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A more accurate half-discrete Hardy-Hilbert-type inequality with the best possible constant factor related to the extended Riemann-Zeta function

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Abstract

By the method of weight coefficients, techniques of real analysis and Hermite-Hadamard's inequality, a half-discrete Hardy-Hilbert-type inequality related to the kernel of the hyperbolic cosecant function with the best possible constant factor expressed in terms of the extended Riemann-zeta function is proved. The more accurate equivalent forms, the operator expressions with the norm, the reverses and some particular cases are also considered.

Keywords: Hardy-Hilbert-type inequality; extended Riemann-zeta function; Hurwitz zeta function; Gamma function; weight function; equivalent form; operator.
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1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$,

$$||f||_p = \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} > 0,$$

and $||g||_q > 0$, then we have the following Hardy-Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1.1}$$

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where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Assuming that

$$a_m, b_n \geq 0, a = \{a_m\}_{m=1}^{\infty} \in l^p, b = \{b_n\}_{n=1}^{\infty} \in l^q, ||a||_p = \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} > 0, ||b||_q > 0,$$

we have the following discrete analogue of (1.1) with the same best constant $\frac{\pi}{\sin(\pi/p)}$ (cf. [1]):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{1.2}$$

Inequalities (1.1) and (1.2) are important in Mathematical Analysis and its applications (cf. [1], [2], [3], [4], [5]).

Suppose that $\mu_i, \upsilon_j > 0$ $(i, j \in \mathbf{N} = \{1, 2, \cdots\})$,

$$U_m := \sum_{i=1}^{m} \mu_i, V_n := \sum_{j=1}^{n} \nu_j \quad (m, n \in \mathbf{N}). \tag{1.3}$$

Then we have the following inequality (cf. [1], Theorem 321, replacing $\mu_m^{1/q} a_m$ and $\upsilon_n^{1/p} b_n$ by a_m and b_n) :

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{\nu_n^{q-1}}\right)^{\frac{1}{q}}. \tag{1.4}$$

For $\mu_i = \upsilon_j = 1$ $(i, j \in \mathbf{N})$, inequality (1.4) reduces to (1.2). We call (1.4) Hardy-Hilbert-type inequality.

Note. The authors of [1] did not prove that (1.4) is valid with the best possible constant factor.

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [6] obtained an extension of (1.1) with the kernel $\frac{1}{(x+y)^{\lambda}}$ for $p = q = 2$. Refining the method applied in [6], Yang [5] provided extensions of (1.1) and (1.2) as follows:

Assuming that $\lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^{\infty} k_\lambda(t, 1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+,$$

$$\phi(x) = x^{p(1-\lambda_1)-1}, \quad \psi(x) = x^{q(1-\lambda_2)-1}, f(x), g(y) \geq 0,$$

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{f; ||f||_{p,\phi} := \{\int_0^{\infty} \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty\right\},$$

where $g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, we have

$$\int_0^{\infty}\int_0^{\infty} k_\lambda(x, y) f(x) g(y) dx dy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{1.5}$$

where, the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x, y)$ keeps finite and

$$k_\lambda(x, y) x^{\lambda_1 - 1} (k_\lambda(x, y) y^{\lambda_2 - 1})$$

is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_{m,} b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{a; ||a||_{p,\phi} := \left(\sum_{n=1}^{\infty} \phi(n)|a_n|^p\right)^{\frac{1}{p}} < \infty\right\},$$

$b = \{b_n\}_{n=1}^{\infty} \in l_{q,\psi}$, $||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{1.6}$$

where, the constant factor $k(\lambda_1)$ is still the best possible.

For $0 < \lambda_1, \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda$, we set

$$k_\lambda(x,y) = \frac{1}{(x+y)^\lambda} \quad ((x,y) \in \mathbf{R}_+^2).$$

Then by (1.6), we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} < B(\lambda_1,\lambda_2)||a||_{p,\phi}||b||_{q,\psi}, \tag{1.7}$$

where, the constant $B(\lambda_1,\lambda_2)$ is the best possible, and

$$B\left(u,v\right) = \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1} dt \quad (u,v > 0)$$

is the beta function. Clearly, for $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (1.7) reduces to (1.2).

In 2015, by adding some conditions, Yang [7] extended (1.7) and (1.4) as follows:

$$\begin{aligned} &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(U_m+V_n)^\lambda} \\ < \ & B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}\frac{U_m^{p(1-\lambda_1)-1}a_m^p}{\mu_m^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\frac{V_n^{q(1-\lambda_2)-1}b_n^q}{\nu_n^{q-1}}\right]^{\frac{1}{q}}, \end{aligned} \tag{1.8}$$

where, the constant $B(\lambda_1,\lambda_2)$ is still the best possible.

Some other results including multidimensional Hilbert-type inequalities are provided in [8]-[30].

Related to the topic of half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy *et al.* provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are the best possible. However, Yang [31] established a result with the kernel $\frac{1}{(1+nx)^\lambda}$ by introducing a variable and proved that the constant factor is the best possible. In 2011 Yang [32] proved the following half-discrete Hardy-Hilbert's inequality with the best possible constant factor $B\left(\lambda_1,\lambda_2\right)$:

$$\int_0^\infty f\left(x\right)\left[\sum_{n=1}^{\infty}\frac{a_n}{\left(x+n\right)^\lambda}\right]dx < B\left(\lambda_1,\lambda_2\right)||f||_{p,\phi}||a||_{q,\psi}, \tag{1.9}$$

where, $\lambda_1 > 0$, $0 < \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$. Zhong *et al.* ([33]–[39]) investigated several half-discrete Hilbert-type inequalities with particular kernels. Applying the method of weight functions, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k\left(\lambda_1\right)$ is obtained as follows:

$$\int_0^\infty f(x)\sum_{n=1}^{\infty} k_\lambda(x,n)a_n dx < k(\lambda_1)||f||_{p,\phi}||a||_{q,\psi}, \tag{1.10}$$

which is an extension of (1.9) (cf. [40]). At the same time, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [41]. In 2012-2014, Yang *et al.* published three books [42], [43] and [44] extensively presenting the framework of half-discrete Hilbert-type inequalities.

In this paper, by the method of weight coefficients, techniques of real analysis and Hermite-Hadamard's inequality, a half-discrete Hardy-Hilbert-type inequality related to the kernel of the hyperbolic cosecant function with a best possible constant factor expressed by the extended Riemann-zeta function is proved, which is an extension of (1.10) for $\lambda = 0$ in the following particular kernel:

$$k_0(x,n) = \frac{\csc h(\rho(\frac{n}{x})^\gamma)}{e^{\alpha(\frac{n}{x})^\gamma}}(\rho > \max\{0,-\alpha\}, 0 < \gamma < 1).$$

Furthermore, the more accurate equivalent forms, the operator expressions with the norm, the reverses and some particular cases are also considered.

2. Some Lemmas

In the sequel, we shall assume that $\nu_n > 0$ $(n \in \mathbf{N}), \{\upsilon_n\}_{n=1}^\infty$ is decreasing, $V_n = \sum_{j=1}^n \nu_j$, $\mu(t)$ is a positive continuous function in $\mathbf{R}_+ = (0,\infty)$,

$$U(0) := 0; \quad U(x) := \int_0^x \mu(t)dt < \infty(x \in (0,\infty)),$$

$$\nu(t) := \nu_n, \ t \in (n-1, n] \ (n \in \mathbf{N}),$$

and

$$V(0) := 0; \quad V(y) := \int_0^y \nu(t)dt(y \in (0,\infty)),$$

$p \neq 0, 1, \ \frac{1}{p} + \frac{1}{q} = 1, \delta \in \{-1, 1\}, \beta \leq \frac{\nu_1}{2}, \ f(x), a_n \geq 0 \ (x \in \mathbf{R}_+, n \in \mathbf{N})$,

$$||f||_{p,\Phi_\delta} = (\int_0^\infty \Phi_\delta(x) f^p(x)dx)^{\frac{1}{p}},$$

$$||a||_{q,\Psi} = (\sum_{n=1}^\infty \Psi_\beta(n) b_n^q)^{\frac{1}{q}},$$

where,

$$\Phi_\delta(x) \ : \ = \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)}(x \in \mathbf{R}_+),$$

$$\Psi_\beta(n) \ : \ = \frac{(V_n - \beta)^{q(1-\sigma)-1}}{\nu_{n+1}^{q-1}}(n \in \mathbf{N}).$$

