Completely Continuous Banach Algebras

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Abstract

For a Banach algebra $\mathfrak{A}$, we introduce $c.c(\mathfrak{A})$, the set of all $\phi \in \mathfrak{A}^*$ such that $\theta_\phi : \mathfrak{A} \to \mathfrak{A}^*$ is a completely continuous operator, where θ_ϕ is defined by $\theta_\phi(a) = a \cdot \phi$ for all $a \in \mathfrak{A}$. We call $\mathfrak{A}$, a completely continuous Banach algebra if $c.c(\mathfrak{A}) = \mathfrak{A}^*$. We give some examples of completely continuous Banach algebras and a sufficient condition for an open problem raised for the first time by J.E Galè, T.J. Ransford and M. C. White: Is there exist an infinite dimensional amenable Banach algebra whose underlying Banach space is reflexive? We prove that a reflexive, amenable, completely continuous Banach algebra with the approximation property is trivial.

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1. Introduction and preliminaries

In 1992, J. E. Galé, T. J. Ransford and M. C. White planned a question whether a reflexive, amenable Banach algebra $\mathfrak{A}$ has to be of the form

$$\mathfrak{A} = \mathbb{M}_{n_1} \oplus \mathbb{M}_{n_2} \oplus ... \oplus \mathbb{M}_{n_k}.$$

with $n_1, ..., n_k$ in $\mathbb{N}$. A Banach algebra with the above representation is usually called trivial.

In that paper [5], they proved that every reflexive, amenable Banach algebra for which all primitive ideals have finite codimension is trivial. In the same year B. E. Johnson proved that every reflexive, amenable Banach algebra whose maximal left ideals are complemented is trivial (see also, [12]).

Later, in 1997, V. Runde found a sufficient condition to answer this question. In [9], V. Runde showed that for a reflexive, amenable Banach algebra $\mathfrak{A}$ with the approximation property such that every bounded linear map from $\mathfrak{A}$ to $\mathfrak{A}^*$ is compact, has to be trivial. More Banach algebras fails such a strong property, however l^p with $p > 2$, whenever be equipped with coordinatewise multiplication, is an example of a Banach algebra for which, by Pitt's theorem, every bounded linear map from it to its dual is compact [6]. Also V. Runde, in [10], proved that:

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Theorem 1.1. *A reflexive, amenable Banach algebra $\mathfrak{A}$ for which every maximal left ideal L satisfying the followings:*

   (i) *the quotient $\mathfrak{A}/L$ has the approximation property,*
   (ii) *the canonical map from $\mathfrak{A}\check{\otimes}L^\perp$ to $(\mathfrak{A}/L)\check{\otimes}L^\perp$ is open.*

is trivial.

After preliminaries in Section 2, completely continuous Banach algebras will be introduced in Section 3 and we will give some examples including p-summing Banach algebras. Finally, in Section 4, we give a sufficient condition for a reflexive, amenable Banach algebra to be trivial.

2. Preliminaries

Let $\mathfrak{A}$ be a Banach algebra and X a Banach $\mathfrak{A}$- bimodule. A continuous derivation of $\mathfrak{A}$ to X or X- derivation is a continuous linear mapping D from $\mathfrak{A}$ into X such that $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in \mathfrak{A}$. For each $x \in X$, the mapping $D_x : \mathfrak{A} \to X$ defined by $D_x(a) = a \cdot x - x \cdot a$ is a bounded X- derivation, the inner derivation associated with x. Let $\mathfrak{Z}^1(\mathfrak{A}, X)$ denote the space of all continuous X- derivation and $\mathfrak{N}^1(\mathfrak{A}, X)$ the subspace of all inner derivations in X. The quotient space $\mathfrak{H}^1(\mathfrak{A}, X) = \mathfrak{Z}^1(\mathfrak{A}, X)/\mathfrak{N}^1(\mathfrak{A}, X)$ is called the first cohomology group of $\mathfrak{A}$ with coefficients in X [7]. Thus the condition that $\mathfrak{H}^1(\mathfrak{A}, X) = \{0\}$ means that every continuous derivation is inner.

If X is a Banach $\mathfrak{A}$- bimodule, then so is the dual X^* with the module actions given by

$$\langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle \ ; \qquad \langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle,$$

$a \in \mathfrak{A}, x \in X, f \in X^*$. One can see that $\mathfrak{A}^*$ is a Banach $\mathfrak{A}$- bimodule with actions of $\mathfrak{A}$ on $\mathfrak{A}^*$ given by $\langle a \cdot f, b \rangle = \langle f, ba \rangle$ and $\langle f \cdot a, b \rangle = \langle f, ab \rangle$, $a, b \in \mathfrak{A}, f \in \mathfrak{A}^*$.

