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Common fixed point theorems with applications to theoretical computer science

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Abstract

Owing to the notion of L-fuzzy mapping, we establish some common L-fuzzy fixed point results for almost Θ-contraction in the setting of complete metric spaces. An application to theoretical computer science is also provided to show the significance of the investigations.

Keywords: Fixed point, Θ-contraction, metric space, L-fuzzy mappings.
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1 Introduction and preliminaries

Answering real-world problems becomes evidently uncomplicated with the initiation of fuzzy set theory in 1965 by Zadeh [35], as it helps in making the explanation of obscurity and inaccuracy fair and more accurate. subsequently, Goguen [17] modified this concept to L-fuzzy set theory by replacing the interval $[0, 1]$ in 1967. There are fundamentally two perceptive of the meaning of L, one is when L is a complete lattice equipped with a multiplication $*$ operator satisfying certain assumptions as shown in the basic paper [17] and the second perceptive of the meaning of L is that L is a completely distributive complete lattice with an order-reversing involution .

Definition 1.1. [17] A partially ordered set $(L, \precsim_L)$ is called

$i)$ a lattice, if $a_1 \vee a_2 \in L$, $a_1 \wedge a_2 \in L$ for each $a_1, a_2 \in L$.

$ii)$ a complete lattice, if $\vee A \in L$, $\wedge A \in L$ for any $A \subseteq L$.

$iii)$ distributive lattice if $a_1 \vee (a_2 \wedge a_3) = (a_1 \vee a_2) \wedge (a_1 \vee a_3)$, $a_1 \wedge (a_2 \vee a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3)$, for any $a_1, a_2, a_3 \in L$.

Definition 1.2. [17] Let L be a lattice with top element 1_L and bottom element 0_L and let $a_1, a_2 \in L$. Then a_2 is said to be a complement of a_1, if $a_1 \vee a_2 = 1_L$, and $a_1 \wedge a_2 = 0_L$. If $a \in L$ has a complement element, then it is unique. It is denoted by $\acute{a}$.

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Definition 1.3. [17] A $L-$fuzzy set A on a nonempty set $\mathcal{S}$ is a function $A : \mathcal{S} \to L$, where L is complete distributive lattice with 1_L and 0_L.

Remark 1.4. An $L-$fuzzy set is a fuzzy set if $L = [0, 1]$, so the family of $L-$fuzzy sets is larger than the family of fuzzy sets.

The α_L-level set of $L-$fuzzy set A, is designated by A_{α_L}, and is given in this way.

$$A_{\alpha_L} = \{u : \alpha_L \precsim_L A(u)\} \text{ if } \alpha_L \in L \backslash \{0_L\},$$

$$A_{0_L} = \overline{\{u : 0_L \precsim_L A(u)\}}.$$

Here $cl(B)$ stands for the closure of the set B. The characteristic function of a L-fuzzy set A is denoted by χ_{L_A} and is defined as follows:

$$\chi_{L_A} := \begin{cases} 0_L & \text{if } u \notin A \\ 1_L & \text{if } u \in A \end{cases}.$$

In 2014, Azam et al. [29] initiated the concept of β_{F_L}-admissible for a pair of L-fuzzy mappings and exploited it to establish a common L-fuzzy fixed point theorem.

Definition 1.5. **[29]** Let $\mathcal{S}_1$ be an arbitrary set, $\mathcal{S}_2$ be a metric space. A mapping $\mathcal{Q}$ is said to be an $L-$fuzzy mapping if $\mathcal{Q}$ is a mapping from $\mathcal{S}_1$ into $\Im_L(\mathcal{S}_2)$. An $L-$fuzzy mapping $\mathcal{Q}$ is a $L-$fuzzy subset on $\mathcal{S}_1 \times \mathcal{S}_2$ with membership function $\mathcal{Q}(u)(v)$. The function $\mathcal{Q}(u)(v)$ is the grade of membership of v in $\mathcal{Q}(u)$.

Definition 1.6. [29] Let $(\mathcal{S}, \sigma)$ be a metric space and $\mathcal{P}, \mathcal{Q}$ be $L-$fuzzy mappings from $\mathcal{S}$ into $\Im_L(\mathcal{S})$. A point $z \in \mathcal{S}$ is called a $L-$fuzzy fixed point of $\mathcal{Q}$ if $u^* \in [\mathcal{Q}u^*]_{\alpha_L}$, where $\alpha_L \in L \backslash \{0_L\}$. The point $u^* \in \mathcal{S}$ is called a common $L-$fuzzy fixed point of $\mathcal{P}$ and $\mathcal{Q}$ if $u^* \in [\mathcal{P}u^*]_{\alpha_L} \cap [\mathcal{Q}u^*]_{\alpha_L}$. When $\alpha_L = 1_L$, it is called a common fixed point of $L-$fuzzy mappings.

