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New Iterative Methods With Seventh-Order Convergence For Solving Nonlinear Equations

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Abstract

In this paper, seventh-order iterative methods for the solution of nonlinear equations are presented. The new iterative methods are developed by using weight function method and using an approximation for the last derivative, which reduces the required number of functional evaluations per step. Several examples are given to illustrate the efficiency and the performance of the new iterative methods.

Keywords: Iterative Methods, Fourth Order, Seventh Order, Newton, Convergence, Nonlinear.
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1. Introduction

The nonlinear equations, often arise from the numerical modeling of problems in many branches of science and engineering [1]. These equations more often are not solved analytically hence resort to numerical solutions. More robust and efficient methods for solving the nonlinear equations are continuously being sought. There are many papers that deal with nonlinear algebraic equations, such as, improving Newton Raphson method for nonlinear equations by modified Adomian decomposition method [2], iterative method improving Newton's method by the decomposition method [3], new family of Iterative methods for nonlinear equations [4], iterative Methods for the solution of Equations

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[5], a third-oredr Newton-type method to solve system of nonlinear equations [6] and and other methods (see [7]-[19]).

In this paper, we recommend iterative methods to solve the nonlinear equations. We show that our proposed methods are of convergence order seven. The obtained results suggested that these new proposed methods introduce a powerful improvement for solving nonlinear equations.

2. Development of Seventh-Order Algorithm

Consider the well-known Traub-Ostrowski's method

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = x_m - \frac{f(y_m)-f(x_m)}{2f(y_m)-f(x_m)} \frac{f(x_m)}{f'(x_m)}. \end{cases} \tag{2.1}$$

It is known that the iterative algorithm defined by (2.1) converges with fourth order [20].

We consider the following three-step iteration scheme by using the method of weight functions (see [16, 17])

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = x_m - \frac{f(y_m)-f(x_m)}{2f(y_m)-f(x_m)} \frac{f(x_m)}{f'(x_m)}. \\ x_{m+1} = z_m - H(\mu_m)\frac{f(z_m)}{f'(z_m)}, \end{cases} \tag{2.2}$$

where $\mu_m = \frac{f(z_m)}{f(x_m)}$ and $H(t)$ represents a real-valued function.

Let us express $f^{'}(z_m)$ as a linear combination of $f[y_m, x_m]$, $f[z_m, y_m]$ and $f[z_m, x_m]$

$$f^{'}(z_m) = \theta_1 f[y_m, x_m] + \theta_2 f[z_m, y_m] + (1 - \theta_1 - \theta_2)f[z_m, x_m], \tag{2.3}$$

where θ_1 and θ_2 are a real numbers.

By using (2.3), we propose the following iterative scheme

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = x_m - \frac{f(y_m)-f(x_m)}{2f(y_m)-f(x_m)} \frac{f(x_m)}{f'(x_m)}, \\ x_{m+1} = z_m - H(\mu_m)\frac{f(z_m)}{\theta_1 f[y_m,x_m]+\theta_2 f[z_m,y_m]+(1-\theta_1-\theta_2)f[z_m,x_m]}. \end{cases} \tag{2.4}$$

3. Convergence Analysis

Theorem 3.1. *Let r be a simple zero of a sufficiently differentiable function $f : D \subseteq \Re \to \Re$. If the initial point x_0 is sufficiently close to r, then the sequence x_m generated by any method of the family (2.4) converges to r. If $H(t)$ is any function with $H(0) = 1$ and $H^{'}(0) = A < \infty$, then the convergence order of any method of the family (2.4) is seven if and only if $\theta_1 = -1$ and $\theta_2 = 1$.*

Proof . Let, $e_m = x_m - r$. Denotes $c_m = \frac{1}{m!}\frac{f^{(m)}(r)}{f^{'}(r)}$, m=2,3,.... Using the Taylor series, we have:

$$f(x_m) = f^{'}(r)[e_m + c_2e_m^2 + c_3e_m^3 + c_4e_m^4 + c_5e_m^5 + c_6e_m^6 + c_7e_m^7 + O(e_m^8)], \tag{3.1}$$

$$\begin{aligned} f^{'}(x_m) = f^{'}(r)[1 + 2c_2e_m + 3c_3e_m^2 &+ 4c_4e_m^3 + 5c_5e_m^4 + 6c_6e_m^5 \\ &+ 7c_7e_m^6 + 8c_8e_m^7 + O(e_m^8)], \end{aligned} \tag{3.2}$$