We denote $\otimes$, $\hat{\otimes}$ and $\check{\otimes}$, respectively, for the algebraic tensor product, the projective tensor product and the injective tensor product. If $\mathfrak{A}$ is a Banach algebra, then $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ and $\mathfrak{A}\check{\otimes}\mathfrak{A}$ are Banach $\mathfrak{A}$- bimodules with the following operations

$$a \cdot (b \otimes c) := ab \otimes c \ ; \qquad (b \otimes c) \cdot a := b \otimes ca \qquad (a, b, c \in \mathfrak{A}).$$

For a Banach algebra $\mathfrak{A}$, the corresponding diagonal operator is defined through

$$\Delta : \mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathfrak{A} \ , \qquad a \otimes b \mapsto ab.$$

It is clear that Δ is a bimodule homomorphism with respect to the defined module structure for $\mathfrak{A}\hat{\otimes}\mathfrak{A}$.

Let $\mathfrak{A}$ be a Banach algebra, $M \in (\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**}$ is called a *virtual diagonal* for $\mathfrak{A}$ if

$$a \cdot M = M \cdot a \ ; \qquad a \cdot \Delta^{**}M = a \qquad (a \in \mathfrak{A}).$$

A bounded net $(m_\alpha)_\alpha$ in $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ is called *approximate diagonal* for $\mathfrak{A}$ if

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0 \ ; \qquad a\Delta m_\alpha \to a \qquad (a \in \mathfrak{A}).$$

Also, $m \in (\mathfrak{A}\hat{\otimes}\mathfrak{A})$ is called *diagonal* for $\mathfrak{A}$ if

$$a \cdot m = m \cdot a \ ; \qquad a\Delta m = a \qquad (a \in \mathfrak{A}).$$

Let $\mathfrak{A}$ be a Banach algebra, $\mathfrak{A}$ is said to be *amenable* if $\mathfrak{H}^1(\mathfrak{A}, X^*) = \{0\}$, for every Banach $\mathfrak{A}$-bimodule X [7]. For a Banach algebra $\mathfrak{A}$, a well known theorem of Johnson is as follow:

Theorem 2.1. *Let $\mathfrak{A}$ be a Banach algebra. Then the followings are equivalent:*

  (i) $\mathfrak{A}$ *is amenable;*
 (ii) $\mathfrak{A}$ *has a virtual diagonal;*
(iii) $\mathfrak{A}$ *has an approximate diagonal.*

Also, the Banach algebra $\mathfrak{A}$ is called *contractible* if $\mathfrak{H}^1(\mathfrak{A}, X) = \{0\}$, for every Banach $\mathfrak{A}$- bimodule X [11]. Certainly, every contractible Banach algebra is amenable. One can check that, a Banach algebra $\mathfrak{A}$ is contractible if and only if it has a diagonal [11].

3. Completely continuous banach algebras

In this section, we will introduce completely continuous Banach algebras. As a result, we will show that a Banach algebra whose underlying Banach space or its dual has *Schur's property* is a completely continuous Banach algebra.

In normed spaces, norm convergent shows that weak convergent, however the converse is not true in general. In 1921, Issai Schur considered the normed spaces for which weak convergence sequences entails convergence in norm.

Definition 3.1. *A normed space X is said to have Schur's property, if for every sequence (x_n) in X such that $x_n \to x$ weakly, it follows that $\| x_n - x \| \to 0$.*

Example 3.2. *Every finite dimensional normed space has Schur's property. To see, let $\{e_1, e_2, ..., e_n\}$ be a basis for a normed space X of dimension n. Let $\{f_1, f_2, ..., f_n\}$ consists of coordinate functionals associated with the above basis. For each $x \in X$, we have*

$$x = \sum_{i=1}^{n} f_i(x)e_i.$$

Therefore, for a sequence (x_n) in X, $\| x_n - x \| \to 0$ if and only if $(f_i(x_n)) \to f_i(x)$ for each $i \in \{1, 2, ..., n\}$. Since, every $f \in X^*$ is a finite combination of $f_1, f_2, ..., f_n$, we have

$$\| x_n - x \| \to 0 \quad \Leftrightarrow \quad x_n \to x \ \text{weakly}$$

Example 3.3. *By the Schur's lemma, every weakly convergent sequence in l^1 is convergent. So, l^1 has Schur's property.*

To see that every normed space does not have Schur's property, consider an infinite dimensional Hilbert space H. Let (x_n) be a sequence of orthonormal elements in H. By the Riesz representation theorem, for each $f \in H^*$ there is a unique element $y \in H$ such that $f(x) = \langle x, y \rangle$ for each $x \in H$. Therefore, by using Bessel's inequality, it follows that

$$f(x_n) = \langle x_n, y \rangle \to 0,$$

as $n \to \infty$. On the other hand, $\| x_n \| = 1$, for each n. It shows that $x_n \nrightarrow 0$, thus H does not have Schur's property.