In 2015, Jleli et al. [24]gave the notion of Θ-contractions and proved some new fixed point results for such contractions in the setting of generalized metric spaces.

Definition 1.7. Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:

(Θ_1) Θ is nondecreasing;

(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty}(\alpha_n) = 0$;

(Θ_3) there exists $0 < r < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha \to 0^+} \frac{\Theta(\alpha)-1}{\alpha^r} = l$.

A mapping $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and a constant $k \in (0, 1)$ such that for all $u, v \in \mathcal{S}$,

$$\sigma(\mathcal{P}u, \mathcal{P}v) > 0 \Longrightarrow \Theta(\sigma(\mathcal{P}u, \mathcal{P}v)) \leq [\Theta(\sigma(u, v))]^k. \tag{1.1}$$

Theorem 1.8. [24] Let $(\mathcal{S}, \sigma)$ be a complete metric space and $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ be a Θ-contraction, then $\mathcal{P}$ has a unique fixed point.

They demonstrated that any Banach contraction is a specific case of Θ-contraction while there are Θ-contractions which are not Banach contractions. We express by Ψ the set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above assertions (Θ_1)-(Θ_3), consistent with Jleli et al. [24],.

Later on Altune et al.[18] modified the above definitions by adding a general condition (Θ_4) which is given as follows.

(Θ_4) $\Theta(\inf A) = \inf \Theta(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

Following Altune et al.[18], we represent the set of all continuous functions $\Theta : \mathbb{R}^+ \to \mathbb{R}$ satisfying $(\Theta_1) - (\Theta_4)$ conditions by $\digamma$. For more details on Θ-contraction, we refer the reader to [4, 27]. For the sake of convenience, we first state the following lemma for subsequent use in the next section. Let $(\mathcal{S}, \sigma)$ be a metric space and $CB(\mathcal{S})$ be the family of nonempty, closed and bounded subsets of $\mathcal{S}$. For$A, B \in CB(\mathcal{S})$, define

$$\mathcal{H}(A, B) = \max \left\{ \sup_{a \in A} \sigma(a, B), \sup_{b \in B} \sigma(b, A) \right\}$$

where

$$\sigma(u, A) = \inf_{v \in A} \sigma(u, v).$$

Lemma 1.9. [29]Let $(\mathcal{S}, \sigma)$ be a metric space and $A, B \in CB(\mathcal{S})$, then for each $a \in A$,

$$\sigma(a, B) \leq \mathcal{H}(A, B).$$

In this paper, we obtain common L-fuzzy fixed point theorems for almost Θ-contraction in the setting of complete metric spaces. A significant example is also given to illustrate the validity of main result.

2 Main Results

In this way, we state and prove a common fixed point theorem for L-fuzzy mappings.

Theorem 2.1. Let $(\mathcal{S}, \sigma)$ be a complete metric space and $\{\mathcal{P}, \mathcal{Q}\}$ be a pair of L-fuzzy mappings from $\mathcal{S}$ into $\Im_L(\mathcal{S})$ and for each $\alpha_L \in L \backslash \{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{Q}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in \digamma$, $k \in (0, 1)$ and $L \geq 0$ such that

$$\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{Q}v]_{\alpha_L(v)}\right) > 0 \Longrightarrow \Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{Q}v]_{\alpha_L(v)}\right)\right) \leq \Theta(\sigma(u, v))^k + Lm(u, v) \tag{2.1}$$

for all $u, v \in \mathcal{S}$, where

$$m(u, v) = \min\left\{\sigma\left(u, [\mathcal{P}u]_{\alpha_L(u)}\right), \sigma\left(v, [\mathcal{Q}v]_{\alpha_L(v)}\right), \sigma\left(u, [\mathcal{Q}v]_{\alpha_L(v)}\right), \sigma\left(v, [\mathcal{P}u]_{\alpha_L(u)}\right)\right\}. \tag{2.2}$$

Then $\mathcal{P}$ and $\mathcal{Q}$ have a common L-fuzzy fixed point.

