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Jensen's inequality for GG-convex functions

G. Zabandan

Department of Mathematics, Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran

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Abstract

In this paper, we obtain Jensen's inequality for GG-convex functions. Also, we get inequalities alike to Hermite-Hadamard inequality for GG-convex functions. Some examples are given.

Keywords: Jensen's inequality, GG-convex, Integral inequality.
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1. Introduction

Let μ be a positive measure on X such that $\mu(X) = 1$. If f is a real-valued function in $L^1(\mu)$, $a < f(x) < b$ for all $x \in X$ and φ is convex on (a, b), then

$$\varphi\left(\int_X f d\mu\right) \leq \int_X (\varphi.f) d\mu \tag{1.1}$$

The inequality (1.1) is known as Jensen's inequality [4], [7].

Definition 1.1. *A function $\varphi : (a, b) \longrightarrow (0, \infty)$, where $0 < a < b \leq \infty$, is called GG-convex or multiplicatively-convex (according to the geometric mean) if the inequality*

$$\varphi(x^\lambda y^{1-\lambda}) \leq \varphi^\lambda(x)\varphi^{1-\lambda}(y) \tag{1.2}$$

holds, where $a < x < b$, $a < y < b$, and $0 \leq \lambda \leq 1$.

In this paper, first we prove Jensen's inequality for GG-convex functions. Then as a result of Jensen's inequality, we prove the geometric mean of positive numbers is not greater than the mean power of the same numbers of order $\alpha > 0$, that is

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \left(\frac{a_1^\alpha a_2^\alpha \cdots a_n^\alpha}{n}\right)^{\frac{1}{\alpha}} \qquad (\alpha > 0, a_1, a_2 \cdots a_n > 0).$$

By GG-convexity of Gamma function on $[1, \infty]$, we obtain several interesting inequalities. Finally, we prove alike to Hermit-Hadamard inequality for GG-convex functions.

*Corresponding author
Email address: zabandan@khu.ac.ir (G. Zabandan)

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2. Main results

First we need the following theorem.

Theorem 2.1. *A function φ is GG-convex on (a,b) if for $0<a<s<t<u<b$ the following inequality holds*

$$\frac{\ln\varphi(t)-\ln\varphi(s)}{\ln t-\ln s}\leq\frac{\ln\varphi(u)-\ln\varphi(t)}{\ln u-\ln t} \tag{2.1}$$

Proof . Let φ be GG-convex and $\lambda=\dfrac{\ln u-\ln t}{\ln u-\ln s}$, then $t=s^{\lambda}u^{1-\lambda}$. Hence

$$\varphi(t)\leq[\varphi(s)]^{\frac{\ln u-\ln t}{\ln u-\ln s}}[\varphi(u)]^{(1-\frac{\ln u-\ln t}{\ln u-\ln s})}$$

It follows that

$$\begin{aligned}\ln\varphi(t)&\leq\frac{\ln u-\ln t}{\ln u-\ln s}\ln\varphi(s)+\frac{\ln t-\ln s}{\ln u-\ln s}\ln\varphi(u)\\ \frac{\ln u-\ln t}{\ln u-\ln s}\ln\varphi(t)+\frac{\ln t-\ln s}{\ln u-\ln s}\ln\varphi(t)&\leq\frac{\ln u-\ln t}{\ln u-\ln s}\ln\varphi(s)+\frac{\ln t-\ln s}{\ln u-\ln s}\ln\varphi(u)\\ \frac{\ln u-\ln t}{\ln u-\ln s}(\ln\varphi(t)-\ln\varphi(s))&\leq\frac{\ln t-\ln s}{\ln u-\ln s}(\ln\varphi(u)-\ln\varphi(t))\end{aligned}$$

since $s<t<u$, we obtain

$$\frac{\ln\varphi(t)-\ln\varphi(s)}{\ln t-\ln s}\leq\frac{\ln\varphi(u)-\ln\varphi(t)}{\ln u-\ln t}$$

Conversely let the inequality (2.1) holds, and $\lambda\in[0,1]$, $a<x<y<b$, then $x\leq x^{\lambda}y^{1-\lambda}\leq y$. By inequality (2.1) we have

$$\begin{aligned}&\frac{\ln\varphi(x^{\lambda}y^{1-\lambda})-\ln\varphi(x)}{\ln x^{\lambda}y^{1-\lambda}-\ln x}\leq\frac{\ln\varphi(y)-\ln\varphi(x^{\lambda}y^{1-\lambda})}{\ln y-\ln x^{\lambda}y^{1-\lambda}}\\ \Longrightarrow&\frac{\ln\varphi(x^{\lambda}y^{1-\lambda})-\ln\varphi(x)}{(1-\lambda)(\ln y-\ln x)}\leq\frac{\ln\varphi(y)-\ln\varphi(x^{\lambda}y^{1-\lambda})}{\lambda(\ln y-\ln x)}\\ &\Longrightarrow\ln\varphi(x^{\lambda}y^{1-\lambda})\leq(1-\lambda)\ln\varphi(y)+\lambda\ln\varphi(x)\\ &\Longrightarrow\varphi(x^{\lambda}y^{1-\lambda})\leq\varphi^{\lambda}(x)\varphi^{1-\lambda}(y)\end{aligned}$$

