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# Analytical Solutions for Spatially Variable Transport-Dispersion of Non-Conservative Pollutants

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**ABSTRACT**

Analytical solutions have been obtained for both conservative and non-conservative forms of one-dimensional transport and transport-dispersion equations applicable for pollution as a result of a non-conservative pollutant-disposal in an open channel with linear spatially varying transport velocity and nonlinear spatially varying dispersion coefficient on account of a steady unpolluted lateral inflow in accordance to the channel. A logarithmic transformation in the space variable has been applied in order to derive a general solution of the transport equation for spatially variable initial pollutant distribution and upstream time-dependent pollutant concentration. The logarithmic transformation reduces both conservative and non-conservative forms of transport-dispersion equation to a form with constant coefficients that is solvable by analytical methods. An analysis of these solutions indicates that only the solution of a conservative form of the governing equation yields appropriate results that are conceptually acceptable in a real physical situation.

The solution lends to analyze the damping effect of such transport on the pollutant with an initial Gaussian profile, in contrast with that of the initial quasi-Gaussian profile available in the literature. It is noteworthy to mention that the solution of conservative form of the transport equation implies that mass of the non-conservative pollutant in the channel decreases with an increase in time, and finally reaches to a constant value that is a ratio of product of the transport velocity coefficient and upstream concentration to the coefficient of decay of the pollutant.

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## 1. Introduction

Rapid industrialization and urbanization have started to grow to number and levels of chemical pollutants, which find their way in the water bodies and move accompanying the water current downstream of their discharge point. The spread of pollutants is governed by the transport and dispersion processes for conservative substances and solutions of the same have been employed by many authors (van Genucheten and Alves [1], Chrysikopoulos and Sim [2], Singh et al. [3-4], Kumar et al. [5]) in connection with ground water flow pollution. Moreover, first-order decay of the pollutant is contemplated for non-conservative pollutants in river streamflow pollution as well (van Genuchten et al. [6]). The decay of organic matter induced from the urban and domestic wastes is one such example which is interpreted by the first-order decay of Biological Oxygen Demand (Streeter and Phelps [7]). Proper prediction of concentration of the pollutants in accordance with the channel at different times can be made with the help of accurate computation of the solution of the basic transport-dispersion equation. Such a prediction is essential for water quality management in open channels from the viewpoint of locating the waste outfalls for disposal of industrial effluents so that they can produce the least detrimental effect on the downstream reach of the channel.

Analytical solutions to the above-mentioned problem are generally available for constant transport velocity and dispersion coefficient with constant upstream and

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downstream boundary conditions. Some efforts have been made in order to acquire analytical solutions for time-dependent and variable diffusion coefficients (Basha and El-Habel [8]; Philip [9]; Zoppou and Knight [10-11]; Pérez and Skaggs [12]). Such solutions have been introduced by Shukla [13] for transport-dispersion of non-conservative pollutants in rivers under time-dependent periodic waste disposal at the upstream end. The solutions of transport and transport-dispersion problems of conservative pollutants for spatially variable dispersion coefficient have been presented by Zoppou and Knight [11] and Pérez and Skaggs [12]. The former derived the solution when the flow velocity is a linear function of distance and the dispersion coefficient is proportional to the square of the velocity as applicable for a channel flow being augmented by steady unpolluted lateral inflow of groundwater and the solution may be applied for flows in circular tubes and between parallel plates due to an existence of similar relationship between flow velocity and dispersion coefficient as well. Pérez and Skaggs [12] developed a solution in a finite spatial domain for linearly increasing dispersion coefficient in groundwater transport-dispersion.

Nevertheless, these solutions are not beneficial for the computation of variation of water quality constituents as BOD (Biochemical Oxygen Demand) concentration as a result of the release of an organic pollutant in open channels or any other non-conservative pollutant because of the first-order decay term being ignored in the equation. Consequently, in this paper, solutions have been obtained for transport-dispersion of non-conservative pollutants for linear transport velocity in distance and dispersion coefficient proportional to the square of the distance. Both conservative and non-conservative forms of transport-dispersion problem applicable for unsteady variation of BOD in open channels have been solved analytically and compared with the results of Zoppou and Knight [11] when BOD decay rate is negligible.

## 2. TRANSPORT EQUATION

Although the transport equation studied here is applicable for the non-conservative pollutant, both conservative and non-conservative forms of the transport equation would be deliberated in order to interpret the role of convection. Moreover, since the upstream boundary condition and initial condition would greatly affect the movement of pollutant front as a result of spatially variable convective velocity, the investigation of all these would be in the line of the present study.