Proof . Let u_0 be an arbitrary point in $\mathcal{S}$, then by hypotheses there exists $\alpha_L(u_0) \in L \backslash \{0_L\}$ such that $[\mathcal{P}u_0]_{\alpha_L(u_0)}$ is a nonempty closed bounded subset of $\mathcal{S}$ and let $u_1 \in [\mathcal{P}u_0]_{\alpha_L(u_0)}$. For this u_1 there exists $\alpha_L(u_1) \in L \backslash \{0_L\}$ such that $[\mathcal{Q}u_1]_{\alpha_L(u_1)}$ is a nonempty, closed and bounded subset of $\mathcal{S}$. By Lemma 1.9, (Θ_1) and (2.1), we have

$$\Theta(\sigma\left(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right) \leq \Theta\left(\mathcal{H}\left([\mathcal{P}u_0]_{\alpha_L(u_0)}, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right)\right) \leq \Theta(\sigma(u_0, u_1))^k + Lm(u_0, u_1) \tag{2.3}$$

where

$$m(u_0, u_1) = \min\left\{\sigma\left(u_0, [\mathcal{P}u_0]_{\alpha_L(u_0)}\right), \sigma\left(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \sigma\left(u_0, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \sigma\left(u_1, [\mathcal{P}u_0]_{\alpha_L(u_0)}\right)\right\}.$$

From (Θ_4), we know that

$$\Theta\left(\sigma\left(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right)\right) = \inf_{v \in [\mathcal{Q}u_1]_{\alpha_L(u_1)}} \Theta(\sigma(u_1, v)).$$

Thus from (2.3), we get

$$\inf_{v \in [\mathcal{Q}u_1]_{\alpha_L(u_1)}} \Theta(\sigma(u_1, v)) \leq \Theta(\sigma(u_0, u_1)^k + \min\left\{\begin{array}{c} \sigma\left(u_0, [\mathcal{P}u_0]_{\alpha_L(u_0)}\right), \sigma\left(u_1, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \\ \sigma\left(u_0, [\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \sigma\left(u_1, [\mathcal{P}u_0]_{\alpha_L(u_0)}\right) \end{array}\right\} \tag{2.4}$$

Then, from (2.4), there exists $u_2 \in [\mathcal{Q}u_1]_{\alpha_L(u_1)}$ such that

$$\Theta(\sigma(u_1,u_2)) \leq [\Theta(\sigma(u_0,u_1)]^k + \min\{\sigma(u_0,u_1),\sigma(u_1,u_2),\sigma(u_0,u_2),\sigma(u_1,u_1)\}.$$

Thus we have

$$\Theta(\sigma(u_1,u_2)) \leq [\Theta(\sigma(u_0,u_1)]^k. \tag{2.5}$$

For this u_2 there exists $\alpha_L(u_2) \in L\backslash\{0_L\}$ such that $[\mathcal{P}u_2]_{\alpha_L(u_2)}$ is a nonempty closed bounded subset of $\mathcal{S}$. By Lemma 1.9, (Θ_1) and (2.1), we have

$$\begin{aligned}\Theta\left(\sigma\left(u_2,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\right) &\leq \Theta(\mathcal{H}\left([\mathcal{Q}u_1]_{\alpha_L(u_1)},[\mathcal{P}u_2]_{\alpha_L(u_2)}\right) = \Theta(\mathcal{H}\left([\mathcal{P}u_2]_{\alpha_L(u_2)},[\mathcal{Q}u_1]_{\alpha_L(u_1)}\right) \\ &\leq [\Theta(\sigma(u_2,u_1)]^k + Lm(u_2,u_1)\end{aligned}$$

thus we get

$$\Theta\left(\sigma\left(u_2,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\right) \leq\leq [\Theta(\sigma(u_2,u_1)]^k + Lm(u_2,u_1) \tag{2.6}$$

where

$$m(u_2,u_1) = \min\left\{\sigma\left(u_2,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right),\sigma\left(u_1,[\mathcal{Q}u_1]_{\alpha_L(u_1)}\right),\sigma\left(u_2,[\mathcal{Q}u_1]_{\alpha_L(u_1)}\right),\sigma\left(u_1,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\right\}$$

which further implies that

$$\Theta\left(\sigma\left(u_2,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\right) \leq \Theta\left[\sigma(u_1,u_2)\right]^k + \min\left\{\begin{array}{c}\sigma\left(u_2,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right),\sigma\left(u_1,[\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \\ \sigma\left(u_2,[\mathcal{Q}u_1]_{\alpha_L(u_1)}\right),\sigma\left(u_1,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\end{array}\right\}. \tag{2.7}$$

