properly C-quasimonotone on D;

(2) for every $y\in D$, $F(y)$ is closed in $K\backslash D$, i.e.,

$$cl(F(y)) \cap (K\backslash D) = F(y) \cap (K\backslash D) = \{x\in K\backslash D : f(y, x) \notin intC\}; \tag{3.2}$$

(3) for every $y\in D$, $F(y)$ is convex on D. Namely, for every $x_1, x_2\in F(y)\cap D$ and $t\in[0, 1]$ such that $\bar{x} = tx_1 + (1 - t)x_2\in D$, then $\bar{x}\in F(y)$.

Then $M_L^D(f,K,C) \neq \emptyset$.

Proof . Since f is not properly C-quasimonotone on D, there exist $x_1, x_2, \ldots, x_n \in D$ and $\bar{x} \in conv\{x_1, x_2, \ldots, x_n\}$ D such that

$$f(x_i, \bar{x}) \in intC, i = 1, 2, \ldots, n.$$

Thus, from $\bar{x} \notin F(x_i) \cap (K \backslash D)$ and condition (2), we derive that $\bar{x} \notin cl(F(x_i)) \cap (K \backslash D)$ for all $i \in \{1, 2, \ldots, n\}$. Hence, for each $i \in \{1, 2, \ldots, n\}$ there exists a neighbourhood U_i of $\bar{x}$ such that $U_i \cap D \subseteq (X \backslash (F(x_i) \cap D))$. If we set $U = \cap_{i=1}^{n} U_i$, then we get

$$f(x_i, y) \in intC, \ \forall y \in U \cap D, i = 1, \ldots, n.$$

Now, the C-quasimonotonicity of f on D implies that

$$f(y, x_i) \notin intC, \ \forall y \in U \cap D, i = 1, \ldots, n.$$

Furthermore, for arbitrary and fixed $y \in U \cap D$, we have $x_i \in F(y)$ for all $i = 1, 2, \ldots, n$. Using the convexity of $F(y)$ on D, we deduce that $f(y, \bar{x}) \notin intC$ for all $y \in U \cap D$. Hence, $\bar{x} \in M_L^D(f, K, C)$. □
Now, the existence of solutions for vector equilibrium problems (VEP) can be obtained.

Corollary 3.2. *Let X be a real Hausdorff locally convex topological vector space, K be a convex subset of X and D be a locally segment-dense set in K, that satisfies the conditions of Theorem 3.1. If f have the C-sequentially sign property, then for every subset convex K of X, that $D \subseteq K$, $S(f, K, C) \neq \emptyset$.*

Remark 3.3. *We note that if f be a real bifunction, the quasiconvexity of f implies the convexity of $F(y)$ [19], but in topological vector spaces we can not conclude the convexity of $F(y)$ even if f is convex. The following examples show that the convexity of $F(y)$ is related to the cone as well as the function. It means that for the convex function f with values in $\mathbb{R}^2$ is convex with respect to the ordering induced by the cone C_1 and is not convex with respect to the ordering induced by the cone C_2.*

Example 3.4. *Let $X = Z = K = D = \mathbb{R}^2$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ with $g(x) =\mid x \mid$ and $C_1 = N_{epig}^P(0,0)$. Let $f : K \times K \longrightarrow Z$ is defined by $f(x, y) = y$. It is easy to check that f is convex on D with respect to the second variable. We claim that $F(y)$ is not convex for every $y \in K$. Let $y \in K$, we have*

$$F(y) := \{x \in K : f(y, x) \notin intC_1\} \quad \forall y \in K.$$

Now suppose that $x_1 = (-1, -1)$ and $x_2 = (1, -1)$. Obviously $x_1, x_2 \in F(y)$, for all $y \in K$. On the other hand, $\bar{x} = (0, -1) \in intC_1$, where $\bar{x} = \frac{1}{2}x_1 + \frac{1}{2}x_2$. Note that in fact $\bar{x} = (1-t)x_1 + tx_2 \in intC_1$ for all $t \in]0, 1[$.

Example 3.5. *In the previous example let cone $C_2 = \{(x, y) : x \in \mathbb{R}, y \in] - \infty, 0]\}$. It is easy to check that $F(y)$ is convex.*

It is worth noting that if $C = [0, +\infty[$ then the example of Jafari et al. in [19] shows that the properly C-quasimonotonicity of f is essential essential in Colorally 3.2 . Until now, we show the nonemptiness of $S(f, K, C)$, where f is C-quasimonotone on D and is not properly C-quasimonotone on D. Now, we follow KKM technique to obtain existence results for vector equilibrium problems when the bifunction f is properly C-quasimonotone on D. Ky Fan in 1984 extended the well-known Fan-KKM theorem [10] in order to relax the compactness condition and obtained the following result, which is known as Fan's Lemma.