Now, from (3.1) and (3.2), we have

$$\begin{aligned} y_m = r + c_2e_m^2 &+ (2c_3 - 2c_2^2)e_m^3 + (3c_4 - 3c_2c_3 - 2(2c_3 - 2c_2^2)c_2)e_m^4 \\ &+ (4c_5 - 10c_2c_4 - 6c_3^2 + 20c_3c_2^2 - 8c_2^4)e_m^5 + (-17c_4c_3 + 28c_4c_2^2 \\ &- 13c_2c_5 + 33c_2c_3^2 + 5c_6 - 52c_3c_2^3 + 16c_2^5)e_m^6 + (-22c_5c_3 + 36c_5c_2^2 \\ &+ 6c_7 - 16c_2c_6 + 92c_4c_2c_3 - 12c_4^2 - 72c_4c_2^3 + 18c_3^3 - 126c_3^2c_2^2 \\ &+ 128c_3c_2^4 - 32c_2^6)e_m^7 + O(e_m^8)], \end{aligned} \tag{3.3}$$

From (3.3), we get

$$\begin{aligned} f(y_m) &= f^{'}(r)[c_2e^2 + (2c_3 - 2c_2^2)e_m^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_m^4 + (-6c_3^2 + 24c_3c_2^2 \\ &- 10c_2c_4 + 4c_5 - 12c_2^4)e^5 + (-17c_4c_3 + 34c_4c_2^2 - 13c_2c_5 + 5c_6 + 37c_2c_3^2 \\ &- 73c_3c_2^3 + 28c_2^5)e_m^6 + (-22c_5c_3 + 44c_5c_2^2 + 6c_7 - 16c_2c_6 - 12c_4^2 + 104c_4c_2c_3 \\ &- 104c_4c_2^3 + 18c_3^3 - 160c_3^2c_2^2 + 206c_3c_2^4 - 64c_2^6)e_m^7 + O(e_m^7)], \end{aligned} \tag{3.4}$$

Combining (3.1), (3.2), (3.3) and (3.4), we have

$$\begin{aligned} z_m &= r + (-c_2c_3 + c_2^3)e_m^4 + (-2c_2c_4 + 8c_3c_2^2 - 4c_2^4 - 2c_3^2)e_m^5 + (-3c_2c_5 + 10c_2^5 \\ &- 7c_4c_3 + 12c_4c_2^2 + 18c_2c_3^2 - 30c_3c_2^3)e_m^6 + (-4c_2c_6 + 12c_3^3 - 20c_2^6 - 10c_3c_5 \\ &- 80c_3^2c_2^2 + 80c_3c_2^4 - 6c_4^2 - 40c_4c_2^3 + 16c_5c_2^2 + 52c_3c_2c_4)e_m^7 + O(e_m^8), \end{aligned} \tag{3.5}$$

From (3.5), we get

$$\begin{aligned} f(z_m) &= f^{'}(r)[(-c_2c_3 + c_2^3)e_m^4 + (-2c_2c_4 + 8c_3c_2^2 - 4c_2^4 - 2c_3^2)e_m^5 + (-3c_2c_5 + 10c_2^5 \\ &- 7c_4c_3 + 12c_4c_2^2 + 18c_2c_3^2 - 30c_3c_2^3)e_m^6 + (-4c_2c_6 + 12c_3^3 - 20c_2^6 - 10c_3c_5 \\ &- 80c_3^2c_2^2 + 80c_3c_2^4 - 6c_4^2 - 40c_4c_2^3 + 16c_5c_2^2 + 52c_3c_2c_4)e_m^7 + O(e_m^8)], \end{aligned} \tag{3.6}$$

From (3.1) and (3.6), we have

$$\begin{aligned}
\frac{f(z_m)}{f(x_m)} &= (c_2^3 - c_2c_3)e_m^3 + (-2c_3^2 + 9c_3c_2^2 - 5c_2^4 - 2c_2c_4)e_m^4 + (-40c_3c_2^3 \\
&+ 21c_2c_3^2 - 3c_2c_5 + 15c_2^5 - 7c_4c_3 + 14c_4c_2^2)e_m^5 + (-55c_4c_2^3 + 62c_3c_2c_4 \\
&- 4c_2c_6 + 14c_3^3 - 35c_2^6 - 10c_3c_5 - 110c_3^2c_2^2 + 125c_3c_2^4 - 6c_4^2 + 19c_5c_2^2)e_m^6 \\
&+ (82c_5c_2c_3 - 294c_4c_3c_2^2 - 5c_2c_7 + 72c_2^7 - 17c_4c_5 + 45c_2c_4^2 + 59c_4c_3^2 + 161c_4c_2^4 \\
&- 71c_5c_2^3 - 13c_6c_3 + 24c_6c_2^2 - 126c_2c_3^3 + 403c_3^2c_2^3 - 320c_3c_2^5)e_m^7 + O(e_m^8) \qquad (3.7)
\end{aligned}$$