From (Θ_4), we know that

$$\Theta\left[\sigma\left(u_2,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\right] = \inf_{v_1\in[\mathcal{P}u_2]_{\alpha_L(u_2)}} \Theta(\sigma(u_2,v_1))$$

$$\inf_{v_1\in[\mathcal{P}u_2]_{\alpha_L(u_2)}} \Theta(\sigma(u_2,v_1)) \leq \Theta\left[\sigma(u_1,u_2)\right]^k + \min\left\{\begin{array}{c}\sigma\left(u_2,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right),\sigma\left(u_1,[\mathcal{Q}u_1]_{\alpha_L(u_1)}\right), \\ \sigma\left(u_2,[\mathcal{Q}u_1]_{\alpha_L(u_1)}\right),\sigma\left(u_1,[\mathcal{P}u_2]_{\alpha_L(u_2)}\right)\end{array}\right\}. \tag{2.8}$$

Then, from (2.8), there exists $u_3 \in [\mathcal{P}u_2]_{\alpha_L(u_2)}$ such that

$$\begin{aligned}\Theta(\sigma(u_2,u_3)) \leq\ & [\Theta(\sigma(u_1,u_2)]^k \\ & + \min\{\sigma(u_2,u_3),\sigma(u_1,u_2),\sigma(u_2,u_2),\sigma(u_1,u_3)\}.\end{aligned}$$

Thus we have

$$\Theta(\sigma(u_2,u_3)) \leq [\Theta(\sigma(u_1,u_2)]^k. \tag{2.9}$$

So, continuing recursively, we obtain a sequence $\{u_n\}$ in $\mathcal{S}$ such that $u_{2n+1} \in [\mathcal{P}u_{2n}]_{\alpha_L(u_{2n})}$ and $u_{2n+2} \in [\mathcal{Q}u_{2n+1}]_{\alpha_L(u_{2n+1})}$ and

$$\Theta(\sigma(u_{2n+1},u_{2n+2})) \leq [\Theta(\sigma(u_{2n},u_{2n+1})]^k \tag{2.10}$$

and

$$\Theta(\sigma(u_{2n+2},u_{2n+3})) \leq [\Theta(\sigma(u_{2n+1},u_{2n+2})]^k \tag{2.11}$$

for all $n \in \mathbb{N}$. From (2.10) and (2.11), we have

$$\Theta(\sigma(u_n,u_{n+1})) \leq [\Theta(\sigma(u_{n-1},u_n)]^k \tag{2.12}$$

which further implies that

$$\Theta(\sigma(u_n,u_{n+1})) \leq [\Theta(\sigma(u_{n-1},u_n)]^k \leq [\Theta(\sigma(u_{n-2},u_{n-1})]^{k^2} \leq ... \leq [\Theta(\sigma(u_0,u_1)]^{k^n} \tag{2.13}$$

for all $n \in \mathbb{N}$. Since $\Theta \in F$, so by taking limit as $n \to \infty$ in (2.13) we have,

$$\lim_{n\to\infty} \Theta(\sigma(u_n, u_{n+1})) = 1 \tag{2.14}$$

which implies that

$$\lim_{n\to\infty} \sigma(u_n, u_{n+1}) = 0 \tag{2.15}$$

by (Θ_2). From the condition (Θ_3), there exist $0 < r < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(\sigma(u_n, u_{n+1})) - 1}{\sigma(u_n, u_{n+1})^r} = l. \tag{2.16}$$

Suppose that $l < \infty$. In this case, let $\beta = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$|\frac{\Theta(\sigma(u_n, u_{n+1})) - 1}{\sigma(u_n, u_{n+1})^r} - l| \leq \beta$$

for all $n > n_0$. This implies that

$$\frac{\Theta(\sigma(u_n, u_{n+1})) - 1}{\sigma(u_n, u_{n+1})^r} \geq l - \beta = \frac{l}{2} = \beta$$

for all $n > n_0$. Then

$$n\sigma(u_n, u_{n+1})^r \leq \alpha n[\Theta(\sigma(u_n, u_{n+1})) - 1] \tag{2.17}$$