Example 3.4. *Let H be a Hilbert space. If for every sequence (x_n) in H such that $x_n \to x$ weakly, we have $\| x_n \| \to \| x \|$. Then H has Schur's property. Indeed, if (x_n) is a sequence in H such that $x_n \to x$ weakly. For every $y \in H$, we have $\langle x_n, y \rangle \to \langle x, y \rangle$. This is true specially for $y = x$, so*

$$\| x_n - x \|^2 = \| x_n \|^2 + \| x \|^2 - 2Re\langle x_n, x \rangle \to 0,$$

whenever $n \to \infty$.

One of the notions plays an important role in the study of geometry of Banach spaces, particular in convergence of sequences is the *Opial condition.*

Definition 3.5. *A normed space X is said to satisfy the Opial condition if whenever a sequence (x_n) in X converges weakly to $x \in X$, then*

$$\liminf_{n \to \infty} \parallel x_n - x \parallel < \liminf_{n \to \infty} \parallel x_n - y \parallel,$$

for all $y \in X$ with $y \neq x$. The above inequality is equivalent to the analogous condition obtained by replacing $\liminf$ by $\limsup$.

Example 3.6. *Every Hilbert space H satisfies the Opial condition. Let $x_n \to x$ weakly in H. Then for all $y \in H$ with $y \neq x$, $\lim_{n \to \infty} \sup \parallel x_n - x \parallel$ and $\lim_{n \to \infty} \sup \parallel x_n - y \parallel$ are finite because every weakly convergent sequence is necessarily bounded. Also we have*

$$\parallel x_n - y \parallel^2 = \parallel x_n - x + x - y \parallel^2 = \parallel x_n - x \parallel^2 + \parallel x - y \parallel^2 + 2Re\langle x_n - x, x - y\rangle \qquad (3.1)$$

since $\langle x_n - x, x - y\rangle \to 0$ whenever $n \to \infty$, we have

$$\limsup_{n \to \infty} \parallel x_n - y \parallel^2 > \limsup_{n \to \infty} \parallel x_n - x \parallel^2.$$

So the assertion is hold.

There exist normed spaces do not have the Opial condition. For every $1 < p < \infty$, $L^p[0, 2\pi]$ is such a normed space. For more about the Opial condition, see [1].

Theorem 3.7. *Let X be a normed space with Schur's property. Then X has the Opial condition and the converse is not true in general.*

Proof . Let $x_n \to x$ weakly in X. Then $\parallel x_n - x \parallel \to 0$, so $\liminf_{n \to \infty} \parallel x_n - x \parallel = 0$. From the uniqueness of a weak limit, for every $y \in X$ such that $y \neq x$, we have

$$\liminf_{n \to \infty} \parallel x_n - y \parallel > 0.$$

So X satisfies the Opial condition. The rest of claim is accomplished by considering an infinite dimensional Hilbert space H. □

Let X and Y be two Banach spaces, we recall that a bounded linear operator $T : X \to Y$ is completely continuous if for each sequence (x_n) in X such that $x_n \to x$ weakly it follows that $\parallel T(x_n) - T(x) \parallel \to 0$. The space of all completely continuous operator from X into Y is denoted by $\mathfrak{L}_{cc}(X, Y)$. It is well known that every compact operator is completely continuous and the converse is true provided that X is reflexive, see Proposition 3.3 of [2].

Let $\mathfrak{A}$ be a Banach algebra. Consider the map $\theta : \mathfrak{A}^* \to \mathfrak{L}(\mathfrak{A}, \mathfrak{A}^*)$ with $\theta(\phi) = \theta_\phi$; where $\mathfrak{L}(\mathfrak{A}, \mathfrak{A}^*)$ is the space of all bounded operators from $\mathfrak{A}$ into $\mathfrak{A}^*$ and θ_ϕ as an element of this space is defined by $\theta_\phi(a) = a \cdot \phi$. The element $a \cdot \phi$ belongs to $\mathfrak{A}^*$ and takes every $b \in \mathfrak{A}$ to $\langle \phi, ba\rangle$.