Thus φ is GG-convex. □

By similar way to the convex functions we can prove that if φ is GG-convex on (a,b), then φ is continuous on (a,b).

Theorem 2.2. *Let μ be a positive measure on a σ-algebra $\mathfrak{m}$ in a set X, so that $\mu(X)=1$. If f is a real function in $L^1(\mu)$, $0<a<f(x)<b$ for all $x\in X$, and if φ is GG-convex on (a,b), then*

$$\varphi\left(e^{\int_X\ln f\,d\mu}\right)\leq e^{\int_X\ln(\varphi\circ f)d\mu} \tag{2.2}$$

Proof . Put $t = e^{\int_X \ln f d\mu}$. Then $a < t < b$. If M is the supremum of quotients on the left side of (2.1), where $a < s < t$, then for any $u \in (t, b)$ we have

$$M \leq \frac{\ln \varphi(u) - \ln \varphi(t)}{\ln u - \ln t}$$

It follows that

$$\frac{\ln \varphi(t) - \ln \varphi(s)}{\ln t - \ln s} \leq M \quad (a < s < b)$$

so

$$\ln \varphi(s) \geq \ln \varphi(t) + M(\ln s - \ln t).$$

Hence, for any $x \in X$, we have

$$\ln \varphi(f(x)) \geq \ln \varphi(t) + M(\ln f(x) - \ln t)$$

since φ is continuous, $\varphi \circ f$ is measureable, and since $f \in L^1(\mu)$, by cancavity of $\psi(x) = \ln x$ and Jensen inequality (1.1) $\ln f \in L^1(\mu)$. By integrating both sides with respect to measure μ we obtain

$$\int_X \ln(\varphi \circ f) d\mu \geq \ln \varphi(t) + M \left(\int_X \ln f d\mu - \ln t \right) \quad (\mu(X) = 1)$$

Now set $t = e^{\displaystyle\int_X \ln f d\mu}$, it follows that

$$\int_X \ln(\varphi \circ f) d\mu \geq \ln \varphi \left(e^{\displaystyle\int_X \ln f d\mu} \right) + M \left(\int_X \ln f d\mu - \ln e^{\displaystyle\int_X \ln d\mu} \right)$$

so

$$\ln \varphi \left(e^{\int_X \ln f d\mu} \right) \leq \int_X \ln(\varphi \circ f) d\mu$$

or

$$\varphi \left(e^{\displaystyle\int_X \ln f d\mu} \right) \leq e^{\displaystyle\int_X \ln(\varphi \circ f) d\mu}.$$

□

In [6], the author proved the following assertion.

Here we prove it in another way and a result of theorem 2.2.

Corollary 2.3. *Let $f : [a, b] \longrightarrow (0, \infty)$ $(b > a > 0)$ be a continuous function and $\varphi : J \longrightarrow (0, \infty)$ be a GG-convex function defined on an interval J which includes the image of f. Then*

$$\varphi \left(e^{\displaystyle\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln f(x)}{x} dx} \right) \leq e^{\displaystyle\frac{1}{\ln b - \ln a} \int_a^b \frac{\ln \varphi(f(x))}{x} dx} \tag{2.3}$$

Proof . In theorem 2.2, put $X = [a, b]$ and $d\mu = \frac{dx}{x}$. □

In the following theorem we prove a version for the inverse of corollary 2.3.

Theorem 2.4. *Let $\varphi:(0,\infty)\longrightarrow(0,\infty)$ be a function such that the inequality (2.3) holds, for every positive real bounded measureable function f. Then φ is GG-convex.*