for all $n > n_0$, where $\alpha = \frac{1}{\beta}$. Now we suppose that $l = \infty$. Let $\beta > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\beta \leq \frac{\Theta(\sigma(u_n, u_{n+1})) - 1}{\sigma(u_n, u_{n+1})^r}$$

for all $n > n_0$. This implies that

$$n\sigma(u_n, u_{n+1})^r \leq \alpha n[\Theta(\sigma(u_n, u_{n+1})) - 1]$$

for all $n > n_0$, where $\alpha = \frac{1}{\beta}$. Thus, in all cases, there exist $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\sigma(u_n, u_{n+1})^r \leq \alpha n[\Theta(\sigma(u_n, u_{n+1})) - 1] \tag{2.18}$$

for all $n > n_0$. Thus by (2.13) and (2.18), we get

$$n\sigma(u_n, u_{n+1})^r \leq \alpha n([(\Theta\sigma(u_0, u_1))]^{r^n} - 1). \tag{2.19}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\sigma(u_n, u_{n+1})^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\sigma(u_n, u_{n+1}) \leq \frac{1}{n^{1/r}} \tag{2.20}$$

for all $n > n_1$. Now we prove that $\{u_n\}$ is a Cauchy sequence. For $m > n > n_1$ we have,

$$\sigma(u_n, u_m) \leq \sum_{i=n}^{m-1} \sigma(u_i, u_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} \leq \sum_{i=1}^{\infty} \frac{1}{i^{1/r}}. \tag{2.21}$$

Since, $0 < r < 1$, $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ converges. Therefore, $\sigma(u_n, u_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{u_n\}$ is a Cauchy sequence in $(\mathcal{S}, \sigma)$. The completeness of $(\mathcal{S}, \sigma)$ ensures that there exists $u^* \in \mathcal{S}$ such that, $\lim_{n\to\infty} u_n \to u^*$. Now, we prove that $u^* \in [\mathcal{Q}u^*]_{\alpha_L(u^*)}$. We suppose on the contrary that $u^* \notin [\mathcal{Q}u^*]_{\alpha_L(u)}$, then there exist a $n_0 \in \mathbb{N}$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\sigma(u_{2n_k+1}, [\mathcal{Q}u^*]_{\alpha_L(u^*)}) > 0$ for all $n_k \geq n_0$. Since $\sigma(u_{2n_k+1}, [\mathcal{Q}u^*]_{\alpha_L(u)}) > 0$ for all $n_k \geq n_0$, so by (Θ_1), we have

$$\begin{aligned} 1 &< \Theta\left[\sigma(u_{2n_k+1}, [\mathcal{Q}u^*]_{\alpha_L(u^*)})\right] \leq \Theta\left[\mathcal{H}([\mathcal{P}u_{2n_k}]_{\alpha_L(u_{2n_k})}, [\mathcal{Q}u^*]_{\alpha_L(u^*)})\right] \\ &\leq [\Theta(\sigma(u_{2n_k}, u^*))]^k + \min\left\{ \begin{array}{c} \sigma\left(u_{2n_k}, [\mathcal{P}u_{2n_k}]_{\alpha_L(u_{2n_k})}\right), \sigma\left(u^*, [\mathcal{Q}u^*]_{\alpha_L(u^*)}\right), \\ \sigma\left(u_{2n_k}, [\mathcal{Q}u^*]_{\alpha_L(u^*)}\right), \sigma\left(u^*, [\mathcal{P}u_{2n_k}]_{\alpha_L(u_{2n_k})}\right) \end{array} \right\} \\ &\leq [\Theta(\sigma(u_{2n_k}, u^*))]^k + L\min\left\{ \begin{array}{c} \sigma\left(u_{2n_k}, u_{2n_k+1}\right), \sigma\left(u^*, [\mathcal{Q}u^*]_{\alpha_L(u^*)}\right), \\ \sigma\left(u_{2n_k}, [\mathcal{Q}u^*]_{\alpha_L(u^*)}\right), \sigma\left(u^*, u_{2n_k+1}\right) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, in above inequality and using the continuity of Θ, we have

$$1 < \Theta\left[\sigma(u^*, [\mathcal{Q}u^*]_{\alpha_L(u^*)})\right] \leq 1$$

which is a conradiction. Hence $u^* \in [\mathcal{Q}u^*]_{\alpha_L(u^*)}$. Similarly, one can easily prove that $u^* \in [\mathcal{P}u^*]_{\alpha_L(u^*)}$. Thus $u^* \in [\mathcal{P}u^*]_{\alpha_L(u^*)} \cap [\mathcal{Q}u^*]_{\alpha_L(u^*)}$. $\square$

The following result is a direct consequence of above theorem by taking $L = 0$.