### 2.1 Conservative form

Firstly, we contemplate the conservative form of the transport equation for non-conservative pollutant which is transported with spatially variable transport velocity u(x) as

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x}[Cu(x)] = -kC, 0 < x < \infty \qquad (1)$$

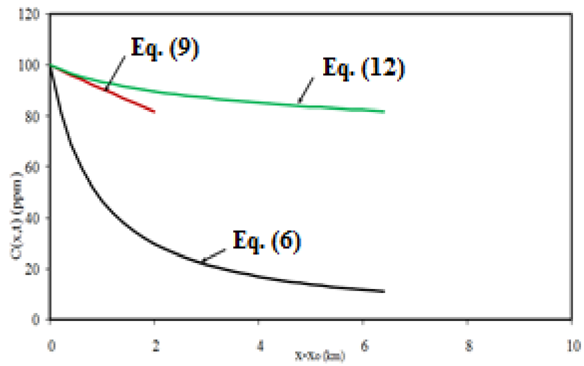


**Figure 1.** Comparison of Analytical Solutions of Transport Equation having Constant Velocity, Eq. (9) with the Non-conservative, Eq. (12) and the Conservative, Eq. (6) both having Spatially Variable Velocity.

where $C(x,t)$ is the cross-sectional averaged concentration of the pollutant at time $t$ and distance $x$ from the upstream boundary; k= first-order decay rate of the pollutant; $u(x)$ is cross-sectional averaged flow velocity, here taken to be a linear function of distance as

$u(x) = u_0\, x$, where $u_0$ is a constant. (2)

The initial and boundary conditions for (1) are prescribed as below:

$$C(x,0) = f(x), x > x_0 \qquad (3)$$

$$C(x_0,t) = g(t), t > 0 \qquad (4)$$

where $f(x)$ is the initial distribution of the pollutant in the channel and $g(t)$ is time-dependent pollution variation at upstream of the channel.

Since equation (1) differs from that of Zoppou and Knight [11] in that it includes also the decay term $-kC$ on the right-hand side, a substitution of $x = e^y$ in (1) can transform it into a transport equation with constant coefficients, which can be solved for transformed initial and upstream conditions (Appendix A). Moreover, substituting $y = ln\, x$ in such a solution, it yields the solution of conservative form (1) under conditions (3) and (4) as

$$C(x,t) = \begin{cases} e^{-(k+u_0)t} f(xe^{-u_0 t}), ln\left(\frac{x}{x_0}\right) > u_0 t \\ \left(\frac{x_0}{x}\right)^{1+\frac{k}{u_0}} g\left(t - \frac{\ln\left(\frac{x}{x_0}\right)}{u_0}\right), ln\left(\frac{x}{x_0}\right) \le u_0 t \end{cases} \qquad (5)$$

In order to deduce some results as the effects of the variable velocity field on step profile-concentration introduced upstream of the channel and initial Gaussian distribution, in the following are discussed briefly two solutions derived as particular cases from (5).

#### *Upstream boundary condition: step profile*

Taking $f(x) = 0, g(t) = C_0$ the solution (5) for non-conservative pollutant is written as

$$C(x,t) = C_0\left(\frac{x_0}{x}\right)^{1+\frac{k}{u_0}} H[u_0 t - ln(x/x_0)] \qquad (6)$$

where Heaviside unit step function is such that $H\left[u_0 t - ln\left(\frac{x}{xo}\right)\right] = 1$ for $x \le x_0\, e^{u_0 t}$, and 0 otherwise.

By setting $x_0 = 1$ and $k = 0$, (6) can be reduced to the solution introduced by Zoppou and Knight [11] under similar conditions for the conservative pollutant.

Figure 1 reveals the concentration profile using (6) for $u_0 = 1, x_0 = 1, C_0 = 100$ and $t = 2$, which decreases along the channel. The concentration front is found to reach a maximum distance of $x = x_0\, e^{u_0 t}$. A rapid decrease in the concentration is attributed to unpolluted lateral inflow to the channel as well as the first-order decay of the non-conservative pollutant.

Taking $x_0 = 1$ where the pollutant enters the channel, the total mass of pollutant say $M_1$ in the channel at time $t$ is given by:

$$M_1 = \int_1^{\infty} C(x,t)dx = \int_1^{e^{u_0 t}} C_0 x^{-1-\frac{k}{u_0}} dx = -\frac{C_0 u_0}{k}\left[x^{-\frac{k}{u_0}}\right]_{x=1}^{e^{u_0 t}} \qquad (7)$$

$$M_1 = \frac{C_0 u_0}{k}(1 - e^{-kt})$$

For a conservative substance, $k$ is negligibly small and therefore, total mass, in this case, becomes as follows:

$$M_1 = C_0 u_0 \lim_{k\to 0} \frac{1 - e^{-kt}}{k} = C_0 u_0 t$$

which is also the mass of pollutants entering the channel at any time t. This indicates that the conservative form of transport equation for conservative pollutant conserves the mass of the pollutant, a result also deduced by Zoppou and Knight [11] for the conservative pollutant. It is observed from (7) that the mass of non-conservative pollutants is not conserved even in the conservative form of the transport-equation as is also expected physically for a non-conservative pollutant. The mass, in this case, keeps on decreasing as time advances and reduces finally to $\frac{C_0 u_0}{k}$ as time becomes infinitely large.