Lemma 2.1. *If $a \in \mathbf{R}$, $f(x)$ is continuous in $[a-\frac{1}{2}, a+\frac{1}{2}]$, $f'(x)$ is strictly increasing in $(a-\frac{1}{2}, a)$ and $(a, a+\frac{1}{2})$ respectively, as well as*

$$\lim_{x \to a-} f'(x) = f'(a-0) \leq f'(a+0) = \lim_{x \to a+} f'(x),$$

then $f(x)$ is strictly convex in $[a-\frac{1}{2}, a+\frac{1}{2}]$, and we have the following Hermite-Hadamard's inequality (cf. [48]*):*

$$f(a) < \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x)dx. \tag{2.1}$$

Proof . Since $f'(a-0)$ $(\leq f'(a+0))$ is finite, we define a function $g(x)$ as follows:

$$g(x) := f'(a-0)(x-a) + f(a), x \in [a-\frac{1}{2}, a+\frac{1}{2}].$$

In view of $f'(x)$ being strictly increasing in $(a-\frac{1}{2}, a)$, then for $x \in (a-\frac{1}{2}, a)$,

$$(f(x) - g(x))' = f'(x) - f'(a-0) < 0.$$

Since $f(a) - g(a) = 0$, it follows that $f(x) - g(x) > 0,\ x \in (a-\frac{1}{2}, a)$. Similarly, we can obtain $f(x) - g(x) > 0,\ x \in (a, a+\frac{1}{2})$. Hence, $f(x)$ is strictly convex in $[a-\frac{1}{2}, a+\frac{1}{2}]$, and therefore

$$\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(x)dx > \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} g(x)dx = f(a),$$

namely, (2.1) follows. □

Example 2.2. For $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1$,

$$\operatorname{csc} h(u) = \frac{2}{e^u - e^{-u}}\ (u > 0)$$

is called hyperbolic cosecant function (cf. [45]), we set

$$h(t) = \frac{\operatorname{csc} h(\rho t^\gamma)}{e^{\alpha t^\gamma}} = \frac{2}{e^{(\alpha+\rho)t^\gamma}(1 - e^{-2\rho t^\gamma})}\ \ (t \in \mathbf{R}_+).$$

(i) Setting $u = \rho t^\gamma$, we find

$$\begin{aligned}
k(\sigma) &:= \int_0^\infty \frac{\operatorname{csc} h(\rho t^\gamma)}{e^{\alpha t^\gamma}} t^{\sigma-1} dt \\
&= \frac{1}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{\operatorname{csc} h(u)}{e^{\frac{\alpha}{\rho}u}} u^{\frac{\sigma}{\gamma}-1} du \\
&= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{e^{-\frac{\alpha}{\rho}u} u^{\frac{\sigma}{\gamma}-1}}{e^u - e^{-u}} du \\
&= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{e^{-(\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1}}{1 - e^{-2u}} du \\
&= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \sum_{k=0}^\infty e^{-(2k+\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1} du.
\end{aligned}$$

By Lebesgue's term by term theorem (cf. [45]), setting $v = (2k + \frac{\alpha}{\rho} + 1)u$, we have

$$\begin{aligned}
k(\sigma) &= \int_0^\infty \frac{\operatorname{csc} h(\rho t^\gamma)}{e^{\alpha t^\gamma}} t^{\sigma-1} dt \\
&= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty \int_0^\infty e^{-(2k+\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1} du \\
&= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty \frac{1}{(2k + \frac{\alpha}{\rho} + 1)^{\sigma/\gamma}} \int_0^\infty e^{-v} v^{\frac{\sigma}{\gamma}-1} dv \\
&= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma (2\rho)^{\sigma/\gamma}} \sum_{k=0}^\infty \frac{1}{(k + \frac{\alpha+\rho}{2\rho})^{\sigma/\gamma}} \\
&= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma (2\rho)^{\sigma/\gamma}} \zeta(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}) \in \mathbf{R}_+, \qquad (2.2)
\end{aligned}$$

where

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (Re(s) > 1, a > 0)$$

is called the extended Riemann-zeta function (also known as the Hurwitz zeta function), and

$$\Gamma(y) := \int_0^{\infty} e^{-v} v^{y-1} dv \ (y > 0)$$

is called Gamma function (cf. [46]).

In particular, for $\alpha = \rho$, we have

$$h(t) = \frac{\csc h(\rho t^{\gamma})}{e^{\rho t^{\gamma}}} \quad \text{and} \quad k(\sigma) = k_1(\sigma) := \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \zeta(\frac{\sigma}{\gamma}).$$

In this case, for $\gamma = \frac{\sigma}{2}$, we have

$$h(t) = \frac{\csc h(\rho t^{\sigma/2})}{e^{\rho t^{\sigma/2}}} \quad \text{and} \quad k(\sigma) = \frac{\pi^2}{6\sigma\rho^2}.$$

(ii) We obtain for $u > 0$ that

$$\frac{1}{1-e^{-2u}} > 0, \quad (\frac{1}{1-e^{-2u}})' = -\frac{2e^{-2u}}{(1-e^{-2u})^2} < 0,$$

and

$$(\frac{1}{1-e^{-2u}})'' = \frac{4e^{-2u}}{(1-e^{-2u})^2} + \frac{8e^{-4u}}{(1-e^{-2u})^3} > 0.$$

(iii) If $g(u) > 0, g'(u) < 0, g''(u) > 0$, then for $0 < \gamma \leq 1$, we find that $g(\rho t^{\gamma}) > 0$, $\frac{d}{dt} g(\rho t^{\gamma}) = \rho\gamma t^{\gamma-1} g'(\rho t^{\gamma}) < 0$, and

$$\frac{d^2}{dt^2} g(\rho t^{\gamma}) = \rho\gamma(\gamma-1)t^{\gamma-2} g'(\rho t^{\gamma}) + (\rho\gamma t^{\gamma-1})^2 g''(\rho t^{\gamma}) > 0.$$

Then we find that for $y \in (n - \frac{1}{2}, n)$,

$$g(V(y)-\beta) > 0, \frac{d}{dy} g(V(y)-\beta) = g'(V(y)-\beta)\nu_n < 0,$$

and

$$\frac{d^2}{dy^2} g(V(y)-\beta) = g''(V(y)-\beta)\nu_n^2 > 0 \ (n \in \mathbf{N});$$

for $y \in (n, n + \frac{1}{2})$,

$$g(V(y)-\beta) > 0, \frac{d}{dy} g(V(y)-\beta) = g'(V(y)-\beta)\nu_{n+1} < 0,$$

and

$$\frac{d^2}{dy^2} g(V(y)-\beta) = g''(V(y)-\beta)\nu_{n+1}^2 > 0 \ (n \in \mathbf{N}).$$

Clearly $\zeta(s,1) = \zeta(s)$, where $\zeta(s)$ is the Riemann-zeta function.

If $g_1(u) > 0, g_1'(u) < 0, g_1''(u) > 0, g_2(u) > 0, g_2'(u) \leq 0, g_2''(u) \geq 0$, then we find for $u > 0$ that

$$g_1(u)g_2(u) > 0, (g_1(u)g_2(u))' = g_1'(u)g_2(u) + g_1(u)g_2'(u) < 0,$$

and

$$(g_1(u)g_2(u))'' = g_1''(u)g_2(u) + 2g_1'(u)g_2'(u) + g_1(u)g_2''(u) > 0.$$

(iv) For $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1$, we have

$$h(t) > 0, h'(t) < 0, \quad h''(t) > 0, \quad \text{with } k(\sigma) \in \mathbf{R}_+,$$

and then for $c > 0, \beta \leq \frac{\nu_1}{2}, y \geq \frac{1}{2}, n \in \mathbf{N}$, we have

$$h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1} > 0, \quad \frac{d}{dy}[h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1}] < 0,$$

and

$$\frac{d^2}{dy^2}[h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1}] > 0 \quad (y \in (n - \frac{1}{2}, n) \cup (n, n + \frac{1}{2})).$$

Setting $f(y) = h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1}$, it follows that $f'(y)(< 0)$ is strictly increasing in $(n - \frac{1}{2}, n)$ and

$$\begin{aligned} \lim_{x \to n-} f'(y) &= f'(n - 0) = [ch'(c(V_n - \beta))(V_n - \beta)^{\sigma-1} \\ &+ (\sigma - 1)h(c(V_n - \beta))(V_n - \beta)^{\sigma-2}]\nu_n. \end{aligned}$$

In the same way, for $x \in (n, n + \frac{1}{2})$, we find that $f'(y)(< 0)$ is strictly increasing and

$$\begin{aligned} \lim_{x \to n+} f'(y) &= f'(n + 0) = [ch'(c(V_n - \beta))(V_n - \beta)^{\sigma-1} \\ &+ (\sigma - 1)h(c(V_n - \beta))(V_n - \beta)^{\sigma-2}]\nu_{n+1}. \end{aligned}$$

In view of $\nu_{n+1} \leq \nu_n$, it follows that

$$\lim_{x \to n+} f'(x) = f'(n + 0) \geq f'(n - 0) = \lim_{x \to n-} f'(x).$$

Then by (2.1), for $n \in \mathbf{N}$, we have

$$f(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(y)dy = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1}dy. \tag{2.3}$$

Lemma 2.3. *If $g(t)(> 0)$ is a strictly decreasing continuous function in $(\frac{1}{2}, \infty)$, which is strictly convex satisfying*

$$\int_{\frac{1}{2}}^{\infty} g(t)dt \in \mathbf{R}_+,$$

then we have

$$\int_1^{\infty} g(t)dt < \sum_{n=1}^{\infty} g(n) < \int_{\frac{1}{2}}^{\infty} g(t)dt. \tag{2.4}$$

Proof . By (2.1) and the decreasing property, we have

$$\int_n^{n+1} g(t)dt < \int_n^{n+1} g(n)dt = g(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t)dt \quad (n \in \mathbf{N}),$$

and for $n_0 \in \mathbf{N}$, it follows that

$$\begin{aligned} \int_1^{n_0+1} g(t)dt &< \sum_{n=1}^{n_0} g(n) < \sum_{n=1}^{n_0} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t)dt = \int_{\frac{1}{2}}^{n_0+\frac{1}{2}} g(t)dt, \\ \int_{n_0+1}^{\infty} g(t)dt &\le \sum_{n=n_0+1}^{\infty} g(n) \le \int_{n_0+\frac{1}{2}}^{\infty} g(t)dt < \infty. \end{aligned}$$

Hence, we obtain (2.4). □

Lemma 2.4. *If $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \le 1$, define the following weight coefficients:*

$$\omega_\delta(\sigma, x) \ :=\sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} \frac{U^{\delta\sigma}(x)\nu_{n+1}}{(V_n-\beta)^{1-\sigma}}, \quad x \in \mathbf{R}_+, \tag{2.5}$$

$$\varpi_\delta(\sigma, n) \ := \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} \frac{(V_n-\beta)^{\sigma}\mu(x)}{U^{1-\delta\sigma}(x)}dx, \quad n \in \mathbf{N}. \tag{2.6}$$

Then, we have the following inequalities:

$$\omega_\delta(\sigma, x) < k(\sigma) \quad (x \in \mathbf{R}_+), \tag{2.7}$$

$$\varpi_\delta(\sigma, n) \le k(\sigma) \quad (n \in \mathbf{N}), \tag{2.8}$$

where, $k(\sigma)$ is indicated by (2.2).