Proof . Let $\lambda\in[0,1]$ and $c,d\in(0,\infty)$. Define

$$f(x)=\begin{cases} c & a\le x<b^\lambda a^{1-\lambda}\\ d & b^\lambda a^{1-\lambda}\le x\le b\end{cases}$$

we have

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(x)}{x}dx&=\frac{1}{\ln b-\ln a}\left[\int_a^{b^\lambda a^{1-\lambda}}(\ln c)\frac{dx}{x}+\int_{b^\lambda a^{1-\lambda}}^b(\ln d)\frac{dx}{x}\right]\\&=\lambda\ln c+(1-\lambda)\ln d\end{aligned}$$

so

$$\varphi\left(e^{\dfrac{1}{\ln b-\ln a}\displaystyle\int_a^b\frac{\ln f(x)}{x}dx}\right)=\varphi\left(e^{\lambda\ln c+(1-\lambda)\ln d}\right)=\varphi(c^\lambda d^{1-\lambda})\qquad(*)$$

on the other hand we have

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln\varphi(f(x))}{x}dx&=\frac{1}{\ln b-\ln a}\left[\int_a^{b^\lambda a^{1-\lambda}}\ln\varphi(c)\frac{dx}{x}+\int_{b^\lambda a^{1-\lambda}}^b\ln\varphi(d)\frac{dx}{x}\right]\\&=\lambda\ln\varphi(c)+(1-\lambda)\ln\varphi(d)\end{aligned}$$

Hence

$$e^{\dfrac{1}{\ln b-\ln a}\displaystyle\int_a^b\frac{\ln\varphi(f(x))}{x}dx}=e^{\lambda\ln\varphi(c)+(1-\lambda)\ln\varphi(d)}=\varphi^\lambda(c)\varphi^{1-\lambda}(d)\qquad(**)$$

Now the (*), (**) and (2.3) show that φ is GG-convex. □

Example 2.5. *(1) Let $X=\{x_1,x_2,\dots,x_n\}$, $\mu(\{x_i\})=\dfrac{1}{n}$ and $f(x_i)=a_i>0$. Then (2.2) becomes*

$$\varphi\left(e^{\dfrac{1}{n}(\ln a_1+\ln a_2+\cdots+\ln a_n)}\right)\le e^{\dfrac{1}{n}(\ln\varphi(a_1)+\ln\varphi(a_2)+\cdots+\ln\varphi(a_n))}$$

Hence

$$\varphi\left(\sqrt[n]{a_1a_2\dots a_n}\right)\le\sqrt[n]{\varphi(a_1)\varphi(a_2)\dots\varphi(a_n)}\qquad(2.4)$$

Now we inestigate this inequality for $\varphi(x)=e^{x^\alpha}$ and $\varphi(x)=\Gamma(x)$

(i) $\varphi(x)=e^{x^\alpha}$ $(\alpha>0)$ is GG-convex on $(0,\infty)$ (see [1]). The inequality (2.4) implies that

$$e^{(\sqrt[n]{a_1a_2\dots a_n})^\alpha}\le\sqrt[n]{e^{a_1^\alpha}e^{a_2^\alpha}\dots e^{a_n^\alpha}}=\left(e^{a_1^\alpha+a_2^\alpha+\cdots+a_n^\alpha}\right)^{\dfrac{1}{n}}$$

$$\Longrightarrow\sqrt[n]{a_1a_2\dots a_n}\le\left(\frac{a_1^\alpha+a_2^\alpha+\cdots+a_n^\alpha}{n}\right)^{\dfrac{1}{\alpha}}\quad(\alpha>0)$$

(ii) $\varphi(x)=\Gamma(x)$ *is GG-convex on* $[1,\infty)$*. The inequality (2.4) follows that*

$$\Gamma\left(\sqrt[n]{\prod_{i=1}^{n} a_i}\right) \le \sqrt[n]{\prod_{i=1}^{n}\Gamma(a_i)}$$

put $a_k = x+\frac{k}{m}$, $k=0,1,2,\ldots,m-1$ $(x\ge 1)$*. Then*

$$\Gamma\left(\sqrt[m]{\prod_{k=0}^{m-1}(x+\frac{k}{m})}\right) \le \sqrt[m]{\prod_{k=0}^{m-1}\Gamma(x+\frac{k}{m})}$$

By Gauss multipication formula $\prod_{k=0}^{m-1}\Gamma(x+\frac{k}{m}) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mx}\Gamma(mx)$ *[8] we obtain*

$$\Gamma\left(\sqrt[m]{x(x+\frac{1}{m})\ldots(x+\frac{m-1}{m})}\right) \le (2\pi)^{\frac{m-1}{2m}} m^{\frac{1}{2m}-x}\sqrt[m]{\Gamma(mx)}$$

Especially for $x=1$ *we have*

$$\Gamma\left(\sqrt[m]{\frac{(2m-1)!}{(m!)m^{m-1}}}\right) \le (2\pi)^{\frac{m-1}{2m}} m^{\frac{1}{2m}-1}\sqrt[m]{(m-1)!}$$

(2) $\Gamma(x)$ *is GG-convex on* $[1,\infty)$*. Hence (2.3) becomes*

$$\Gamma\left(e^{\frac{1}{\ln b-\ln a}\int_a^b \ln f(t)\frac{dt}{t}}\right) \le e^{\frac{1}{\ln b-\ln a}\int_a^b \ln\Gamma(f(t))\frac{dt}{t}}$$