Using the Taylor expansion and (3.7), we get

$$\begin{aligned}
H(\mu_m) &= H(0) + H^{'}(0)\mu_m + O(\mu_m^2) = H(0) + H^{'}(0)(-c_3 + c_2^2)c_2e_m^3 \\
&- H^{'}(0)(2c_2c_4 - 9c_3c_2^2 + 5c_2^4 + 2c_3^2)e_m^4 + H^{'}(0)(-3c_2c_5 + 15c_2^5 \\
&- 7c_4c_3 + 14c_4c_2^2 + 21c_2c_3^2 - 40c_3c_2^3)e_m^5 + O(e_m^6) \qquad (3.8)
\end{aligned}$$

if $H(0) = 1$ and $H^{'}(0) = A < \infty$, by (3.1)-(3.8), we have

$$\begin{aligned}
e_{m+1} &= -c_2^2(-c_3 + c_2^2)(-1 + \theta_2)e_m^5 + c_2(-3c_3^2 + 3\theta_2c_3^2 - 12\theta_2c_3c_2^2 - \theta_1c_3c_2^2 \\
&+ 10c_3c_2^2 + 7\theta_2c_2^4 + \theta_1c_2^4 - 2c_2c_4 + 2\theta_2c_2c_4 - 5c_2^4 + \theta_2^2c_3c_2^2 - \theta_2^2c_2^4)e_m^6 \\
&+ (-2\theta_1c_2^3c_4 + 15\theta_1c_3c_2^4 - 5\theta_1c_3^2c_2^2 + 3\theta_2c_5c_2^2 + 10\theta_2c_3c_2c_4 + \theta_2^3c_3c_2^4 - 2c_3^3 \\
&+ 15c_2^6 + 30c_3^2c_2^2 - 45c_3c_2^4 + 15c_4c_2^3 - 3c_5c_2^2 - 19c_4c_2^3 + 71\theta_2c_3c_2^4 \\
&- 38\theta_2c_3^2c_2^2 - 8\theta_1c_2^6 - \theta_2^3c_2^6 + 9\theta_2^2c_2^6 + 2\theta_2c_3^3 - 29\theta_2c_2^6 \\
&+ 4\theta_2^2c_3^2c_2^2 - 15\theta_2^2c_3c_2^4 + 2\theta_2c_2^6\theta_1 + 2\theta_2^2c_4c_2^3 - 2\theta_2c_3c_2^4\theta_1 \\
&- 10c_3c_2c_4 - Ac_2^2c_3^2 + 2Ac_2^4c_3 - Ac_2^6)e_m^7 + O(e_m^8). \qquad (3.9)
\end{aligned}$$

Which shows that the convergence order of any method of the family (2.4) is seven if $\theta_1 = -1$ and $\theta_2 = 1$, and the error equation is

$$e_{m+1} = -c_2^2(-c_3 + c_2^2)(-Ac_3 + Ac_2^2 + c_3)e_m^7 + O(e_m^8). \qquad (3.10)$$

The proof is completed. □

4. Efficiency Index

We consider the definition of efficiency index as $P^{\frac{1}{d}}$, where P is the order of the method and d is the number of functional evaluations per iteration required by the method. Any method of the family (2.4) has the efficiency index equals to $7^{\frac{1}{4}} \approx 1.627$, which is better than the Newton's method with efficiency index equals to $2^{\frac{1}{2}} \approx 1.414$.

5. The Concrete Iterative Methods

In what follows, we give some concrete iterative methods of (2.4).

Example 5.1. ***M1***. *For the function H defined by $H(t) = 1$. It can easily be seen that $H(0) = 1$ and $H^{'}(0) = 0 < \infty$. Hence we get a new seventh-order method*

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f^{'}(x_m)}, \\ z_m = x_m - \frac{f(y_m)-f(x_m)}{2f(y_m)-f(x_m)} \frac{f(x_m)}{f^{'}(x_m)}, \\ x_{m+1} = z_m - \frac{f(z_m)}{\theta_1 f[y_m,x_m]+\theta_2 f[z_m,y_m]+(1-\theta_1-\theta_2)f[z_m,x_m]}, \end{cases} \tag{5.1}$$

Example 5.2. ***M2***. *For the function H defined by*

$$H(t) = 1 + \alpha t \tag{5.2}$$

where $\alpha \in \Re$. It can easily be seen that $H(0) = 1$ and $H^{'}(0) = 1 < \infty$. Hence we get a new seventh-order method

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f^{'}(x_m)}, \\ z_m = x_m - \frac{f(y_m)-f(x_m)}{2f(y_m)-f(x_m)} \frac{f(x_m)}{f^{'}(x_m)}, \\ x_{m+1} = z_m - (1+\alpha\frac{f(z_m)}{f(x_m)})\frac{f(z_m)}{\theta_1 f[y_m,x_m]+\theta_2 f[z_m,y_m]+(1-\theta_1-\theta_2)f[z_m,x_m]}, \end{cases} \tag{5.3}$$