Proof . Since $V_n = V(n)$, and for $t \in (n-\frac{1}{2}, n)$,

$$\nu_{n+1} \le \nu_n = V'(t);$$

for $t \in (n, n+\frac{1}{2})$,

$$\nu_{n+1} = V'(t),$$

by (2.3) (for $c = U^\delta(x)$), we have

$$\begin{aligned} & \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} \frac{U^{\delta\sigma}(x)}{(V_n-\beta)^{1-\sigma}} \\ = & \frac{\csc h(\rho U^{\delta\gamma}(x)(V(n)-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(n)-\beta)^\gamma}} \frac{U^{\delta\sigma}(x)}{(V(n)-\beta)^{1-\sigma}} \\ < & \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t)-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t)-\beta)^\gamma}} \frac{U^{\delta\sigma}(x)}{(V(t)-\beta)^{1-\sigma}}dt \quad (n \in \mathbf{N}), \end{aligned}$$

$$\begin{aligned} \omega_\delta(\sigma, x) &< \sum_{n=1}^{\infty} \nu_{n+1} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t)-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t)-\beta)^\gamma}} \frac{U^{\delta\sigma}(x)dt}{(V(t)-\beta)^{1-\sigma}} \\ &\le \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t)-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t)-\beta)^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{(V(t)-\beta)^{1-\sigma}}dt \\ &= \int_{\frac{1}{2}}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V(t)-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t)-\beta)^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{(V(t)-\beta)^{1-\sigma}}dt. \end{aligned}$$

Setting $u = U^{\delta}(x)(V(t) - \beta)$, by (2.2), we obtain

$$\begin{aligned}\omega_{\delta}(\sigma, x) &< \int_{U^{\delta}(x)(\frac{\nu_1}{2}-\beta)}^{U^{\delta}(x)V(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} \frac{U^{\delta\sigma}(x)U^{-\delta}(x)}{(uU^{-\delta}(x))^{1-\sigma}} du \\ &\leq \int_0^{\infty} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du = k(\sigma).\end{aligned}$$

Hence, (2.7) follows.

Setting $u = (V_n - \beta)U^{\delta}(x)$ in (2.6), we find $du = \delta(V_n - \beta)U^{\delta-1}(x)\mu(x)dx$ and

$$\begin{aligned}\varpi_{\delta}(\sigma, n) &= \frac{1}{\delta} \int_{(V_n-\beta)U^{\delta}(0)}^{(V_n-\beta)U^{\delta}(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} \frac{(V_n - \beta)^{\sigma-1}[(V_n - \beta)^{-1}u]^{\frac{1}{\delta}-1}}{[(V_n - \beta)^{-1}u]^{\frac{1}{\delta}-\sigma}} du \\ &= \frac{1}{\delta} \int_{(V_n-\beta)U^{\delta}(0)}^{(V_n-\beta)U^{\delta}(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du.\end{aligned}$$

If $\delta = 1$, then

$$\varpi_1(\sigma, n) = \int_0^{(V_n-\beta)U(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du \leq \int_0^{\infty} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du;$$

if $\delta = -1$, then

$$\varpi_{-1}(\sigma, n) = -\int_{\infty}^{(V_n-\beta)U^{-1}(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du \leq \int_0^{\infty} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du.$$

Hence, by (2.2), we have (2.8). □

Remark 2.5. We do not need the condition of $\sigma \leq 1$ in obtaining (2.8). If $U(\infty) = \infty$, then we have

$$\varpi_{\delta}(\sigma, n) = k(\sigma) \quad (n \in \mathbf{N}). \tag{2.9}$$

For example, we set $\mu(t) = \frac{1}{(1+t)^a}$ $(t > 0; 0 \leq a \leq 1)$, then for $x \geq 0$, we find

$$U(x) = \int_0^x \frac{dt}{(1+t)^a} = \begin{cases} \frac{(1+x)^{1-a}-1}{1-a}, 0 \leq a < 1 \\ \ln(1+x), a = 1 \end{cases} < \infty,$$

$U(0) = 0$ and $U(\infty) = \int_0^{\infty} \frac{dt}{(1+t)^a} = \infty$.

Lemma 2.6. *If $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1, V(\infty) = \infty$, then, (i) for $x \in \mathbf{R}_+$, we have*

$$k(\sigma)(1 - \theta_{\delta}(\sigma, x)) < \omega_{\delta}(\sigma, x), \tag{2.10}$$

where,

$$\begin{aligned}\theta_{\delta}(\sigma, x) &:= \frac{1}{k(\sigma)} \int_0^{U^{\delta}(x)(\nu_1-\beta)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du \\ &= O((U(x))^{\frac{\delta}{2}(\sigma-\gamma)}) \in (0, 1);\end{aligned}$$

(ii) for any $b > 0$, we have

$$\sum_{n=1}^{\infty} \frac{\nu_{n+1}}{(V_n - \beta)^{1+b}} = \frac{1}{b} \left[\frac{1}{(\nu_1 - \beta)^b} + bO(1) \right]. \tag{2.11}$$

Proof . By (2.4), we find

$$\begin{aligned}
\omega_\delta(\sigma,x) &> \sum_{n=1}^{\infty}\nu_{n+1}\int_n^{n+1}\frac{\csc h(\rho U^{\delta\gamma}(x)(V(t)-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t)-\beta)^\gamma}}\frac{U^{\delta\sigma}(x)dt}{(V(t)-\beta)^{1-\sigma}}\\
&= \sum_{n=1}^{\infty}\int_n^{n+1}\frac{\csc h(\rho U^{\delta\gamma}(x)(V(t)-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t)-\beta)^\gamma}}\frac{U^{\delta\sigma}(x)V'(t)}{(V(t)-\beta)^{1-\sigma}}dt\\
&= \int_1^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V(t)-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V(t)-\beta)^\gamma}}\frac{U^{\delta\sigma}(x)V'(t)}{(V(t)-\beta)^{1-\sigma}}dt.
\end{aligned}$$

Setting $u = U^\delta(x)(V(t)-\beta)$, in view of $V(\infty)=\infty$, by (2.2), we find

$$\begin{aligned}
\omega_\delta(\sigma,x) &> \int_{U^\delta(x)(V(1)-\beta)}^{\infty}\frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}}u^{\sigma-1}du\\
&= k(\sigma)-\int_0^{U^\delta(x)(\nu_1-\beta)}\frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}}u^{\sigma-1}du\\
&= k(\sigma)(1-\theta_\delta(\sigma,x)).
\end{aligned}$$

Since

$$F(u)=\frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}}$$

is continuous in $(0,\infty)$ satisfying $u^{\frac{1}{2}(\sigma+\gamma)}F(u)\to 0$ $(u\to 0^+)$, and $u^{\frac{1}{2}(\sigma+\gamma)}F(u)\to 0$ $(u\to\infty)$, there exists a constant $L>0$, such that $u^{\frac{1}{2}(\sigma+\gamma)}F(u)\le L$, namely,

$$\frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}}\le Lu^{\frac{-1}{2}(\sigma+\gamma)}\quad (u\in(0,\infty)).$$

Hence we find

$$\begin{aligned}
0 &< \theta_\delta(\sigma,x)\le\frac{L}{k(\sigma)}\int_0^{U^\delta(x)(\nu_1-\beta)}u^{\frac{1}{2}(\sigma-\gamma)-1}du\\
&= \frac{2L[U^\delta(x)(\nu_1-\beta)]^{\frac{1}{2}(\sigma-\gamma)}}{k(\sigma)(\sigma-\gamma)}\quad (x\in\mathbf{R}_+),
\end{aligned}$$

and then (2.10) follows.

For $b>0$, we find

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1+b}} &< \frac{\nu_2}{(V_1-\beta)^{1+b}}+\sum_{n=2}^{\infty}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\frac{V'(x)}{(V(x)-\beta)^{1+b}}dx\\
&= \frac{\nu_2}{(\nu_1-\beta)^{1+b}}+\int_{\frac{3}{2}}^{\infty}\frac{V'(x)}{(V(x)-\beta)^{1+b}}dx\\
&= \frac{\nu_2}{(\nu_1-\beta)^{1+b}}+\int_{\nu_1+\frac{1}{2}\nu_2-\beta}^{\infty}\frac{du}{u^{1+b}}\\
&\le \frac{1}{b}\left[\frac{1}{(\nu_1-\beta)^b}+b\frac{\nu_2}{(\nu_1-\beta)^{1+b}}\right],
\end{aligned}$$

$$\sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1+b}} = \sum_{n=1}^{\infty}\int_n^{n+1}\frac{\nu_{n+1}}{(V(n)-\beta)^{1+b}}dx > \sum_{n=1}^{\infty}\int_n^{n+1}\frac{V'(x)dx}{(V(x)-\beta)^{1+b}}$$
$$= \int_1^{\infty}\frac{V'(x)dx}{(V(x)-\beta)^{1+b}} = \frac{1}{b(\nu_1-\beta)^b}.$$

Hence we have (2.11). □

Note. For example, $\nu_n = \frac{1}{n^a}$ $(n \in \mathbf{N}; 0 \le a \le 1)$ satisfies the condition that $\nu_n > 0$ $(n \in \mathbf{N})$, $\{\nu_n\}_{n=1}^{\infty}$ is decreasing, and $V(\infty) = \infty$.