Definition 3.8. *Let $\mathfrak{A}$ be a Banach algebra. We call $\mathfrak{A}$ a completely continuous Banach algebra if $\theta(\mathfrak{A}^*) \subseteq \mathfrak{L}_{cc}(\mathfrak{A}, \mathfrak{A}^*)$; in other words, $\mathfrak{A}$ is completely continuous if for every $\phi \in \mathfrak{A}^*$, the map $a \mapsto a \cdot \phi$ from $\mathfrak{A}$ into $\mathfrak{A}^*$ is a completely continuous operator. By setting*

$$c.c(\mathfrak{A}) = \{\phi \in \mathfrak{A}^* : \theta_\phi \in \mathfrak{L}_{cc}(\mathfrak{A}, \mathfrak{A}^*)\},$$

we see that $\mathfrak{A}$ is a completely continuous Banach algebra whenever $c.c(\mathfrak{A}) = \mathfrak{A}^*$.

Theorem 3.9. *A Banach algebra $\mathfrak{A}$ is completely continuous if one of the following holds:*

(i) $\mathfrak{A}$ *has Schur's property;*

(ii) $\mathfrak{A}^*$ *has Schur's property.*

Proof . Let $\phi \in \mathfrak{A}^*$. To see (i), let $a_n \to a$ weakly. Since $\mathfrak{A}$ has Schur's property, $a_n \to a$ in norm. θ_ϕ is continuous, so $\parallel \theta_\phi(a_n) - \theta_\phi(a) \parallel \to 0$.

Now let $\mathfrak{A}^*$ has Schur's property and let $a_n \to a$ weakly. θ_ϕ is continuous, so takes weakly convergent sequences to weakly convergent sequences. Thus $\theta_\phi(a_n) \to \theta_\phi(a)$ weakly; $\mathfrak{A}^*$ has Schur's property so $\parallel \theta_\phi(a_n) - \theta_\phi(a) \parallel \to 0$. Therefore, in two cases, θ_ϕ is a completely continuous operator, that is, $\phi \in c.c.(\mathfrak{A})$. $\square$

Example 3.10. l^1 *is a completely continuous Banach algebra. Indeed, l^1 has Schur's property and by applying the above theorem $c.c.(l^1) = l^\infty$, so the claim is proved.*

Example 3.11. c_0 *is a completely continuous Banach algebra. We have $c_0^* = l^1$; applying the above theorem shows that $c.c.(c_0) = l^1$.*

For the next example, we need to bring some definitions. Let $1 \leq p < \infty$ and let $T : X \to Y$ be a linear operator between Banach spaces. T is called a p-summing operator if there is a constant $c \geq 0$ such that regardless of the natural number m and regardless of the choice of $x_1, ..., x_m$ in X we have

$$(\sum_{i=1}^m \parallel Tx_i \parallel^p)^{1/p} \leq c \cdot \sup\{(\sum_{i=1}^m |\langle x^*, x_i \rangle|^p)^{1/p} : x^* \in B_{X^*}\}, \tag{3.2}$$

where B_{X^*} is the ball of X^*. The space of all p-summing operators from X into Y is denoted by $\mathfrak{L}_p(X,Y)$. One can easily see that $\mathfrak{L}_p(X,Y)$ is a subspace of $\mathfrak{L}(X,Y)$.

A norm is defined on $\mathfrak{L}_p(X,Y)$, by setting $\parallel T \parallel_p$ equals to the least c the inequality (3.2) always holds. One can check that $\parallel T \parallel \leq \parallel T \parallel_p$ holds for every T in $\mathfrak{L}_p(X,Y)$.

By Proposition 18.1 of [4], for $1 \leq p < \infty$, a Banach algebra $\mathfrak{A}$ is a p-summing Banach algebra if for every $\phi \in \mathfrak{A}^*$, the map θ_ϕ by $a \mapsto a \cdot \phi$ from $\mathfrak{A}$ into $\mathfrak{A}^*$ is a p-summing operator. In this case, there is a constant $c \geq 0$ such that

$$\parallel \theta_\phi \parallel_p \ \leq c \parallel \phi \parallel .$$

Some examples of p-summing Banach algebras are as follows:

Example 3.12. *Let K be a compact Hausdorff space. Then $C(K)$, the space of all continuous $\mathbb{C}$-valued functions, is a p-summing Banach algebra for every $1 \leq p < \infty$, see Example 18.2 of [4].*