Moreover, the transport equation with constant transport velocity for a non-conservative pollutant is given by:

$$\frac{\partial C}{\partial t} + u_0 \frac{\partial C}{\partial x} = -kC, 0 < x < \infty\, , t > 0$$

which has the general solution as follows:

$$C(x,t) = \begin{cases} e^{-kt} f(x - u_0 t), x - x_0 > u_0 t \\ e^{\frac{-k(x-x_0)}{u_0}} g\left(t - \frac{x - x_0}{u_0}\right), x - x_0 \le u_0 t \end{cases} \qquad (8)$$

For constant step profile condition $C(x_0, t) = C_0$ and initial condition $C(x, 0)$=0, the solution (8) reduces to

$$C(x,t) = C_0\, e^{\frac{-k(x-x_0)}{u_0}} H(u_0 t - x + x_0) \qquad (9)$$

where $H(u_0 t - x + x_0) = 1$ for $x < x_0 + u_0 t$, and 0 otherwise.

The solution profile (9), as plotted in Fig. 1 reveals that the step profile introduced at upstream of the channel decreases exponentially as it advances with transport velocity $u_0$, and travels a distance $x = x_0 + u_0 t$ in time $t$. The decrease in the concentration with an increase in the distance is caused by the first-order decay of the pollutant. As it is reported from Fig. 1 that the concentration obtained from the solution of transport equation in conservative form decreases much faster with its front also traveling faster than that of the concentration predicted by transport equation with constant velocity.

## *2.2 Non-conservative form*

The non-conservative form of the transport equation for non-conservative pollutant with transport velocity increases linearly with distance is given by:

$$\frac{\partial C}{\partial t} + u_0 x \frac{\partial C}{\partial x} = -kC, 0 < x < \infty \qquad (10)$$

Proceeding on the same lines as for the conservative form of the transport equation, the solution of non-conservative form (10) under conditions (3) and (4) is derived as follows:

$$C(x,t) = \begin{cases} e^{-kt} f(x e^{-u_0 t}), ln\left(\frac{x}{x_0}\right) > u_0 t \\ \left(\frac{x_0}{x}\right)^{\frac{k}{u_0}} g\left(t - \frac{\ln\left(\frac{x}{x_0}\right)}{u_0}\right), ln\left(\frac{x}{x_0}\right) \le u_0 t \end{cases} \qquad (11)$$

***Upstream boundary condition: step profile***

For the step profile condition $C_0$ at upstream boundary and initially unpolluted situation, Eq. (11) gives the solution as follows:

$$C(x,t) = C\left(\frac{x_0}{x}\right)^{\frac{k}{u_0}} H\left[u_0 t - ln\left(\frac{x}{x_0}\right)\right] \qquad (12)$$

from which one can also get the solution of Zoppou and Knight (1997) by setting $k = 0$.

The solution profile for (12) is manifested in Fig. 1. It decreases with an increase in distance, and its front is located at $x = x_0\, e^{u_0 t}$. The decrease being only as a result of the first-order decay is not as pronounced as that of the conservative form.

The mass of pollutants say $M_2$ in the channel for this case at time $t$ is obtained as follows:

$$M_2 = \int_1^{\infty} C(x,t)dx = \int_1^{e^{u_0 t}} C_0 x^{-k/u_0} dx = \frac{C_0}{1 - \frac{k}{u_0}} x^{1-\frac{k}{u_0}}\Big|_1^{e^{u_0 t}} \qquad (13)$$

$$= \frac{C_0 u_0}{k - u_0}\left[1 - e^{-(k-u_0)t}\right] \ne C_0 u_0 t, \text{ for } k \ne u_0.$$

which displays that mass is not conserved. However, if $k = u_0$, it can be verified that $M_2 = C_0 u_0 t$, i.e., mass is conserved. In case $u_0 > k$, it is seen from (13) that mass $M_2$ keeps on increasing exponentially with time. Since $k = 0$ for conservative substance and $u_0 > 0$, we note that this case is not physically feasible for conservative pollutants. Furthermore, the value of $k$ is generally small for most of the decaying pollutants; consequently it is most unlikely that $u_0 < k$. This indicates that even for a non-conservative pollutant, the non-conservative form of transport equation cannot give physically plausible solutions.

Having analyzed the effect of the upstream step profile on the propagation of pollution front, in the following, we inspected the effect of convective velocity on the initial Gaussian profile for both conservative and non-conservative form of the transport equation.

## *2.3 Initial condition: Gaussian Profile*

In the absence of any pollutant concentration at upstream, now we analyze the effect of transport-dispersion on the initial condition taken as a Gaussian profile.

$$C(x,0) = f(x) = \frac{M_0}{\sigma\sqrt{2\pi}} e^{\left[-\frac{(x-x_0)^2}{2\sigma^2}\right]} \tag{14}$$

in the $x - t$ plane with a peak concentration located at $x = x_0$. Then, the solution given by (5) takes the form:

$$C(x,t) = \frac{M_0}{\sigma\sqrt{2\pi}} e^{\left[-\frac{(k+u_0)t+(xe^{-u_0t}-x_0)^2}{2\sigma^2}\right]} \tag{15}$$

This equation indicates that peak concentration $C(x_p,t)$ occurs at $x_p = x_0\, e^{u_0t}$ and is given by

$$C(x,t) = \frac{M_0}{\sigma\sqrt{2\pi}} e^{-(k+u_0)t} \tag{16}$$

This also indicates that the peak concentration decays exponentially with an increase in time even for conservative pollutants ($k = o$).