Especially for $f(t)=\ln t$ $(e\le a<t<b)$ *we have*

$$\Gamma\left(e^{\frac{1}{\ln b-\ln a}\int_a^b \ln(\ln t)\frac{dt}{t}}\right) \le e^{\frac{1}{\ln b-\ln a}\int_a^b \ln\Gamma(\ln t)\frac{dt}{t}}$$

By change of variable $\ln t = x$, $\frac{dt}{t}=dx$,

$$\Gamma\left(e^{\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} \ln x dx}\right) \le e^{\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} \ln\Gamma(x) dx}$$

Now put $a=e^p$ *and* $b=e^{p+1}$ $(p\ge 1)$

$$\Gamma\left(e^{\int_p^{p+1} \ln x dx}\right) \le e^{\int_p^{p+1} \ln\Gamma(x) dx}$$

By easy calculations we see that

$$\int_p^{p+1} \ln x dx = \ln \frac{(p+1)^{p+1}}{p^p} - 1 \quad and \quad \int_p^{p+1} \ln \Gamma(x) dx = -p + p\ln p + \ln\sqrt{2\pi}$$

so

$$\Gamma\left(e^{\ln \frac{(p+1)^{p+1}}{p^p} - 1} \right) \leq e^{-p + p\ln p + \ln\sqrt{2\pi}}$$

or

$$\Gamma\left(\frac{(p+1)^{p+1}}{ep^p} \right) \leq \sqrt{2\pi} p^p e^{-p}$$

In the following theorem we obtain inequalities alike to Hermite-Hadamard inequality for GG-convex functions.

Theorem 2.6. *Let $f : [a,b] \longrightarrow (0,\infty)$ be a GG-convex function $(b > a > 0)$. Then the following inequalities hold:*

$$f(\sqrt{ab}) \leq e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(x)}{x} dx} \leq \frac{1}{\ln b - \ln a}\int_a^b \sqrt{f(x)f(\frac{ab}{x})}\frac{dx}{x} \leq \sqrt{f(a)f(b)} \tag{2.5}$$

Proof . Since f is GG-convex, the corollary 2.3 implies that

$$e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(x)}{x} dx} \geq f\left(e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln x}{x} dx} \right)$$
$$= f\left(e^{\frac{1}{2(\ln b - \ln a}(\ln^2 b - \ln^2 a)} \right) = f(\sqrt{ab})$$

For the proof of middle part, since $\varphi(f) = \ln t$ is concave, by Jensen's inequality (1.1) we get

$$\ln\left(\frac{1}{\ln b - \ln a}\int_a^b \sqrt{f(x)f(\frac{ab}{x})}\frac{dx}{x} \right) \geq \frac{1}{\ln b - \ln a}\int_a^b \left[\frac{1}{2}\ln f(x) + \frac{1}{2}\ln f(\frac{ab}{x}) \right] \frac{dx}{x}$$
$$= \frac{1}{\ln b - \ln a}\int_a^b \ln f(x) \frac{dx}{x}$$

Because by change of variable, $\frac{ab}{x} = t$, $dx = \frac{-ab}{t^2}dt$ we see that

$$\int_a^b \ln f(\frac{ab}{x})\frac{dx}{x} = \int_a^b \ln f(t) \frac{dt}{t}$$

so

$$e^{\frac{1}{\ln b - \ln a}\int_a^b \ln f(x)\frac{dx}{x}} \leq \frac{1}{\ln b - \ln a}\int_a^b \sqrt{f(x)f(\frac{ab}{x})}\frac{dx}{x}$$

For the proof of right side of (2.5), by change of variable $x = a^{1-t}b^t = a(\frac{b}{a})^t$, $dx = a\ln\frac{b}{a}(\frac{b}{a})^t dt$ and GG-convexity of f we obtain

$$\frac{1}{\ln b - \ln a}\int_a^b \sqrt{f(x)f(\frac{ab}{x})}\frac{dx}{x} = \frac{1}{\ln b - \ln a}\int_0^1 \sqrt{f(a^{1-t}b^t)f(a^t b^{1-t})}\frac{a\ln\frac{b}{a}(\frac{b}{a})^t}{a(\frac{b}{a})^t}dt$$
$$= \int_0^1 \sqrt{f(a^{1-t}b^t)f(a^t b^{1-t})}dt$$
$$\leq \int_0^1 \sqrt{f^{1-t}(a)f^t(b)f^t(a)f^{1-t}(b)}dt = \sqrt{f(a)f(b)}.$$

□

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