3. Main results and operator expressions

Theorem 3.1. *If $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \le 1$, $k(\sigma)$ is indicated by (2.2), then for $p > 1$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities:*

$$I := \sum_{n=1}^{\infty}\int_0^{\infty}\frac{\operatorname{csc} h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}a_n f(x)dx < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \tag{3.1}$$

$$J_1 := \sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_0^{\infty}\frac{\operatorname{csc} h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}f(x)dx\right]^p$$
$$< k(\sigma)||f||_{p,\Phi_\delta}, \tag{3.2}$$

$$J_2 := \left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\operatorname{csc} h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}}$$
$$< k(\sigma)||a||_{q,\Psi_\beta}. \tag{3.3}$$

Proof . By the weighted Hölder inequality (cf. [48]), we have

$$\left[\int_0^{\infty}\frac{\operatorname{csc} h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}f(x)dx\right]^p$$
$$= \left[\int_0^{\infty}\frac{\operatorname{csc} h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}\right.$$
$$\left.\times\frac{U^{\frac{1-\delta\sigma}{q}}(x)f(x)}{(V_n-\beta)^{\frac{1-\sigma}{p}}\mu^{\frac{1}{q}}(x)}\cdot\frac{(V_n-\beta)^{\frac{1-\sigma}{p}}\mu^{\frac{1}{q}}(x)}{U^{\frac{1-\delta\sigma}{q}}(x)}dx\right]^p$$
$$\le \int_0^{\infty}\frac{\operatorname{csc} h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}\left[\frac{U^{\frac{p(1-\delta\sigma)}{q}}(x)f^p(x)}{(V_n-\beta)^{1-\sigma}\mu^{\frac{p}{q}}(x)}\right]dx$$
$$\times\left[\int_0^{\infty}\frac{\operatorname{csc} h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}\frac{(V_n-\beta)^{(1-\sigma)(p-1)}\mu(x)}{U^{1-\delta\sigma}(x)}dx\right]^{p-1}$$
$$= \frac{(\varpi_\delta(\sigma,n))^{p-1}}{(V_n-\beta)^{p\sigma-1}\nu_{n+1}}$$
$$\times\int_0^{\infty}\frac{\operatorname{csc} h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}\frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_{n+1}f^p(x)}{(V_n-\beta)^{1-\sigma}\mu^{p-1}(x)}dx. \tag{3.4}$$

In view of (2.8) and the Lebesgue term by term integration theorem (cf. [47]), we find

$$\begin{aligned} J_1 &\leq (k(\sigma))^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_{n+1}}{(V_n-\beta)^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}} \\ &= (k(\sigma))^{\frac{1}{q}} \left[\int_0^{\infty} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_{n+1}}{(V_n-\beta)^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}} \\ &= (k(\sigma))^{\frac{1}{q}} \left[\int_0^{\infty} \omega_\delta(\sigma,x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}}. \end{aligned} \tag{3.5}$$

Then by (2.7), we derive (3.2).

By Hölder's inequality (cf. [48]), we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \left[\frac{\nu_{n+1}^{\frac{1}{p}}}{(V_n-\beta)^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} f(x)dx \right] \\ &= \left[\frac{(V_n-\beta)^{\frac{1}{p}-\sigma} a_n}{\nu_{n+1}^{\frac{1}{p}}} \right] \leq J_1 ||a||_{q,\Psi_\beta}. \end{aligned} \tag{3.6}$$

Then by (3.2), we obtain (3.1). On the other hand, assuming that (3.1) is valid, we set

$$a_n := \frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} f(x)dx \right]^{p-1}, \quad n \in \mathbf{N}.$$

Then we find $J_1^p = ||a||_{q,\Psi_\beta}^q$. If $J_1 = 0$, then (3.2) is trivially valid; if $J_1 = \infty$, then (3.2) is still not valid. Suppose that $0 < J_1 < \infty$. By (3.1), we have

$$\begin{aligned} ||a||_{q,\Psi_\beta}^q &= J_1^p = I < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \\ ||a||_{q,\Psi_\beta}^{q-1} &= J_1 < k(\sigma)||f||_{p,\Phi_\delta}, \end{aligned}$$

and then (3.2) follows, which is equivalent to (3.1).

Still by the weighted Hölder inequality (cf. [48]), we have

$$\begin{aligned} &\left[\sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} a_n \right]^q \\ &= \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} \cdot \frac{U^{\frac{1-\delta\sigma}{q}}(x)\nu_{n+1}^{\frac{1}{p}}}{(V_n-\beta)^{\frac{1-\sigma}{p}}} \cdot \frac{(V_n-\beta)^{\frac{1-\sigma}{p}} a_n}{U^{\frac{1-\delta\sigma}{q}}(x)\nu_{n+1}^{\frac{1}{p}}} \right]^q \\ &\leq \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_{n+1}}{(V_n-\beta)^{1-\sigma}} \right]^{q-1} \\ &\quad \times \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} \frac{(V_n-\beta)^{\frac{q(1-\sigma)}{p}}}{U^{1-\delta\sigma}(x)\nu_{n+1}^{q-1}} a_n^q \\ &= \frac{(\omega_\delta(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} \frac{(V_n-\beta)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_{n+1}^{q-1}} a_n^q. \end{aligned} \tag{3.7}$$

Then by (2.7) and the Lebesgue term by term integration theorem (cf. [47]), it follows that

$$J_2 < (k(\sigma))^{\frac{1}{p}} \left[\int_0^\infty \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} \frac{(V_n-\beta)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_{n+1}^{q-1}} a_n^q dx \right]^{\frac{1}{q}}$$

$$\begin{aligned} &= (k(\sigma))^{\frac{1}{p}} \left[\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} \frac{(V_n-\beta)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_{n+1}^{q-1}} a_n^q dx \right]^{\frac{1}{q}} \\ &= (k(\sigma))^{\frac{1}{p}} \left[\sum_{n=1}^\infty \varpi_\delta(\sigma,n) \frac{(V_n-\beta)^{q(1-\sigma)-1}}{\nu_{n+1}^{q-1}} a_n^q \right]^{\frac{1}{q}}. \end{aligned} \tag{3.8}$$

Then by (2.8), we derive (3.3).

By Hölder's inequality (cf. [48]), we have

$$\begin{aligned} I &= \int_0^\infty \left(\frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x) \right) \left[\frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} a_n \right] dx \\ &\leq ||f||_{p,\Phi_\delta} J_2. \end{aligned} \tag{3.9}$$

Then by (3.3), we obtain (3.1). On the other hand, assuming that (3.3) is valid, we set

$$f(x) := \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} a_n \right]^{q-1}, \quad x \in \mathbf{R}_+.$$

Then we find $J_2^q = ||f||_{p,\Phi_\delta}^p$. If $J_2 = 0$, then (3.3) is trivially valid; if $J_2 = \infty$, then (3.3) remains impossible. Suppose that $0 < J_2 < \infty$. By (3.1), we have

$$\begin{aligned} ||f||_{p,\Phi_\delta}^p &= J_2^q = I < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \\ ||f||_{p,\Phi_\delta}^{p-1} &= J_2 < k(\sigma)||a||_{q,\Psi_\beta}, \end{aligned}$$

and then (3.3) follows, which is equivalent to (3.1).