Corollary 2.2. Let $(\mathcal{S}, \sigma)$ be a complete metric space and $\{\mathcal{P}, \mathcal{Q}\}$ be a pair of L-fuzzy mappings from $\mathcal{S}$ into $\Im_L(\mathcal{S})$ and for each $\alpha_L \in L\backslash\{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{Q}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in F$ and $k \in (0, 1)$ such that

$$\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{Q}v]_{\alpha_L(v)}\right) > 0 \Longrightarrow \Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{Q}v]_{\alpha_L(v)}\right)\right) \leq \Theta(\sigma(u, v))^k$$

for all $u, v \in \mathcal{S}$. Then $\mathcal{P}$ and $\mathcal{Q}$ have a common L-fuzzy fixed point.

If we take a single L-fuzzy mapping, we get the following result.

Corollary 2.3. Let $(\mathcal{S}, \sigma)$ be a complete metric space and let $\mathcal{P}$ be L-fuzzy mapping from $\mathcal{S}$ into $\Im_L(\mathcal{S})$ and for each $\alpha_L \in L\backslash\{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{P}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in F$, $k \in (0, 1)$ and $L \geq 0$ such that

$$\Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{P}v]_{\alpha_L(v)}\right)\right) \leq \Theta(\sigma(u, v))^k + Lm(u, v)$$

where

$$m(u, v) = \min\left\{\sigma\left(u, [\mathcal{P}u]_{\alpha_L(u)}\right), \sigma\left(v, [\mathcal{P}v]_{\alpha_L(v)}\right), \sigma\left(u, [\mathcal{P}v]_{\alpha_L(v)}\right), \sigma\left(v, [\mathcal{P}u]_{\alpha_L(u)}\right)\right\}.$$

for all $u, v \in \mathcal{S}$ with $\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{P}v]_{\alpha_L(v)}\right) > 0$. Then $\mathcal{P}$ has an L-fuzzy fixed point.

Corollary 2.4. Let $(\mathcal{S}, \sigma)$ be a complete metric space and let $\mathcal{P}$ be L-fuzzy mapping from $\mathcal{S}$ into $\Im_L(\mathcal{S})$ and for each $\alpha_L \in L\backslash\{0_L\}$, $[\mathcal{P}u]_{\alpha_L(u)}$, $[\mathcal{P}v]_{\alpha_L(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in F$ and $k \in (0, 1)$ such that

$$\Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{P}v]_{\alpha_L(v)}\right)\right) \leq \Theta(\sigma(u, v))^k$$

for all $u, v \in \mathcal{S}$ with $\mathcal{H}\left([\mathcal{P}u]_{\alpha_L(u)}, [\mathcal{P}v]_{\alpha_L(v)}\right) > 0$. Then $\mathcal{P}$ has an L-fuzzy fixed point.

L-fuzzy fixed point results are real generalization of fuzzy fixed point theorems. It can be shown in the following Theorem.

Theorem 2.5. Let $(\mathcal{S}, \sigma)$ be a complete metric space and let $\mathcal{P}, \mathcal{Q}$ be fuzzy mappings from $\mathcal{S}$ into $\Im(\mathcal{S})$ and for each $\alpha(u) \in (0, 1]$, $[\mathcal{P}u]_{\alpha(u)}$, $[\mathcal{Q}v]_{\alpha(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta \in F$, $k \in (0, 1)$ and $L \geq 0$ such that

$$\Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha(u)}, [\mathcal{Q}v]_{\alpha(v)}\right)\right) \leq \Theta(\sigma(u, v))^k + Lm(u, v)$$

where

$$m(u,v)=\min\left\{\sigma\left(u,[\mathcal{P}u]_{\alpha(u)}\right),\sigma\left(v,[\mathcal{Q}v]_{\alpha(v)}\right),\sigma\left(u,[\mathcal{Q}v]_{\alpha(v)}\right),\sigma\left(v,[\mathcal{P}u]_{\alpha(u)}\right)\right\}.$$

for all $u,v\in\mathcal{S}$ with $\mathcal{H}\left([\mathcal{P}u]_{\alpha(u)},[\mathcal{Q}v]_{\alpha(v)}\right)>0$. Then $\mathcal{P}$ and $\mathcal{Q}$ have a common fuzzy fixed point.