Total mass under profile (15) is given by

$$M = \int_0^{\infty} C(x,t)\, dx = \frac{M_0 e^{-(k+u_0)t}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left[-\frac{(xe^{-u_0t}-x_0)^2}{2\sigma^2}\right]} dx = M_0 e^{-kt}$$

Since for $k = 0$, this becomes a constant $M_0$, it persuades that the mass remains conserved for the conservative pollutant. The same is not true for the non-conservative pollutant ($k \neq 0$) as the mass keeps on decreasing as time advances.

The first moment of the profile is given by

$$M_3 = \int_0^{\infty} xC(x,t)\, dx$$

$$= \frac{M_0 e^{-(k+u_0)t}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{\left[-\frac{(xe^{-u_0t}-x_0)^2}{2\sigma^2}\right]} dx$$

$$= M_0 x_0 e^{-(u_0-k)t}.$$

Thus, the centroid of the concentration profile is given by

$$x = \frac{\int_0^{\infty} xC(x,t)\, dx}{\int_0^{\infty} C(x,t)\, dx} = x_0 e^{u_0t}$$

The above expression indicates that the centroid moves faster than the peak concentration (given by Eq. (16)) of the profile for both conservative and non-conservative pollutants. The centroid has the same location as the peak concentration for a Gaussian profile, as a similar result derived by Zoppou and Knight (1997) for a quasi-Gaussian profile in case of the conservative pollutant.

The concentration profiles at $t = 2, 4, 6, 8$, and $10$ as computed by applying Eq. (15) with $u_0 = 0.2, x_0 = 1, C_0 = 20$ and $\sigma = 0.2$ for an initial Gaussian distribution are compared in Fig.2 which indicates that the profiles and corresponding mass decay exponentially as time increases, as mentioned in (15) and the expression for mass M given above as well. Here, the decay of the profiles is more pronounced as a result of both transport velocity and the first-order decay playing their roles simultaneously than the decay of pollutant-mass that is attributed only to the first-order decay of the pollutant.

Moreover, for the Gaussian type initial condition (14), the solution (11) of the non-conservative form of transport equation (10) is written as follows:

$$C(x,t) = \frac{M_0}{\sigma\sqrt{2\pi}} e^{\left[-\frac{kt+(xe^{-u_0t}-x_0)^2}{2\sigma^2}\right]} \tag{17}$$

The concentration profile for Eq. (17) has a peak concentration say $C'(x_p,t)$ at $x_p = x_0 e^{u_0t}$ given by:

$$C'(x,t) = \frac{M_0}{\sigma\sqrt{2\pi}} e^{-kt} \tag{18}$$

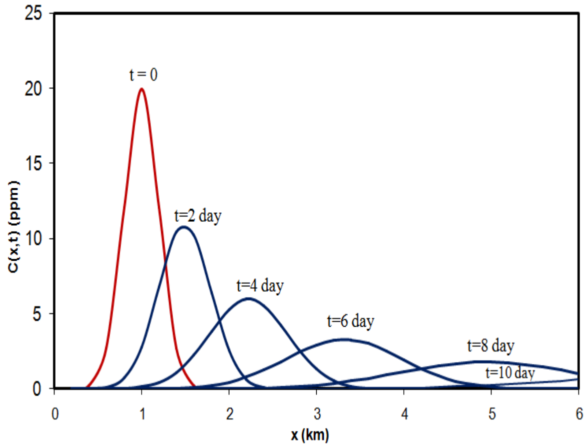


**Figure 2.** Concentration Profiles for Transport of a Gaussian Profile in a spatially variable flow field using Conservative form of Transport Eq. (1).

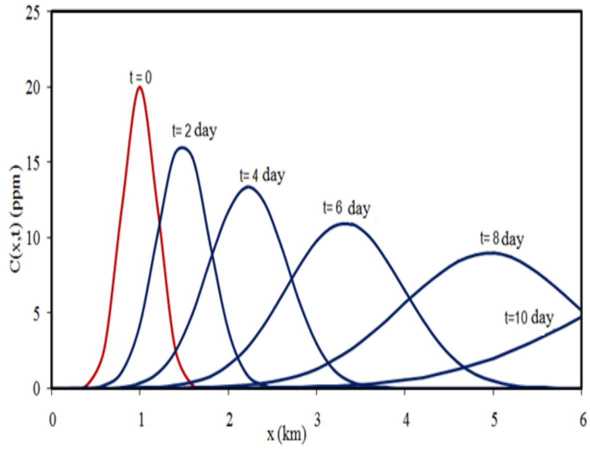


**Figure 3**. Concentration Profiles obtained by Eq. (17) for Transport of an Initial Gaussian Profile in a spatially variable flow field using non-conservative form of Transport Equation (10).

This peak concentration is attenuated (exponentially due to $e^{-kt}$) with the advancement in time for non-conservative substances, but it is not affected by conservative substances (k=0).