Therefore, (3.1), (3.2) and (3.3) are equivalent. □

Theorem 3.2. *With the assumptions of Theorem 3.1, if $U(\infty) = V(\infty) = \infty$, then the constant factor $k(\sigma)$ in (3.1), (3.2) and (3.3) is the best possible.*

Proof . For $\varepsilon \in (0, \frac{q(\sigma-\gamma)}{2}))$, we set $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{q}$, and $\widetilde{f} = \widetilde{f}(x),\ x \in \mathbf{R}_+,\ \widetilde{a} = \{\widetilde{a}_n\}_{n=1}^\infty$,

$$\widetilde{f}(x) = \begin{cases} U^{\delta(\widetilde{\sigma}+\varepsilon)-1}(x)\mu(x), \ 0 < x^\delta \leq 1 \\ 0, \ x^\delta > 0 \end{cases}, \tag{3.10}$$

$$\widetilde{a}_n = (V_n-\beta)^{\widetilde{\sigma}-1}\nu_{n+1} = (V_n-\beta)^{\sigma-\frac{\varepsilon}{q}-1}\nu_{n+1}, \ n \in \mathbf{N}. \tag{3.11}$$

Then for $\delta = \pm 1$, since $U(\infty) = \infty$, we find

$$\int_{\{x>0;0<x^\delta\leq 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx = \frac{1}{\varepsilon} U^{\delta\varepsilon}(1). \tag{3.12}$$

By (2.11), (3.12) and (2.10), we obtain

$$\begin{aligned} ||\widetilde{f}||_{p,\Phi_\delta}||\widetilde{a}||_{q,\Psi_\beta} &= \left(\int_{\{x>0;0<x^\delta\leq 1\}} \frac{\mu(x)dx}{U^{1-\delta\varepsilon}(x)}\right)^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \frac{\nu_{n+1}}{(V_n-\beta)^{1+\varepsilon}}\right]^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} U^{\frac{\delta\varepsilon}{p}}(1) \left[\frac{1}{(\nu_1-\beta)^\varepsilon} + \varepsilon O(1)\right]^{\frac{1}{q}}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \widetilde{I} \ : &= \int_0^\infty \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} \widetilde{a}_n \widetilde{f}(x)dx \\ &= \int_{\{x>0;0<x^\delta\leq 1\}} \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} \frac{(V_n-\beta)^{\widetilde{\sigma}-1}\nu_{n+1}\mu(x)}{U^{1-\delta(\widetilde{\sigma}+\varepsilon)}(x)}dx \\ &= \int_{\{x>0;0<x^\delta\leq 1\}} \omega_\delta(\widetilde{\sigma},x)\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\ &\geq k(\widetilde{\sigma})\int_{\{x>0;0<x^\delta\leq 1\}} (1-\theta_\delta(\widetilde{\sigma},x))\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\ &= k(\widetilde{\sigma})\int_{\{x>0;0<x^\delta\leq 1\}} (1-O((U(x))^{\delta\frac{\sigma-\frac{\varepsilon}{q}-\gamma}{2}}))\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\ &= k(\widetilde{\sigma})\left[\int_{\{x>0;0<x^\delta\leq 1\}} \frac{\mu(x)dx}{U^{1-\delta\varepsilon}(x)} - \int_{\{x>0;0<x^\delta\leq 1\}} O\left(\frac{\mu(x)}{U^{1-\delta(\varepsilon+\frac{\sigma-\frac{\varepsilon}{q}-\gamma}{2})}(x)}\right)dx\right] \\ &= \frac{1}{\varepsilon}k(\sigma-\frac{\varepsilon}{q})(U^{\delta\varepsilon}(1)-\varepsilon O_1(1)). \end{aligned}$$

If there exists a positive constant $K \leq k(\sigma)$, such that (3.1) is valid when replacing $k(\sigma)$ to K, then in particular, by the Lebesgue term by term integration theorem, we have

$$\varepsilon\widetilde{I} < \varepsilon K||\widetilde{f}||_{p,\Phi_\delta}||\widetilde{a}||_{q,\Psi_\beta},$$

namely,

$$k(\sigma-\frac{\varepsilon}{q})(U^{\delta\varepsilon}(1)-\varepsilon O_1(1)) < K\cdot U^{\frac{\delta\varepsilon}{p}}(1)\left[\frac{1}{(\nu_1-\beta)^\varepsilon}+\varepsilon O(1)\right]^{\frac{1}{q}}.$$

It follows that $k(\sigma) \leq K(\varepsilon \to 0^+)$. Hence, $K = k(\sigma)$ is the best possible constant factor of (3.1).

The constant factor $k(\sigma)$ in (3.2) ((3.3)) is still the best possible. Otherwise, we would reach a contradiction by (3.6) ((3.9)) that the constant factor in (3.1) is not the best possible. □

For $p > 1$, we obtain

$$\Psi_\beta^{1-p}(n) = \frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}} \quad (n\in\mathbf{N}), \quad \Phi_\delta^{1-q}(x) = \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \quad (x\in\mathbf{R}_+),$$

and define the following real normed spaces:

$$\begin{aligned} L_{p,\Phi_\delta}(\mathbf{R}_+) &= \{f; f=f(x), x\in\mathbf{R}_+, ||f||_{p,\Phi_\delta}<\infty\}, \\ l_{q,\Psi_\beta} &= \{a; a=\{a_n\}_{n=1}^\infty, ||a||_{q,\Psi_\beta}<\infty\}, \\ L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+) &= \{h; h=h(x), x\in\mathbf{R}_+, ||h||_{q,\Phi_\delta^{1-q}}<\infty\}, \\ l_{p,\Psi_\beta^{1-p}} &= \{c; c=\{c_n\}_{n=1}^\infty, ||c||_{p,\Psi_\beta^{1-p}}<\infty\}. \end{aligned}$$

Assuming that $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$ and setting

$$c = \{c_n\}_{n=1}^{\infty}, \quad c_n := \int_0^\infty \frac{\csc h(\rho[U^\delta(x)(V_n-\beta)]^\gamma)}{e^{\alpha[U^\delta(x)(V_n-\beta)]^\gamma}} f(x)dx, \ n \in \mathbf{N},$$

we can rewrite (3.2) as

$$||c||_{p,\Psi_\beta^{1-p}} < k(\sigma)||f||_{p,\Phi_\delta} < \infty,$$

namely, $c \in l_{p,\Psi_\beta^{1-p}}$.

Definition 3.3. Define a half-discrete Hardy-Hilbert-type operator $T_1 : L_{p,\Phi_\delta}(\mathbf{R}_+) \to l_{p,\Psi_\beta^{1-p}}$ as follows: For any $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$, there exists a unique representation $T_1 f = c \in l_{p,\Psi_\beta^{1-p}}$. Define the formal inner product of $T_1 f$ and $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\Psi_\beta}$ as follows:

$$(T_1 f, a) := \sum_{n=1}^{\infty} \left[\int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} f(x)dx \right] a_n. \tag{3.14}$$

Then we can rewrite (3.1) and (3.2) as follows:

$$(T_1 f, a) < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \tag{3.15}$$
$$||T_1 f||_{p,\Psi_\beta^{1-p}} < k(\sigma)||f||_{p,\Phi_\delta}. \tag{3.16}$$

Define the norm of operator T_1 as follows:

$$||T_1|| := \sup_{f(\neq\theta)\in L_{p,\Phi_\delta}(\mathbf{R}_+)} \frac{||T_1 f||_{p,\Psi_\beta^{1-p}}}{||f||_{p,\Phi_\delta}}.$$

Then by (3.16), it follows that $||T_1|| \leq k(\sigma)$. Since by Theorem 3.2, the constant factor in (3.16) is the best possible, we have

$$||T_1|| = k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \zeta(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}). \tag{3.17}$$

Assuming that $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\Psi_\beta}$ and setting

$$h(x) := \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} a_n, \ x \in \mathbf{R}_+,$$

we can rewrite (3.3) as $||h||_{q,\Phi_\delta^{1-q}} < k(\sigma)||a||_{q,\Psi_\beta} < \infty$, namely, $h \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$.

Definition 3.4. Define a half-discrete Hardy-Hilbert-type operator $T_2 : l_{q,\Psi_\beta} \to L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$ as follows: For any $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\Psi_\beta}$, there exists a unique representation

$$T_2 a = h \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+).$$

Define the formal inner product of $T_2 a$ and $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$ as follows:

$$(T_2 a, f) := \int_0^\infty \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} a_n \right] f(x)dx. \tag{3.18}$$

Then we can rewrite (3.1) and (3.3) as follows:

$$(T_2 a, f) < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \quad (3.19)$$

$$||T_2 a||_{q,\Phi_\delta^{1-q}} < k(\sigma)||a||_{q,\Psi_\beta}. \quad (3.20)$$

Define the norm of operator T_2 as follows:

$$||T_2|| := \sup_{a(\neq\theta)\in l_{q,\Psi}} \frac{||T_2 a||_{q,\Phi_\delta^{1-q}}}{||a||_{q,\Psi_\beta}}.$$

Then by (3.20), we find $||T_2|| \leq k(\sigma)$. Since by Theorem 3.2, the constant factor in (3.20) is the best possible, we obtain

$$||T_2|| = k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}}\zeta(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}) = ||T_1||. \quad (3.21)$$

4. Some Equivalent Reverse Inequalities

In the following, we also set

$$\widetilde{\Phi}_\delta(x) := (1-\theta_\delta(\sigma,x))\frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} \ (x \in \mathbf{R}_+).$$

For $0<p<1$ or $p<0$, we still use the formal symbols $||f||_{p,\Phi_\delta}$, $||f||_{p,\widetilde{\Phi}_\delta}$ and $||a||_{q,\Psi_\beta}$ *et al.*

Theorem 4.1. *If* $\rho > \max\{0,-\alpha\}, 0<\gamma<\sigma\leq 1, k(\sigma)$ *is indicated by* (2.1), *and* $U(\infty)=V(\infty)=\infty$, *then for* $p<0$, $0<||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta}<\infty$, *we have the following equivalent inequalities with the best possible constant factor* $k(\sigma)$*:*

$$I = \sum_{n=1}^{\infty}\int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} a_n f(x)dx > k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \quad (4.1)$$

$$J_1 = \sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} f(x)dx\right]^p > k(\sigma)||f||_{p,\Phi_\delta}, \quad (4.2)$$

$$J_2 = \left\{\int_0^\infty \frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi_\beta}. \quad (4.3)$$

Proof . By the reverse weighted Hölder inequality (cf. [48]), since $p<0$, similarly to the way we obtained (3.4) and (3.5), we have

$$\left[\int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} f(x)dx\right]^p \leq \frac{(\varpi_\delta(\sigma,n))^{p-1}}{(V_n-\beta)^{p\sigma-1}\nu_{n+1}}\int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}}\frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_{n+1}}{(V_n-\beta)^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx.$$

Then by (2.9) and the Lebesgue term by term integration theorem, it follows that

$$\begin{aligned} J_1 &\geq (k(\sigma))^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_{n+1}}{(V_n-\beta)^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}} \\ &= (k(\sigma))^{\frac{1}{q}} \left[\int_0^{\infty} \omega_\delta(\sigma,x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}}. \end{aligned}$$

Then by (2.7), we have (4.2).