Proof . Consider an L-fuzzy mapping $\mathcal{J}:\mathcal{S}\to\Im_L(\mathcal{S})$ defined by

$$\mathcal{J}u=\chi_{L_{\mathcal{P}(u)}}.$$

Then for $\alpha_L\in L\backslash\{0_L\}$, we have

$$[\mathcal{J}u]_{\alpha_L(u)}=\mathcal{P}u.$$

Hence by Theorem 2.1 we follow the result. □

Taking $L=0$ in above result, we have following corollary.

Corollary 2.6. Let $(\mathcal{S},\sigma)$ be a complete metric space and let $\mathcal{P},\mathcal{Q}$ be fuzzy mappings from $\mathcal{S}$ into $\Im(\mathcal{S})$ and for each $\alpha(u)\in(0,1]$, $[\mathcal{P}u]_{\alpha(u)}$, $[\mathcal{Q}v]_{\alpha(v)}$ are nonempty closed bounded subsets of $\mathcal{S}$. If there exist some $\Theta\in\digamma$ and $k\in(0,1)$ such that

$$\Theta\left(\mathcal{H}\left([\mathcal{P}u]_{\alpha(u)},[\mathcal{Q}v]_{\alpha(v)}\right)\right)\leq\Theta(\sigma(u,v))^k$$

for all $u,v\in\mathcal{S}$ with $\mathcal{H}\left([\mathcal{P}u]_{\alpha(u)},[\mathcal{Q}v]_{\alpha(v)}\right)>0$. Then $\mathcal{P}$ and $\mathcal{Q}$ have a common fuzzy fixed point.

Example 2.7. Let $\mathcal{S}=[0,1]$, $\sigma(u,v)=|u-v|$, whenever $u,v\in\mathcal{S}$. Then $(\mathcal{S},\sigma)$ is a complete metric space. Let $L=\{\eta,\omega,\tau,\kappa\}$ with $\eta\preceq_L\omega\preceq_L\kappa$ and $\eta\preceq_L\tau\preceq_L\kappa$, where ω and τ are not comparable, then $(L,\preceq_L)$ is a complete distributive lattice. Define $\mathcal{P},\mathcal{Q}:\mathcal{S}\to\Im_L(\mathcal{S})$ as follows:

$$\mathcal{P}(u)(t)=\begin{cases}\kappa \text{ if } 0\leq t\leq\frac{u}{6}\\ \omega \text{ if } \frac{u}{6}<t\leq\frac{u}{3}\\ \tau \text{ if } \frac{u}{3}<t\leq\frac{u}{2}\\ \eta \text{ if } \frac{u}{2}<t\leq 1\end{cases},$$

$$\mathcal{Q}(u)(t)=\begin{cases}\kappa \text{ if } 0\leq t\leq\frac{u}{12}\\ \eta \text{ if } \frac{u}{12}<t\leq\frac{u}{8}\\ \omega \text{ if } \frac{u}{8}<t\leq\frac{u}{4}\\ \tau \text{ if } \frac{u}{4}<t\leq 1\end{cases}.$$

Let $\Theta(t)=e^{\sqrt{t}}\in\digamma$ for $t>0$. And for all $u\in\mathcal{S}$, there exists $\alpha_L(u)=\kappa$, such that

$$[\mathcal{P}u]_{\alpha_L(u)}=\left[0,\frac{u}{6}\right],\qquad [\mathcal{Q}u]_{\alpha_L(u)}=\left[0,\frac{u}{12}\right].$$

and all conditions of Theorem 2.1 are satisfied. And 0 is a common fixed point of $\mathcal{P}$ and $\mathcal{Q}$.