Lemma 3.6. *[11] Let K be a nonempty subset of a Hausdorff topological vector space X and $\Gamma : K \rightrightarrows X$ be a set-valued mapping such that*

(1) Γ has closed values;

(2) Γ is a KKM mapping, that is, for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in K$

$$conv\{x_1, \ldots, x_n\} \subseteq \cup_{i=1}^{n} \Gamma(x_i);$$

(3) there exists a nonempty compact convex subset B of K such that $\cap_{x \in B} \Gamma(x)$ is compact.

Then $\cap_{x \in C} \Gamma(x) \neq \emptyset$.

We use following definitions and lemma to prove the vector version of theorem 3.2 in [19].

Definition 3.7. *[30] Let Λ be a nonempty set and X be a topological space. A set-valued mapping $F : \Lambda \rightrightarrows X$ is said to be intersectionally closed on Λ iff*

$$\cap_{y \in \Lambda} cl(F(y)) = cl(\cap_{y \in \Lambda} F(y)).$$

Definition 3.8. *[30] Let Λ be a nonempty set and X a topological space. A set-valued mapping $F : \Lambda \rightrightarrows X$ is said to be transfer closed on Λ iff*

$$\cap_{y \in \Lambda} cl(F(y)) = \cap_{y \in \Lambda} F(y).$$

Definition 3.9. *[30] Let Λ be a nonempty set and X a topological space. A set-valued mapping $F : \Lambda \rightrightarrows X$ is said to be unionly open on Λ iff*

$$\cup_{y \in \Lambda} int(F(y)) = int(\cup_{y \in \Lambda} F(y)).$$

It is clear that transfer closed maps are intersectionally closed. The converse is not true in general. For instance, $F(\lambda) =]0, 1[$ for every $\lambda \in [0, 1]$ is a constant set-valued map from $\Lambda = [0, 1]$ to $X = [0, 1]$ and is intersectionally closed, but not transfer closed.

Proposition 3.10. *[30] The set-valued map F is intersectionally closed iff its complement F^c is unionly open. Denote by F^c the complement of F, that is $F^c(\lambda) = X \setminus F(\lambda)$ for $\lambda \in \Lambda$.*

Definition 3.11. *The function $f : K \times K \longrightarrow Z$ is C-transfer lower semicontinuous on B if for every $x \in B$ and $y \in K$, $f(x, y) \in intC$ implies that there exist some point $\bar{x} \in B$ and some neighbourhood U of y such that $f(\bar{x}, z) \in intC$ for all $z \in U \cap K$.*

Remark 3.12. *It is easy to check that, if f is C-lower semicontinuous on B, then it is C-transfer lower semicontinuous on B.*

Lemma 3.13. *If $f : K \times K \longrightarrow Z$ is C-transfer lower semicontinuous on B, then the set-valued mapping F in relation* (3.1) *is intersectionally closed on B.*

Proof . It sufficient to show that the set $F^c(y) = \{x \in K : f(y,x) \in intC\}$ is unionly open. Let $\bar{x}$ be an element from the interior of the set $\cup_{\lambda\in\Lambda} F^c(\lambda)$. Then, there exists a neighbourhood V of $\bar{x}$ such that for every $x \in V$, there is some $\lambda \in \Lambda$ for which we have $f(\lambda, x) \in intC$. Since f is C-transfer lower semicontinuous, there are some λ_0 and a neighbourhood V_0 of $\bar{x}$ such that $V_0 \subseteq F^c(\lambda_0)$. Consequently,

$$int(\cup_{\lambda\in\Lambda} F^c(\lambda)) \subseteq \cup_{\lambda\in\Lambda} int(F^c(\lambda)).$$

The converse inclusion being evident, we obtain equality, by which F^c is transfer open. Due to Proposition 3.10, the map F is intersectionally closed. □ Note that according to Proposition 2.3 in [30], if the set-valued mapping F is transfer closed-valued on B in the sense of Tian [32] or $F(y)$ is closed for every $y \in B$, then the set-valued mapping F is intersectionally closed on B.

Theorem 3.14. *Let K a convex subset of X and D be a locally segment-dense set in K and $f : K \times K \longrightarrow Z$ be a properly C-quasimonotone bifunction on D satisfying the following conditions:*

(1) there exist a nonempty compact subset $K_0 \subseteq K$ and nonempty convex compact subset $B \subseteq D$ such that for every $x \in K \backslash K_0$, there exists $y \in B$ such that $f(y,x) \in intC$ and F is intersectionally closed on B;

(2) there exists a nonempty subset $A \subseteq D$ such that for every $x \in K \backslash A$, there exists $y \in A$ such that $f(y,x) \in intC$;

(3) for every $x \in D, f(x,\cdot)$ is C-lower semicontinuous on K_0.

Then $M^D(f,K,C) \neq \emptyset$.