By the reverse Hölder inequality (cf. [48]), we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \left[\frac{\nu_{n+1}^{\frac{1}{p}}}{(V_n-\beta)^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} f(x)dx \right] \left[\frac{(V_n-\beta)^{\frac{1}{p}-\sigma}a_n}{\nu_{n+1}^{\frac{1}{p}}} \right] \\ &\geq J_1||a||_{q,\Psi_\beta}. \end{aligned} \tag{4.4}$$

Then by (4.2), we derive (4.1). On the other hand, assuming that (4.1) is valid, we set a_n as in Theorem 3.1. Then we obtain

$$J_1^p = ||a||_{q,\Psi_\beta}^q.$$

If $J_1 = \infty$, then (4.2) is trivially valid. If $J_1 = 0$, then (4.2) is still not valid. Suppose that $0 < J_1 < \infty$. By (4.1), it follows that

$$\begin{aligned} ||a||_{q,\Psi_\beta}^q &= J_1^p = I > k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \\ ||a||_{q,\Psi_\beta}^{q-1} &= J_1 > k(\sigma)||f||_{p,\Phi_\delta}, \end{aligned}$$

and then (4.2) follows, which is equivalent to (4.1).

Applying again the weighted reverse Hölder inequality (cf. [48]), since $0 < q < 1$, similarly to how we obtained (3.7) and (3.8), we have

$$\begin{aligned} & \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} a_n \right]^q \\ \geq & \frac{(\omega_\delta(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} \frac{(V_n-\beta)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_{n+1}^{q-1}} a_n^q. \end{aligned}$$

Then, by (2.7) and the Lebesgue term by term integration theorem, it follows that

$$\begin{aligned} J_2 &> (k(\sigma))^{\frac{1}{p}} \left[\int_0^{\infty} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} \frac{(V_n-\beta)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_{n+1}^{q-1}} a_n^q dx \right]^{\frac{1}{q}} \\ &= (k(\sigma))^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \varpi_\delta(\sigma,n) \frac{(V_n-\beta)^{q(1-\sigma)-1}}{\nu_{n+1}^{q-1}} a_n^q \right]^{\frac{1}{q}}. \end{aligned}$$

Hence, by (2.9), we have (4.3).

By the reverse Hölder inequality (cf. [48]), we get

$$\begin{aligned} I &= \int_0^{\infty} \left(\frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x) \right) \left[\frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}} a_n \right] dx \\ &\geq ||f||_{p,\Phi_\delta} J_2. \end{aligned} \tag{4.5}$$

Thus by (4.3), we obtain (4.1). On the other hand, assuming that (4.3) is valid, we set $f(x)$ as in Theorem 4.1. Then we derive that

$$J_2^q = ||f||_{p,\Phi_\delta}^p.$$

If $J_2 = \infty$, then (4.3) is trivially valid. If $J_2 = 0$, then (4.3) remains impossible. Suppose that $0 < J_2 < \infty$. By (4.1), it follows that

$$\begin{aligned} ||f||_{p,\Phi_\delta}^p &= J_2^q = I > k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \\ ||f||_{p,\Phi_\delta}^{p-1} &= J_2 > k(\sigma)||a||_{q,\Psi_\beta}, \end{aligned}$$

and then (4.3) follows, which is equivalent to (4.1).

Therefore, inequalities (4.1), (4.2) and (4.3) are equivalent.

For $\varepsilon \in (0, \frac{q(\sigma-\gamma)}{2})$, we set $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{q}$, and $\widetilde{f} = \widetilde{f}(x),\ x \in \mathbf{R}_+,\ \widetilde{a} = \{\widetilde{a}_n\}_{n=1}^\infty$,

$$\begin{aligned} \widetilde{f}(x) &= \begin{cases} U^{\delta(\widetilde{\sigma}+\varepsilon)-1}(x)\mu(x), 0 < x^\delta \le 1 \\ 0, x^\delta > 0 \end{cases}, \\ \widetilde{a}_n &= (V_n - \beta)^{\widetilde{\sigma}-1}\nu_{n+1} = (V_n - \beta)^{\sigma-\frac{\varepsilon}{q}-1}\nu_{n+1},\ n \in \mathbf{N}. \end{aligned}$$

By (2.11), (3.12) and (2.7), we obtain

$$||\widetilde{f}||_{p,\Phi_\delta}||\widetilde{a}||_{q,\Psi_\beta} = \frac{1}{\varepsilon}U^{\frac{\delta\varepsilon}{p}}(1)\left[\frac{1}{(\nu_1-\beta)^\varepsilon} + \varepsilon O(1)\right]^{\frac{1}{q}},$$

$$\begin{aligned} \widetilde{I} &= \sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}}\widetilde{a}_n\widetilde{f}(x)dx \\ &= \int_{\{x>0;0<x^\delta\le 1\}} \omega_\delta(\widetilde{\sigma}, x)\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\ &\le k(\widetilde{\sigma})\int_{\{x>0;0<x^\delta\le 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\ &= \frac{1}{\varepsilon}k(\sigma - \frac{\varepsilon}{q})U^{\delta\varepsilon}(1). \end{aligned}$$

If there exists a positive constant $K \ge k(\sigma)$, such that (4.1) is valid when replacing $k(\sigma)$ to K, then in particular, we have

$$\varepsilon\widetilde{I} > \varepsilon K||\widetilde{f}||_{p,\Phi_\delta}||\widetilde{a}||_{q,\Psi\beta},$$

namely,

$$k(\sigma - \frac{\varepsilon}{q})U^{\delta\varepsilon}(1) > K \cdot U^{\frac{\delta\varepsilon}{p}}(1)\left[\frac{1}{(\nu_1-\beta)^\varepsilon} + \varepsilon O(1)\right]^{\frac{1}{q}}.$$

It follows that $k(\sigma) \ge K(\varepsilon \to 0^+)$. Hence, $K = k(\sigma)$ is the best possible constant factor of (4.1).

The constant factor $k(\sigma)$ in (4.2) ((4.3)) is still the best possible. Otherwise, we would reach a contradiction by (4.4) ((4.5)) that the constant factor in (4.1) is not the best possible. □

Theorem 4.2. *With the assumptions of Theorem* 4.1*, if*

$$0 < p < 1,\ 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty,$$

then we have the following equivalent inequalities with the best possible constant factor $k(\sigma)$:

$$I = \sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}a_n f(x)dx > k(\sigma)||f||_{p,\widetilde{\Phi}_\delta}||a||_{q,\Psi_\beta}, \tag{4.6}$$

$$J_1 = \sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}f(x)dx\right]^p > k(\sigma)||f||_{p,\widetilde{\Phi}_\delta}, \tag{4.7}$$

$$J := \left\{\int_0^{\infty}\frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}}$$
$$> k(\sigma)||a||_{q,\Psi_\beta}. \tag{4.8}$$

Proof . By the reverse weighted Hölder inequality (cf. [48]), since $0 < p < 1$, similarly to as we obtained (3.4) and (3.5), we have

$$\left[\int_0^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}f(x)dx\right]^p$$
$$\geq \frac{(\varpi_\delta(\sigma,n))^{p-1}}{(V_n-\beta)^{p\sigma-1}\nu_{n+1}}\int_0^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}\frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_{n+1}}{(V_n-\beta)^{1-\sigma}\mu^{p-1}(x)}f^p(x)dx.$$

In view of (2.9) and the Lebesgue term by term integration theorem, we find

$$J_1 \geq (k(\sigma))^{\frac{1}{q}}\left[\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}\frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_{n+1}}{(V_n-\beta)^{1-\sigma}\mu^{p-1}(x)}f^p(x)dx\right]^{\frac{1}{p}}$$
$$= (k(\sigma))^{\frac{1}{q}}\left[\int_0^{\infty}\omega_\delta(\sigma,x)\frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)}f^p(x)dx\right]^{\frac{1}{p}}.$$

Then by (2.10), we have (4.7).

By the reverse Hölder inequality (cf. [48]), we have

$$I = \sum_{n=1}^{\infty}\left[\frac{\nu_{n+1}^{\frac{1}{p}}}{(V_n-\beta)^{\frac{1}{p}-\sigma}}\int_0^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}f(x)dx\right]\left[\frac{(V_n-\beta)^{\frac{1}{p}-\sigma}a_n}{\nu_{n+1}^{\frac{1}{p}}}\right]$$
$$\geq J_1||a||_{q,\Psi_\beta}. \tag{4.9}$$

Then by (4.7), we have (4.6). On the other hand, assuming that (4.6) is valid, we set a_n as in Theorem 3.1. Then we find $J_1^p = ||a||_{q,\Psi_\beta}^q$. If $J_1 = \infty$, then (4.7) is trivially valid; if $J_1 = 0$, then (4.7) keeps impossible. Suppose that $0 < J_1 < \infty$. By (4.6), it follows that

$$||a||_{q,\Psi}^q = J_1^p = I > k(\sigma)||f||_{p,\widetilde{\Phi}_\delta}||a||_{q,\Psi_\beta},$$
$$||a||_{q,\Psi}^{q-1} = J_1 > k(\sigma)||f||_{p,\widetilde{\Phi}_\delta},$$

and then (4.7) follows, which is equivalent to (4.6).