Total mass under the profile (17) obtained by integrating the whole profile with respect to x is

$$\int_0^{\infty} C(x,t)\, dx = \frac{M_0 e^{-kt}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left[-\frac{(xe^{-u_0t}-x_0)^2}{2\sigma^2}\right]} dx = M_0 e^{(u_0-kt)}.$$

The expression indicates that although the mass is conserved initially, generally, it is not conserved even for conservative pollutants. Furthermore, for most of the non-conservative substances we have $u_0 > k$, due to which total mass will keep on rising exponentially with time, thus demonstrating that non-conservative form does not yield physically feasible results for non-conservative pollutants as well.

The solution profiles obtained by applying Eq. (17) with $u_0 = 0.2$, $x_0 = 1$, $C_0 = 20$ and $\sigma = 0.2$ at $t = 0, 2, 4, 6, 8$ and $10$ are plotted as Fig. 3. The profiles are observed decreasing, and the mass is increasing exponentially with an increase in time, as is evident from (17) and the discussion above about mass as well.

## 3. TRANSPORT-DISPERSION QUATION

The unsteady state transport-dispersion equation for non-conservative pollutants such as BOD uniformly distributed across the cross-section and varying along the longitudinal direction $x > o$ in open channel is given by:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D\frac{\partial^2}{\partial x^2} - kC, 0 < x < \infty \qquad (19)$$

where $u$ = cross-sectional averaged velocity; $D$ = dispersion constant and $C(x,t)$ = cross-sectional averaged concentration of a pollutant at time $t$ and distance $x$ from the upstream boundary.

In case, there exists a lateral entry of steady unpolluted flow distributed in accordance with the entire length of the channel (e.g., a steady inflow of an aquifer), the conservation of the mass of the non-conservative pollutant can be expressed in both conservative and non-conservative forms of transport-dispersion equation as given below.

### *3.1 Conservative form of transport-dispersion equation*

The conservative form of transport-dispersion equation is given by

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x}[Cu(x)] = \frac{\partial}{\partial x}\left[D(x)\frac{\partial}{\partial x}C\right] - k\,C, x_0 \leq x \leq \infty \qquad (20)$$

where $u(x)$ and $D(x)$ for lateral inflow into the prismatic channel are given by

$$u(x) = u_0 x,\; D(x) = D_0\, x^2 \qquad (21)$$

both $u_0$ and $D_0$ being the positive constants having dimension $T^{-1}$.

Deliberating that a slug of pollutant is released at the upstream end such that a constant length (say $x_0$) of the channel from the upstream end has always a constant concentration say Co in an initially unpolluted river, the following initial and boundary conditions are prescribed for Eq.(20).

$$C(x,0) = 0 \text{ for } x > x_0 \qquad (22)$$

$$C(x_0,t) = C_0 \text{ for } t > 0 \qquad (23)$$

$$C(x,t) = 0 \text{ as } x \to \infty \text{ for } t \geq 0 \qquad (24)$$

A substitution $x = e^y$ into (20) with $u(x)$ and $D(x)$ as given by (21) transforms it into a transport-dispersion equation with constant coefficients (Zoppou and Knight, [11]) that can be solved by Fourier transform method (Sneddon [15]), and a back substitution $y = ln\,x$ in it (Appendix B) gives the solution of equation (20) under conditions (22)-(24) as follows:

$$C(x,t) = \frac{C_0}{2}\left\{\left(\frac{x_0}{x}\right)^{\mu_1} erfc\left[\frac{ln\frac{x}{x_0} - t\sqrt{(u_0+D_0)^2+4D_0k}}{\sqrt{4D_0t}}\right] + \left(\frac{x}{x_0}\right)^{\mu_2} erfc\left[\frac{ln\frac{x}{x_0} + t\sqrt{(u_0+D_0)^2+4D_0k}}{\sqrt{4D_0t}}\right]\right\} \qquad (25)$$

$$\text{where } \mu_1, \mu_2 = -\frac{-\sqrt{(u_0+D_0)^2+4D_0k} \pm (u_0-D_0)}{2D_0} \qquad (26)$$

It is observed that for $k = 0$ that $\mu_1 = 1,\ \mu_2 = \frac{u_0}{D_0}$ and that the solution (25) reduces to the solution for transport-dispersion of conservative contaminants.

The concentration profile obtained by applying (25) with $u_0 = 1,\ D_0 = 0.02,\ x_0 = 1,\ C_0 = 100$ and $t = 2$ is plotted in Fig. 4 that exhibits a continuous decrease of concentration in conformity with the direction of the flow in the channel. The decrease is more pronounced here than that of the corresponding case of no dispersion (compare with equation 6 in Fig.1). This implies that the variable dispersion field plays a significant role in the fast spread and reduction of the pollutant level.