Proof . Define a set-valued mapping $cl(F) : D \rightrightarrows K$ by

$$cl(F)(y) := cl(F(y)), \forall y \in D.$$

Obviously, cl(F)(y) is closed for every $y \in D$. It is easy to conclude from the assumption (1) that $\cap_{y\in B} F(y) \subseteq K_0$ and

$$\cap_{y\in B} cl(F)(y) = cl(\cap_{y\in B} F(y)) \subseteq K_0.$$

Hence, $\cap_{y\in B}\, cl(F)(y)$ is compact. To verify that cl(F) is a KKM mapping, let $y_1, y_2, \ldots, y_n \in D$ and $\overline{y} \in conv\{y_1, y_2, \ldots, y_n\} \cap D$. Since f is properly C-quasimonotone on D, there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $f(y_{i_0}, \overline{y}) \notin intC$, which yields $\overline{y} \in \cup_{i=1}^n F(y_i)$. This means that

$$conv\{y_1, y_2, \ldots, y_n\} \cap D \subseteq \cup_{i=1}^n F(y_i),$$

and then,

$$cl(conv\{y_1, y_2, \ldots, y_n\} \cap D) \subseteq cl(\cup_{i=1}^n F(y_i) = \cup_{i=1}^n cl(F)(y_i).$$

It follows from Lemma 2.8 that

$$conv\{y_1, y_2, \ldots, y_n\} \subseteq \cup_{i=1}^n cl(F)(y_i),$$

which means that cl(F) is a KKM mapping. Now it follows from lemma 3.6 that $\cap_{y\in D} cl(F)(y) \neq \emptyset$. Since $\cap_{y\in B}\, F(y) \subseteq K_0$, we imply that

$$\cap_{y\in D} cl(F)(y) = (\cap_{y\in D} cl(F)(y)) \cap K_0 = \cap_{y\in D}(cl(F)(y) \cap K_0).$$

Moreover, since for every $y \in D, f(y,.)$ is C-lower semicontinuous on K_0, by employing Lemma 2.3, we have $cl(F)(y) \cap K_0 = F(y) \cap K_0$. Hence, $\cap_{y\in D} F(y) = \cap_{y\in D}(F(y) \cap K_0) \neq \emptyset$. In addition, the assumption (2) allows us to find $\bar{x} \in (\cap_{y\in D} F(y)) \cap D$. This means that $M_L^D(f,K,C) \neq \emptyset$. □ In the following theorem, by eliminating the condition involving $\cap_{y\in B}\, cl(F(y)) = \cap_{y\in B} F(y)$, the same result as Theorem 3.14 is achieved.

Theorem 3.15. *Let K a convex subset of X and D be a locally segment-dense set in K, and $f : X \times X \longrightarrow Z$ a properly C-quasimonotone bifunction on D satisfying the following conditions:*

(1) there exist a nonempty compact subset $K_0 \subseteq K$ and $y_0 \in D$ such that

$$f(y_0, x) \in intC, \forall x \in K \backslash K_0;$$

(2) there exists a nonempty subset $A \subseteq D$ such that for every $x \in K \backslash A$, there exists $y \in A$ such that $f(y, x) \in intC$;

(3) for every $x \in D, f(x, \cdot)$ is C-lower semicontinuous on K_0.

Then $M_L^D(f, K, C) \neq \emptyset$.

Proof . Let $B = \{y_0\}$ in the condition (1) of Theorem 3.14. □

Corollary 3.16. *Let K a convex subset of X and D be a locally segment-dense set in K. Let $f : K \times K \longrightarrow Z$ be a bifunction satisfying all conditions of Theorems 3.14 or 3.15. If f has C-sequentially sign property, then $S(f, K, C) \neq \emptyset$.*

Proof . The assertion immediately follows from Theorems 3.14 or 3.15 and then Lemma 2.16. □
Now, we are in a position to state our main result for C-quasimonotone equilibrium problems.

Theorem 3.17. *Let K a convex subset of X and D be a locally segment-dense set in K. Let $f : K \times K \longrightarrow Z$ be a C-quasimonotone bifunction on D satisfying the following conditions:*

(1) for every $y \in D$, $F(y)$ is closed in $K \backslash D$ and convex on D;

(2) there exist a nonempty compact subset $K_0 \subseteq K$ and $y_0 \in D$ such that

$$f(y_0, x) \in intC, \forall x \in K \backslash K_0;$$

(3) there exists a nonempty subset $A \subseteq D$ such that for every $x \in K \backslash A$, there exists $y \in A$ such that $f(y, x) \in intC$;

(4) for every $x \in D$, $f(x, \cdot)$ is C-lower semicontinuous on K_0;

(5) f has sequentially sign property.

Then $S(f, K, C) \neq \emptyset$.
Proof . *We split the proof into the following two cases.*
Case 1: f is C-quasimonotone on D and not properly C-quasimonotone on D. It follows from Corollary 3.2 that $S(f, K, C) \neq \emptyset$.
Case 2: f is properly C-quasimonotone on D. It follows from Corollary 3.16 that $S(f, K, C) \neq \emptyset$. □

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