Similarly, by the reverse weighted Hölder inequality (cf. [48]), since $q < 0$, we have

$$\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}a_n\right]^q$$
$$\leq \frac{(\omega_\delta(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)}\sum_{n=1}^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}\frac{(V_n-\beta)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_{n+1}^{q-1}}a_n^q.$$

Therefore, by (2.10) and the Lebesgue term by term integration theorem, it follows that

$$\begin{aligned} J &> (k(\sigma))^{\frac{1}{p}} \left[\int_0^\infty \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} \frac{(V_n-\beta)^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_{n+1}^{q-1}} a_n^q dx \right]^{\frac{1}{q}} \\ &= (k(\sigma))^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \varpi_\delta(\sigma,n) \frac{(V_n-\beta)^{q(1-\sigma)-1}}{\nu_{n+1}^{q-1}} a_n^q \right]^{\frac{1}{q}}. \end{aligned}$$

Hence, by (2.9), we have (4.8).

By the reverse Hölder inequality (cf. [48]), we have

$$I = \int_0^\infty \left[(1-\theta_\delta(\sigma,x))^{\frac{1}{p}} \frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x) \right]$$

$$\times \left[\frac{(1-\theta_\delta(\sigma,x))^{\frac{-1}{p}} \mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} a_n \right] dx \geq ||f||_{p,\widetilde{\Phi}_\delta} J. \tag{4.10}$$

Then by (4.8), we have (4.6). On the other hand, assuming that (4.6) is valid, we set $f(x)$ as in Theorem 3.1. Then we derive that $J^q = ||f||_{p,\widetilde{\Phi}_\delta}^p$. If $J=\infty$, then (4.8) is trivially valid; if $J=0$, then (4.8) is still not valid. Suppose that $0<J<\infty$. By (4.6), it follows that

$$\begin{aligned} ||f||_{p,\widetilde{\Phi}_\delta}^p &= J^q = I > k(\sigma)||f||_{p,\widetilde{\Phi}_\delta}||a||_{q,\Psi_\beta}, \\ ||f||_{p,\widetilde{\Phi}_\delta}^{p-1} &= J > k(\sigma)||a||_{q,\Psi_\beta}, \end{aligned}$$

and then (4.8) follows, which is equivalent to (4.6). Therefore, inequalities (4.6), (4.7) and (4.8) are equivalent.

For $\varepsilon \in (0, \frac{p(\sigma-\gamma)}{2})$, we set $\widetilde{\sigma} = \sigma + \frac{\varepsilon}{p}$, and $\widetilde{f} = \widetilde{f}(x), x \in \mathbf{R}_+, \widetilde{a} = \{\widetilde{a}_n\}_{n=1}^\infty$,

$$\begin{aligned} \widetilde{f}(x) &= \begin{cases} U^{\delta\widetilde{\sigma}-1}(x)\mu(x), 0<x^\delta\leq 1 \\ 0, x^\delta > 0 \end{cases}, \\ \widetilde{a}_n &= (V_n-\beta)^{\widetilde{\sigma}-\varepsilon-1}\nu_{n+1} = (V_n-\beta)^{\sigma-\frac{\varepsilon}{q}-1}\nu_{n+1}, n \in \mathbf{N}. \end{aligned}$$

By (2.10), (2.11) and (3.12), we obtain

$$\begin{aligned} &||\widetilde{f}||_{p,\widetilde{\Phi}_\delta}||\widetilde{a}||_{q,\Psi_\beta} \\ &= \left[\int_{\{x>0;0<x^\delta\leq 1\}} (1-O((U(x))^{\frac{\delta}{2}(\sigma-\gamma)})) \frac{\mu(x)dx}{U^{1-\delta\varepsilon}(x)} \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \frac{\nu_{n+1}}{(V_n-\beta)^{1+\varepsilon}} \right]^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} \left(U^{\delta\varepsilon}(1) - \varepsilon O_1(1) \right)^{\frac{1}{p}} \left[\frac{1}{(\nu_1-\beta)^\varepsilon} + \varepsilon O(1) \right]^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}\widetilde{a}_n\widetilde{f}(x)dx \\
&= \sum_{n=1}^{\infty}\left[\int_{\{x>0;0<x^{\delta}\le 1\}}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}\frac{(V_n-\beta)^{\widetilde{\sigma}}\mu(x)}{U^{1-\delta\widetilde{\sigma}}(x)}dx\right]\frac{\nu_{n+1}}{(V_n-\beta)^{1+\varepsilon}} \\
&\le \sum_{n=1}^{\infty}\left[\int_{0}^{\infty}\frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\delta\gamma}(x)(V_n-\beta)^{\gamma}}}\frac{(V_n-\beta)^{\widetilde{\sigma}}\mu(x)}{U^{1-\delta\widetilde{\sigma}}(x)}dx\right]\frac{\nu_{n+1}}{(V_n-\beta)^{1+\varepsilon}} \\
&= \sum_{n=1}^{\infty}\varpi_{\delta}(\widetilde{\sigma},n)\frac{\nu_{n+1}}{(V_n-\beta)^{1+\varepsilon}} = k(\widetilde{\sigma})\sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1+\varepsilon}} \\
&= \frac{1}{\varepsilon}k(\sigma+\frac{\varepsilon}{p})\left[\frac{1}{(\nu_1-\beta)^{\varepsilon}}+\varepsilon O(1)\right].
\end{aligned}$$

If there exists a positive constant $K \ge k(\sigma)$, such that (4.1) is valid when replacing $k(\sigma)$ by K, then in particular, we have

$$\varepsilon\widetilde{I} > \varepsilon K||\widetilde{f}||_{p,\widetilde{\Phi}_{\delta}}||\widetilde{a}||_{q,\Psi_{\beta}},$$

namely,

$$\begin{aligned}
& k(\sigma+\frac{\varepsilon}{p})\left[\frac{1}{(\nu_1-\beta)^{\varepsilon}}+\varepsilon O(1)\right] \\
> & K\left(U^{\delta\varepsilon}(1)-\varepsilon O_1(1)\right)^{\frac{1}{p}}\left[\frac{1}{(\nu_1-\beta)^{\varepsilon}}+\varepsilon O(1)\right]^{\frac{1}{q}}.
\end{aligned}$$

It follows that $k(\sigma) \ge K(\varepsilon \to 0^{+})$. Hence, $K = k(\sigma)$ is the best possible constant factor of (4.6).

The constant factor $k(\sigma)$ in (4.7) ((4.8)) is still the best possible. Otherwise, we would reach a contradiction by (4.9) ((4.10)) that the constant factor in (4.6) is not the best possible. □

5. Some Corollaries

For $\delta = 1$ in Theorem 3.2, Theorem 4.1 and Theorem 4.2, the following inequalities with the non-homogeneous kernel hold true:

Corollary 5.1. If $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \le 1, k(\sigma)$ is indicated by (2.2), and $U(\infty) = V(\infty) = \infty$, then
(i) for $p > 1$, $0 < ||f||_{p,\Phi_1}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{\csc h(\rho U^{\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\gamma}(x)(V_n-\beta)^{\gamma}}}a_n f(x)dx < k(\sigma)||f||_{p,\Phi_1}||a||_{q,\Psi_{\beta}}, \tag{5.1}$$

$$\sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_{0}^{\infty}\frac{\csc h(\rho U^{\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\gamma}(x)(V_n-\beta)^{\gamma}}}f(x)dx\right]^{p} < k(\sigma)||f||_{p,\Phi_1}, \tag{5.2}$$

$$\left\{\int_{0}^{\infty}\frac{\mu(x)}{U^{1-q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho U^{\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\gamma}(x)(V_n-\beta)^{\gamma}}}a_n\right]^{q}dx\right\}^{\frac{1}{q}} < k(\sigma)||a||_{q,\Psi_{\beta}}; \tag{5.3}$$

(ii) for $p<0$, $0<||f||_{p,\Phi_1}, ||a||_{q,\Psi}<\infty$, we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho U^{\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\gamma}(x)(V_n-\beta)^{\gamma}}}a_n f(x)dx > k(\sigma)||f||_{p,\Phi_1}||a||_{q,\Psi_\beta}, \tag{5.4}$$

$$\sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho U^{\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\gamma}(x)(V_n-\beta)^{\gamma}}}f(x)dx\right]^p > k(\sigma)||f||_{p,\Phi_1}, \tag{5.5}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho U^{\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\gamma}(x)(V_n-\beta)^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi_\beta}; \tag{5.6}$$

(iii) for $0<p<1$, $0<||f||_{p,\Phi_1}, ||a||_{q,\Psi}<\infty$, we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho U^{\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\gamma}(x)(V_n-\beta)^{\gamma}}}a_n f(x)dx > k(\sigma)||f||_{p,\widetilde{\Phi}_1}||a||_{q,\Psi_\beta}, \tag{5.7}$$

$$\sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho U^{\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\gamma}(x)(V_n-\beta)^{\gamma}}}f(x)dx\right]^p > k(\sigma)||f||_{p,\widetilde{\Phi}_1}, \tag{5.8}$$

$$\left\{\int_0^{\infty}\frac{(1-\theta_1(\sigma,x))^{1-q}\mu(x)}{U^{1-q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho U^{\gamma}(x)(V_n-\beta)^{\gamma})}{e^{\alpha U^{\gamma}(x)(V_n-\beta)^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}}$$
$$> \quad k(\sigma)||a||_{q,\Psi_\beta}. \tag{5.9}$$

The above inequalities involve the best possible constant factor $k(\sigma)$.