The analytical solution for corresponding transport-dispersion equation (19) for non-conservative pollutant with constant transport velocity $u = u_0$ and dispersion coefficient $D = D_0$ is given by:

$$C(x,t) = \frac{C_0}{2}\left\{e^{\lambda_1(-x+x_0)} erfc\left[\frac{x-x_0-t\sqrt{u_0^2+4D_0k}}{\sqrt{4D_0t}}\right] + e^{\lambda_2(x-x_0)} erfc\left[\frac{x-x_0+t\sqrt{u_0^2+4D_0k}}{\sqrt{4D_0t}}\right]\right\} \qquad (27)$$

$$\text{where } \lambda_1, \lambda_2 = -\frac{-\sqrt{u_0^2+4D_0k} \pm u_0}{2D_0} \qquad (28)$$

For $k = 0$ as applicable to conservative pollutants, Eq. (27) yields the same solution as achieved by Zoppou and Knight [11] and de Marsily [4], The concentration profile using Eq. (27) with $u_0 = 1, x_0 = 1,\ C_0 = 100$ and $t = 2$ is also plotted in Fig 4, which indicates a rapid decrease of the concentration and farther spread of the pollutant accompanying channel than that of the case when no dispersion is contemplated (see the profile for equation 9 in Fig. 1).

### *3.2 Non-conservative form of transport-dispersion equation*

The non-conservative form of transport- dispersion equation for non-conservative pollutant can be written as follows:

$$\frac{\partial C}{\partial t} + u_0 x\frac{\partial C}{\partial x} = D_0 x^2\frac{\partial^2 C}{\partial x^2} - k\,C, < x_0 \leq x \leq \infty, t > 0 \qquad (29)$$

Substituting $x = e^y$ in (29), we get

$$\frac{\partial C}{\partial t} + (u_0 + D_0)\frac{\partial C}{\partial y} = D_0\frac{\partial^2 C}{\partial y^2} - k\,C, 0 < x_0 \leq x \leq \infty, t > 0 \qquad (30)$$

This equation is in contrast with that of Zoppou and Knight ([11]) who computed the second term as $u_0\frac{\partial C}{\partial y}$ only probably by missing the term $D_0\frac{\partial C}{\partial y}$ while substituting $y = ln\,x$ in $D_0 x^2\frac{\partial^2 C}{\partial x^2}$ to get $D_0\frac{\partial^2 C}{\partial y^2} - D_0\frac{\partial C}{\partial y}$. Thus, proceeding in the same way as previously, the solution of (30) under transformed initial and boundary conditions corresponding to (23) and (24) is derived as follows:

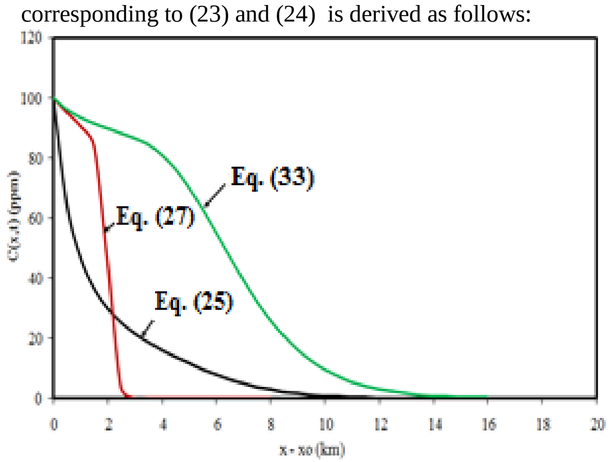


**Figure 4.** Comparison of Analytical Solutions of Transport-dispersion Equation having Constant Velocity and Dispersion Coefficient, Eq. (27) with the Non-conservative, Eq. (33); and the Conservative, Eq. (25) both having Spatially Variable Velocity and Dispersion Coefficient.

$$C(x,t) = \frac{C_0}{2}\left\{e^{v_1(-y+y_0)} erfc\left[\frac{y-y_0-t\sqrt{(u_0+D_0)^2+4D_0k}}{\sqrt{4D_0t}}\right] + e^{v_2(y-y_0)} erfc\left[\frac{y-y_0+t\sqrt{(u_0+D_0)^2+4D_0k}}{\sqrt{4D_0t}}\right]\right\} \quad (31)$$

$$where\ v_1, v_2 = -\frac{-\sqrt{(u_0+D_0)^2+4D_0k} \pm (u_0+D_0)}{2D_0} \quad (32)$$

On substituting $y = ln\, x$, we get the analytical solution of (29) under conditions (22)-(24) as follows:

$$C(x,t) = \frac{C_0}{2}\left\{\left(\frac{x_0}{x}\right)^{v_1} erfc\left[\frac{ln\frac{x}{x_0} - t\sqrt{(u_0+D_0)^2+4D_0k}}{\sqrt{4D_0t}}\right] + \left(\frac{x}{x_0}\right)^{v_2} erfc\left[\frac{ln\frac{x}{x_0}+t\sqrt{(u_0+D_0)^2+4D_0k}}{\sqrt{4D_0t}}\right]\right\} \quad (33)$$

For $k = 0$, we get $v_1$ and $v_2 = \frac{u_0+D_0}{D_0}$ for which Eq. (33) reduces to the solution for conservative pollutants (as given by Eq. (21) in Zoppou and Knight [11] with the correction that $u_0$ in their solution is to be replaced by $u_0$+$D_0$).