Example 5.3. ***M3***. *For the function H defined by*

$$H(t) = (1 + \beta t)^{\gamma} \tag{5.4}$$

where $\beta, \gamma \in \Re$. It can easily be seen that $H(0) = 1$ and $H^{'}(0) = \beta.\gamma < \infty$. Hence we get a new seventh-order method

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f^{'}(x_m)}, \\ z_m = x_m - \frac{f(y_m)-f(x_m)}{2f(y_m)-f(x_m)} \frac{f(x_m)}{f^{'}(x_m)}, \\ x_{m+1} = z_m - (1+\beta\frac{f(z_m)}{f(x_m)})^{\gamma}\frac{f(z_m)}{\theta_1 f[y_m,x_m]+\theta_2 f[z_m,y_m]+(1-\theta_1-\theta_2)f[z_m,x_m]}, \end{cases} \tag{5.5}$$

Example 5.4. ***M4***. *For the function H defined by*

$$H(t) = 1 + \frac{\lambda_1 t}{1 + \lambda_2 t} \tag{5.6}$$

where $\beta, \gamma \in \Re$. It can easily be seen that $H(0) = 1$ and $H^{'}(0) = \lambda_1 < \infty$. Hence we get a new seventh-order method

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f^{'}(x_m)}, \\ z_m = x_m - \frac{f(y_m)-f(x_m)}{2f(y_m)-f(x_m)} \frac{f(x_m)}{f^{'}(x_m)}, \\ x_{m+1} = z_m - (1+\frac{\lambda_1\frac{f(z_m)}{f(x_m)}}{1+\lambda_2\frac{f(z_m)}{f(x_m)}})\frac{f(z_m)}{\theta_1 f[y_m,x_m]+\theta_2 f[z_m,y_m]+(1-\theta_1-\theta_2)f[z_m,x_m]}, \end{cases} \tag{5.7}$$

f_i, x_0	HM	CM	NM	RWB	$NETA$	CH	WKL	$M1-M2-M3-M4$
$f_1, 0.9$:	33	36	40	44	36	36	44	$5-4-5-4$
$f_2, 1$:	36	39	20	20	20	20	20	$5-5-5-5$
$f_3, 0.5$:	21	21	18	20	16	16	20	$4-4-4-4$
$f_4, 0.85$:	div	54	20	20	20	20	20	$5-5-5-5$
$f_5, -0.45$:	24	24	20	20	20	20	20	$5-5-5-5$
$f_6, 0.5$:	24	27	26	20	20	20	20	$5-5-5-5$

Table 1.Number of functional evaluations for various iterative methods.

6. Numerical Implementations

We present some examples to illustrate the efficiency of the iterative algorithm, see Table 1. We compare the Chebyshev method (CM) (see [21]-[22]); the Halley method (HM) (see [21]-[22]); the Newton iterative method (see [21]-[22]); RWB method proposed by Ren et al. [23]; NETA method proposed by Neta et al. [24]; the method developed by Chun et al. (CH) [15]; the method developed by Wang et. al. (WKL) [25], and M1, M2($\alpha = 1$), M3($\beta = 1, \gamma = \frac{1}{2}$) and M4($\lambda_1 = 1, \lambda_2 = 1$).
Free parameters are randomly selected as: for the method RWB $a = b = c = 1$, in the method by Chun et al. (CH) $\beta = 1$, in the method WKL $\alpha = \beta = 1$ and in the method NETA $a = 10$.

Example 6.1. $f_1(x) = \sin(x) - \frac{x}{100}; r = 0;$

Example 6.2. $f_2(x) = x^3 + 4x^2 - 10; r = 1.365;$

Example 6.3. $f_3(x) = \arctan(x); r = 0.0;$

Example 6.4. $f_4(x) = x^4 + \sin(\frac{\pi}{x^2}) - 5; r = \sqrt{2};$

Example 6.5. $f_5(x) = e^{-x^2+x+2} - 1; r = -1.0;$

Example 6.6. $f_6(x) = \frac{1}{3}x^4 - x^2 - \frac{1}{3}x + 1; r = 1.0.$

All the computations were done using MAPLE. The stopping criteria are

$$i.\ \| x_{n+1} - x_n \| \leq 10^{-320},\ ii.\ \| f(x_n) \| \leq 10^{-320}.$$

7. Conclusions

In this work we presented an approach which can be used to constructing seventh-order iterative methods that do not require the computation of second or higher derivatives. According to obtained results, the iterative methods that were introduced in this paper perform better than the CM method; the HM method; the Newton's method; the RWB method; the NETA method; the CH method; the WKL method for solving nonlinear equations.

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