For $\delta=-1$ in Theorem 3.2, Theorem 4.1 and Theorem 4.2, we have the following inequalities with the homogeneous kernel of degree 0:

Corollary 5.2. If $\rho>\max\{0,-\alpha\}, 0<\gamma<\sigma\le 1, k(\sigma)$ is indicated by (2.2), and $U(\infty)=V(\infty)=\infty$, then
(i) for $p>1$, $0<||f||_{p,\Phi_{-1}}, ||a||_{q,\Psi_\beta}<\infty$, we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(\frac{V_n-\beta}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}}a_n f(x)dx < k(\sigma)||f||_{p,\Phi_{-1}}||a||_{q,\Psi_\beta}, \tag{5.10}$$

$$\sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho(\frac{V_n-\beta}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n-\beta}{U(x)})^{\gamma}}}f(x)dx\right]^p < k(\sigma)||f||_{p,\Phi_{-1}}, \tag{5.11}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1+q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho(\frac{V_n-\beta}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n-\beta}{U(x)})^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}} < k(\sigma)||a||_{q,\Psi_\beta}; \tag{5.12}$$

(ii) for $p<0$, $0<||f||_{p,\Phi_{-1}}, ||a||_{q,\Psi_\beta}<\infty$, we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(\frac{V_n-\beta}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n-\beta}{U(x)})^{\gamma}}}a_n f(x)dx > k(\sigma)||f||_{p,\Phi_{-1}}||a||_{q,\Psi_\beta}, \tag{5.13}$$

$$\sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho(\frac{V_n-\beta}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n-\beta}{U(x)})^{\gamma}}}f(x)dx\right]^p > k(\sigma)||f||_{p,\Phi_{-1}}, \tag{5.14}$$

$$\left\{\int_0^\infty \frac{\mu(x)}{U^{1+q\sigma}(x)}\left[\sum_{n=1}^\infty \frac{\csc h(\rho(\frac{V_n-\beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n-\beta}{U(x)})^\gamma}}a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi_\beta}; \quad (5.15)$$

(iii) for $0 < p < 1$, $0 < ||f||_{p,\Phi_{-1}}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho(\frac{V_n-\beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n-\beta}{U(x)})^\gamma}}a_n f(x)dx > k(\sigma)||f||_{p,\widetilde{\Phi}_{-1}}||a||_{q,\Psi_\beta}, \quad (5.16)$$

$$\sum_{n=1}^\infty \frac{\nu_n}{V_n^{1-p\sigma}}\left[\int_0^\infty \frac{\csc h(\rho(\frac{V_n-\beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n}{U(x)})^\gamma}}f(x)dx\right]^p > k(\sigma)||f||_{p,\widetilde{\Phi}_{-1}}, \quad (5.17)$$

$$\left\{\int_0^\infty \frac{(1-\theta_{-1}(\sigma,x))^{1-q}\mu(x)}{U^{1+q\sigma}(x)}\left[\sum_{n=1}^\infty \frac{\csc h(\rho(\frac{V_n-\beta}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n-\beta}{U(x)})^\gamma}}a_n\right]^q dx\right\}^{\frac{1}{q}}$$
$$> k(\sigma)||a||_{q,\Psi_\beta}. \quad (5.18)$$

The above inequalities involve the best possible constant factor $k(\sigma)$.

For $\alpha = \rho$ in Theorem 3.2, Theorem 4.1 and Theorem 4.2, we have

Corollary 5.3. If $\rho > 0, 0 < \gamma < \sigma \leq 1$, and $U(\infty) = V(\infty) = \infty$, then
(i) for $p > 1$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities with the best possible constant factor

$$k_1(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} :$$

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma}}a_n f(x)dx < k_1(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \quad (5.19)$$

$$\sum_{n=1}^\infty \frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma}}f(x)dx\right]^p < k_1(\sigma)||f||_{p,\Phi_\delta}, \quad (5.20)$$

$$\left\{\int_0^\infty \frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma}}a_n\right]^q dx\right\}^{\frac{1}{q}} < k_1(\sigma)||a||_{q,\Psi_\beta}; \quad (5.21)$$

(ii) for $p < 0, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities with the best possible constant factor $k_1(\sigma)$:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma}}a_n f(x)dx > k_1(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \quad (5.22)$$

$$\sum_{n=1}^\infty \frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_0^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma}}f(x)dx\right]^p > k_1(\sigma)||f||_{p,\Phi_\delta}, \quad (5.23)$$

$$\left\{\int_0^\infty \frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma}}a_n\right]^q dx\right\}^{\frac{1}{q}} > k_1(\sigma)||a||_{q,\Psi_\beta}; \quad (5.24)$$

(iii) for $0 < p < 1, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities with the best possible constant factor $k_1(\sigma)$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} a_n f(x)dx > k_1(\sigma)||f||_{p,\widetilde{\Phi}_\delta}||a||_{q,\Psi_\beta}, \tag{5.25}$$

$$\sum_{n=1}^{\infty} \frac{\nu_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} f(x)dx\right]^p > k_1(\sigma)||f||_{p,\widetilde{\Phi}_\delta}, \tag{5.26}$$

$$\left\{\int_0^{\infty} \frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{\rho U^{\delta\gamma}(x)(V_n-\beta)^\gamma}} a_n\right]^q dx\right\}^{\frac{1}{q}}$$
$$> \quad k_1(\sigma)||a||_{q,\Psi_\beta}. \tag{5.27}$$

For $\gamma = \frac{\sigma}{2}$ in Corollary 5.3, we obtain the following:

Corollary 5.4. If $\rho > 0, 0 < \sigma \leq 1$, and $U(\infty) = V(\infty) = \infty$, then
(i) for $p > 1$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2}}} a_n f(x)dx < \frac{\pi^2}{6\sigma\rho^2}||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \tag{5.28}$$

$$\sum_{n=1}^{\infty} \frac{\nu_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2}}} f(x)dx\right]^p < \frac{\pi^2}{6\sigma\rho^2}||f||_{p,\Phi_\delta}, \tag{5.29}$$

$$\left\{\int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2}}} a_n\right]^q dx\right\}^{\frac{1}{q}} < \frac{\pi^2}{6\sigma\rho^2}||a||_{q,\Psi_\beta}; \tag{5.30}$$

(ii) for $p < 0, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2}}} a_n f(x)dx > \frac{\pi^2}{6\sigma\rho^2}||f||_{p,\Phi_\delta}||a||_{q,\Psi_\beta}, \tag{5.31}$$

$$\sum_{n=1}^{\infty} \frac{\nu_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2}}} f(x)dx\right]^p > \frac{\pi^2}{6\sigma\rho^2}||f||_{p,\Phi_\delta}, \tag{5.32}$$

$$\left\{\int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2}}} a_n\right]^q dx\right\}^{\frac{1}{q}} > \frac{\pi^2}{6\sigma\rho^2}||a||_{q,\Psi_\beta}; \tag{5.33}$$

(iii) for $0 < p < 1, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n - \beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2}}} a_n f(x)dx > \frac{\pi^2}{6\sigma\rho^2}||f||_{p,\widetilde{\Phi}_\delta}||a||_{q,\Psi_\beta}, \tag{5.34}$$

$$\sum_{n=1}^{\infty}\frac{\nu_{n+1}}{(V_n-\beta)^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2}}}f(x)dx\right]^p > \frac{\pi^2}{6\sigma\rho^2}||f||_{p,\widetilde{\Phi}_\delta}, \tag{5.35}$$

$$\left\{\int_0^{\infty}\frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2})}{e^{\rho U^{\delta\sigma/2}(x)(V_n-\beta)^{\sigma/2}}}a_n\right]^q dx\right\}^{\frac{1}{q}}$$
$$> \frac{\pi^2}{6\sigma\rho^2}||a||_{q,\Psi_\beta}. \tag{5.36}$$

Remark 5.5. (i) For $\beta=0$ in (3.1), the following inequality holds true:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}}a_nf(x)dx < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi_0}. \tag{5.37}$$

Hence, (3.1) is a more accurate inequality of (5.37) for $0<\beta\le\frac{\nu_1}{2}$.

(ii) For $\mu(x)=\nu_n=1$ in (5.37), we have the following inequality with the best possible constant factor $k(\sigma)$:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(x^\delta n)^\gamma)}{e^{\alpha(x^\delta n)^\gamma}}a_nf(x)dx$$
$$< k(\sigma)\left[\int_0^{\infty}x^{p(1-\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{5.38}$$

In particular, for $\delta=1$, we have the following inequality with the non-homogeneous kernel:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(xn)^\gamma)}{e^{\alpha(xn)^\gamma}}a_nf(x)dx$$
$$< k(\sigma)\left[\int_0^{\infty}x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}; \tag{5.39}$$

for $\delta=-1$, we have the following inequality with the homogeneous kernel of degree 0:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(\frac{n}{x})^\gamma)}{e^{\alpha(\frac{n}{x})^\gamma}}a_nf(x)dx$$
$$< k(\sigma)\left[\int_0^{\infty}x^{p(1+\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{5.40}$$

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