The concentration profile using Eq. (33) is exhibited in Fig. 4, which displays a faster spread of pollutants than observed from the profiles for conservative form and the one having constant coefficients. However, the mass of pollutants in the channel is not conserved, as well as it noticed from (13) for the case of no dispersion.

## 4. CONCLUSIONS

Analytical solutions have been given for the problem of spatially variable transport and transport-dispersion of non-conservative pollutants, which include and generalize the solutions derived by Zoppou and Knight ([11]) for the conservative pollutant. The solution for the transport equation accepts spatially variable initial and time-dependent boundary conditions.

By observing the solution of conservative form of the transport equation for non-conservative pollutant entering the channel with constant upstream concentration, it is noted that mass of the pollutant in the channel decreases with an increase in time; and finally reaches to a constant value that is a ratio of product of the transport velocity coefficient and upstream concentration to the coefficient of decay of the pollutant. For the case of an initial Gaussian distribution of the pollutant and no upstream discharge of the pollutant, it is observed that the mass decreases as time advances and tends to zero as time tends to become infinitely large. This concurs with the similar result introduced by Zoppou and Knight ([11]) from the solution of a conservative form of transport-equation for quasi-Gaussian distribution of a conservative pollutant. The results for non-conservative form of transport equation manifest that although the pollutant intrusion length is the same as in the case of conservative form, mostly the mass of the pollutant would keep on rising with time; accordingly implying that the non-conservative form of the basic equation does not govern a physically plausible situation even for the non-conservative pollutant.

The solutions of the transport-dispersion equation for some particular cases tend to those obtained by Zoppou and Knight [11] for negligibly small decay coefficient, and reiterate the same qualitative results regarding application of the appropriate form viz. conservative form of the governing equation for non-conservative pollutants as well. However, by considering the solution, it is likely to be concluded that variable dispersion field in conservative form plays a dominant role in the fast-spreading of the pollutant and reduction of the concentration of non-conservative pollutants along the channel.

## APPENDIX A. SOLUTION OF TRANSPORT EQUATION

In view of $u(x) = u_0x$, a substitution of $x = e^y$ in Eq. (1) transforms it to the following form:

$$\frac{\partial C}{\partial t} + u_0\frac{\partial C}{\partial y} = -(k+u_0)C, 0 < y < \infty \quad (34)$$

This is well-known as Lagrange's equation, and can be written in the form of subsidiary equations as follows:

$$dt = \frac{dy}{u_0} = -\frac{dC}{(k+u_0)C} \quad (35)$$

From the first two terms in (35), the following formula is obtained:

$$y - y_0 = u_0t + K \quad (36)$$

where $K$ is an arbitrary constant to be obtained with the help of the transformed initial and boundary conditions as given as below:

$$C(y,0) = f(e^y), \qquad y > y_0 = \ln x_0 \quad (37)$$

$$C(y_0,t) = g(t), \qquad t > o \quad (38)$$

depending on whether $y > u_0t$ or $y < u_0t$.

Case 1: When $y - y_0 > u_0t$.

In this case, $K > 0$ for $y > y_0$ and $t > 0$. Then, the following equation is derived from (35),

$$\frac{dC(y,\ t)}{dt} = -(k+u_0)C(y,t)$$

Using Eq. (36), y in the above equation can be replaced with $y_0 + u_0t + K$. Then the equation takes the form:

$$\frac{dC(y_0+u_0t+K,\ t)}{dt} = -(k+u_0)C(y_0+u_0t+K,t)$$

The above ordinary differential equation has the solution as follows:

$$C(y_0+u_0t+K,t) = C(y_0+K,0)e^{-(k+u_0)t}$$

Now using Eqns. (36) and (37), the above equation can be written as below:

$$C(y,t) = f(e^{y-u_0t})e^{-(k+u_0)t}.$$

Case 2: When $y - y_0 < u_0t$.

Here $K < 0$ for $y > y_0,\ t > o$. Considering the second equality in the subsidiary equation (35), the following is computed:

$$\frac{d}{dy}C(y,t) = -(1+k/u_0)C(y,t)$$

which in this case is rewritten as follows:

$$\frac{d}{dy}C\left(y, \frac{y-y_0-K}{u_0}\right) = -(1+\frac{k}{u_0})C\left(y,\frac{y-y_0-K}{u_0}\right)$$

has the solution

$$C\left(y,\frac{y-y_0-K}{u_0}\right) = C\left(y_0,\frac{-K}{u_0}\right)e^{-\left(1+\frac{k}{u_0}\right)(y-y_0)}$$

that can be written (after using equations (36) and (38) as below:

$$C(y,t) = g(t - \frac{y-y_0}{u_0})e^{-\left(1+\frac{k}{u_0}\right)(y-y_0)}.$$

This shows that the solution of (34) under conditions (37) and (38) is:

$$C(x,t) = \begin{cases} e^{-(k+u_0)t} f(e^{y-u_0 t}), & y-y_0 > u_0 t \\ e^{-\left(1+\frac{k}{u_0}\right)(y-y_0)} g\left(t-\frac{y-y_0}{u_0}\right), & y-y_0 \le u_0 t \end{cases} \quad (39)$$