3 Applications to domain of words

Suppose Ω be a nonempty alphabet and Ω^∞ be the collection of all finite and infinite sequences ("words") over Ω, where we adopt the convention that the empty sequence $\emptyset$ is an element of Ω^∞. Moreover, on Ω^∞, we consider the prefix order $\precapprox$ given by:

$$u\precapprox v \quad \text{if and only if } u \text{ is a prefix of } v.$$

For each nonempty $u\in\Omega^\infty$ denote by $l(u)$ the length of u. Then $l(u)\in[0,\infty]$, whenever $u\neq\emptyset$ and $l(\emptyset)=0$. For each $u,v\in\Omega^\infty$, let $u\sqcap v$ be the common prefix of u and v. Clearly, $u=v$ if and only if $u\precapprox v$ and $v\precapprox u$ and $l(u)=l(v)$. Then, the the Baire metric $\sigma_{\precapprox}$ is defined on $\Omega^\infty\times\Omega^\infty$ by

$$\begin{cases}\sigma_{\precapprox}(u,v)=0, \text{if u=v}\\ \sigma_{\precapprox}(u,v)=2^{-l(u\sqcap v)}, \text{otherwise}\end{cases}$$

such that the metric space $(\Omega^\infty, \sigma_{\Leftrightarrow})$ is complete. Certainly, we assign to the average case time complexity analysis of the Quicksort divide-and-conquer sorting algorithm in [32]. Exactly, we deal with the following recurrence relation:

$$\Re(1) = 0 \quad \text{and} \quad \Re(n) = \frac{2(n-1)}{n} + \frac{n+1}{n}\Re(n-1), \quad n \geq 2. \tag{3.1}$$

Consider as an alphabet Ω the set of nonnegative real numbers, i.e., $\Omega = \mathbb{R}^+$. We accomplice to $\Re$ the functional $\Phi : \Omega^\infty \to \Omega^\infty$ given by

$$(\Phi(u))_1 = \Re(1)$$

and

$$(\Phi(u))_n = \frac{2(n-1)}{n} + \frac{n+1}{n}u_{n-1}$$

for all $n \geq 2$ (if $u \in \Omega^\infty$ has length $n < \infty$, we write $u := u_1u_2...u_n$, and if u is an infinite word we write $u := u_1u_2\ldots$). It follows by the construction that $l(\Phi(u)) = l(u)+1$ for all $u \in \Omega^\infty$ and $l(\Phi(u)) = +\infty$ whenever $l(u) = +\infty$. We will prove that the functional Φ has an L-fuzzy fixed point by an application of 2.4. Let $\mathcal{P} : \Omega^\infty \to \Im(\Omega^\infty)$ be the L-fuzzy mapping given by

$$\mathcal{P}_u = (\Phi(u))_{\alpha_L} \text{ for all } u \in \Omega^\infty \text{ and } \alpha_L \in L\backslash\{0_L\}.$$

and analyze the following two cases:

Case 01: If $u = v$, then we have

$$\mathcal{H}_{\Leftrightarrow}((\Phi(u))_{\alpha_L}, (\Phi(u))_{\alpha_L}) = 0 = \sigma_{\Leftrightarrow}(u,u).$$

Case 02: If $u \neq v$, then we write

$$\begin{aligned}
\mathcal{H}_{\Leftrightarrow}((\Phi(u))_{\alpha_L}, (\Phi(v))_{\alpha_L}) &= \sigma_{\Leftrightarrow}((\Phi(u))_{\alpha_L}, (\Phi(v))_{\alpha_L}) = 2^{-(l(\Phi(u))_{\alpha_L} \sqcap (\Phi(v))_{\alpha_L})} \\
&\leq 2^{-(l(\Phi(u \sqcap v))_{\alpha_L})} = 2^{-(l(u \sqcap v)+1)} \\
&= \frac{1}{2}2^{-l(u\sqcap v)} = (\frac{1}{\sqrt{2}})^2 \sigma_{\Leftrightarrow}(u,v).
\end{aligned}$$

It is immediate to achieve that all the assertions of the Corollary 2.4 are satisfied with $\Theta(t) = e^{\sqrt{t}}$ and $k = \frac{1}{\sqrt{2}}$. Consequently, the L- fuzzy mapping $\mathcal{P}$ has a L- fuzzy fixed point $u = u_1u_2... \in \Omega^\infty$ that is, $u \in (\mathcal{P}_u)_{\alpha_L}$. Also, in the light of the definition of $\mathcal{P}, u$ is a fixed point of Φ, and hence, u solves the recurrence relation (3.1). We have

$$u_1 = 0,$$

$$u_n = \frac{2(n-1)}{n} + \frac{n+1}{n}u_{n-1}, \quad n \geq 2.$$

4 Conclusions

We proved some common L-fuzzy fixed point results for almost Θ-contraction in the setting of complete metric spaces by using the notion of L-fuzzy mappings.We also presented an application to domain of words which shows the significance of the investigation of this paper.

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