Substituting $y = ln\, x$ in (39), we get the solution of (1) under conditions (3) and (4) as:

$$C(x,t) = \begin{cases} e^{-(k+u_0)t} f(xe^{-u_0 t}), ln\left(\frac{x}{x_0}\right) > u_0 t \\ \left(\frac{x_0}{x}\right)^{1+\frac{k}{u_0}} g\left(t-\frac{ln\left(\frac{x}{x_0}\right)}{u_0}\right), ln\left(\frac{x}{x_0}\right) \le u_0 t \end{cases}$$

## APPENDIX B. SOLUTION FOR TRANSPORT-DISPERSION EQUATION

Substituting $x = e^y$ in (20) with $u(x)$ and $D(x)$ as given by Eq. (21) and simplifying, we get

$$\frac{\partial C}{\partial t} + (u_0 - D_0)\frac{\partial C}{\partial y} = D_0 \frac{\partial^2 C}{\partial x^2} - (k+u_0)\, C,\ y_0 \le y \le \infty \quad (40)$$

with the initial and boundary conditions as

$$C(y,0) = 0 \text{ for } y > y_0 \quad (41)$$

$$C(y_0,t) = C_0 \text{ for } t > t \quad (42)$$

$$C(y,t) = 0 \text{ as } y \to \infty \text{ for } t \ge 0 \quad (43)$$

Substituting

$$C(y,t) = c(y,t)\, e^{\frac{(u_0-D_0)y}{D_0} - \lambda t} \quad (44)$$

$$\text{where } \lambda = u_0 + k + \frac{(u_0-D_0)^2}{4D_0} \quad (45)$$

in (40)-(43), we obtain the diffusion equation

$$\frac{\partial c}{\partial t} = D_0 \frac{\partial^2 c}{\partial y^2}, \qquad y_0 < y < \infty \quad (46)$$

$$c(y,0) = 0\ for\ y > y_0 \quad (47)$$

$$c(y_0,t) = C_0 e^{-\frac{(u_0-D_0)y}{2D_0} - \lambda t}, \qquad t > o \quad (48)$$

$$c(y,t) = o \text{ as } y \to \infty, \qquad t > o \quad (49)$$

Applying semi-infinite Fourier sine transform (Sneddon, [15]) with respect to y on Eq. (46), the following is computed:

$$\frac{d}{dt} c_F(s,t) = -s^2 D_0 c_F(s,t) + s D_0 c(y_0,t) \quad (50)$$

$$\text{where } c_F(s,t) = \int_0^\infty c(y,t)\sin(sY)\, dY,\ Y = y - y_0 \quad (51)$$

and $s$ is the Fourier sine transform parameter.

The solution of (50) with transformed initial condition $c_F(s,t) = 0$ is given by:

$$c_F(s,t) = c_0\, e^{-\frac{(u_0-D_0)y_0}{2D_0}} \frac{se^{\lambda t}}{s^2 + \frac{\lambda}{D_0}} - \frac{se^{-s^2 D_0 t}}{s^2 + \frac{\lambda}{D_0}} \quad (52)$$

Taking inverse Fourier sine transform with respect to s, the solution $c(y,t)$ of Eq. (46) is obtained as follows:

$$c(y,t) = \frac{2C_0}{\pi} e^{-\frac{(u_0-D_0)y_0}{2D_0}} \left\{ e^{\lambda t} \int_0^\infty \frac{s \sin[s(y-y_0)]}{s^2+\frac{\lambda}{D_0}} ds \right\} - \int_0^\infty \frac{se^{-D_0 t s^2} \sin[s(y-y_0)]}{s^2+\frac{\lambda}{D_0}}\, ds \quad (53)$$

which on applying Eq.(44) and simplifying gives the solution of Eq.(40) under conditions (41) and (42) as below:

$$C(x,t) = \frac{C_0}{2}\left\{ e^{\mu_1(-y+y_0)} erfc\left[\frac{y-y_0-t\sqrt{(u_0+D_0)^2+4D_0 k}}{\sqrt{4D_0 t}}\right] \right. \quad (54)$$

$$\text{where } \mu_1, \mu_2 = -\frac{-\sqrt{(u_0+D_0)^2+4D_0 k} \pm (u_0 - D_0)}{2D_0}$$

Substituting back $y = ln\, x$ in Eq. (54), the analytical solution of Eq. (20) with Eq. (21) under initial and boundary conditions (22)-(24) is obtained as below:

$$C(x,t) = \frac{C_0}{2}\left\{ \left(\frac{x_0}{x}\right)^{\mu_1} erfc\left[\frac{ln\frac{x}{x_0} - t\sqrt{(u_0+D_0)^2+4D_0 k}}{\sqrt{4D_0 t}}\right] + \left(\frac{x}{x_0}\right)^{\mu_2} erfc\left[\frac{ln\frac{x}{x_0} + t\sqrt{(u_0+D_0)^2+4D_0 k}}{\sqrt{4D_0 t}}\right] \right\} \quad (55)$$

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