Example 3.13. l^p *is a 1-summing Banach algebra for $p \leq 2$, and a p^*-summing Banach algebra for $2 < p < \infty$; where $1/p + 1/p^* = 1$, see Example 18.3 of [4]. (Consider l^p with coordinatewise multiplication)*

Example 3.14. *Let H be an infinite dimensional Hilbert space, then $\mathfrak{L}(H)$ can not be a p-summing Banach algebra for any $1 \leq p < \infty$, see 18.22 of [4].*

Theorem 3.15. *Regardless of $1 \leq p < \infty$, every p-summing Banach algebra is a completely continuous Banach algebra.*

Proof . Let $1 \leq p < \infty$ and let $\mathfrak{A}$ be a p-summing Banach algebra. Then, for every $\phi \in \mathfrak{A}^*$, the map θ_ϕ by $a \mapsto a \cdot \phi$ from $\mathfrak{A}$ into $\mathfrak{A}^*$ is a p-summing operator. By Theorem 2.17 of [4], θ_ϕ is completely continuous (and weakly compact) operator; So $c.c(\mathfrak{A}) = \mathfrak{A}^*$. $\square$

We saw that a Banach algebra whose underlying Banach space has Schur's property is a completely continuous Banach algebra, but what about the converse?

By Example 3.13 and Theorem 3.15, for $1 < p < \infty$, l^p is a completely continuous Banach algebra, but it does not have Schur's property. Indeed, for $1 < p < \infty$, let (x_n) in l^p with $1 < p < \infty$ be defined by $x_n(i) = 1/n$ if $1 \leq i \leq n$ and $x_n(i) = 0$ if $i > n$. One can see that $x_n \to 0$ weakly and it does not converge to 0 in norm.

One should note that the separability plays a strong role in the proof of that l^1 has Schur's property, but this is not sufficient for a normed space to have Schur's property. We see that l^p is separable and does not have Schur's property.

Theorem 3.16. *Let $\mathfrak{A}$ be a Banach algebra with the character space $\Phi_{\mathfrak{A}}$. Then $c.c(\mathfrak{A})$ is a linear subspace of $\mathfrak{A}^*$ and*

$$\Phi_{\mathfrak{A}} \subseteq c.c(\mathfrak{A}).$$

Proof . To see that $c.c(\mathfrak{A})$ is a linear subspace of $\mathfrak{A}^*$ is straightforward. Let $\phi \in \Phi_{\mathfrak{A}}$ and let (a_n) be a sequence in $\mathfrak{A}$ such that $a_n \to a$ weakly $(a \in \mathfrak{A})$. We have

$$
\begin{aligned}
\| \theta_\phi(a_n) - \theta_\phi(a) \| &= \| a_n \cdot \phi - a \cdot \phi \| \\
&= \sup\{|\langle a_n \cdot \phi - a \cdot \phi, b\rangle| : \quad \| b \| \leq 1 \} \\
&= \sup\{|\langle \phi, b(a_n - a)\rangle| : \quad \| b \| \leq 1 \} \\
&= \sup\{|\langle \phi, b\rangle||\langle \phi, a_n - a\rangle| : \quad \| b \| \leq 1 \} \\
&= \| \phi \| |\langle \phi, a_n - a\rangle| \\
&\to 0,
\end{aligned}
$$

where $n \to \infty$. It shows that $\phi \in c.c(\mathfrak{A})$. $\square$

4. A sufficient condition

We recall that a Banach space X is said to have the approximation property, if for every compact set $K \subseteq X$ and every $\epsilon > 0$, there is an operator $T : X \to X$ of finite rank so that for every $x \in K$, $\| Tx - x \| < \epsilon$. Examples of Banach spaces have the approximation property including Hilbert spaces and Banach spaces with a Schuder basis.

Theorem 4.1. *Let $\mathfrak{A}$ be a reflexive, amenable and completely continuous Banach algebra with the approximation property. Then there are $n_1, n_2, ..., n_k \in \mathbb{N}$ such that*

$$\mathfrak{A} = \mathbb{M}_{n_1} \oplus \mathbb{M}_{n_2} \oplus ... \oplus \mathbb{M}_{n_k}.$$

Proof . For each $\phi \in \mathfrak{A}^*$, θ_ϕ is a completely continuous operator. Since $\mathfrak{A}$ is reflexive, θ_ϕ is a compact operator. So, $\theta_\phi \in \mathfrak{K}(\mathfrak{A}, \mathfrak{A}^*)$ for each $\phi \in \mathfrak{A}^*$. We may also see, by Theorem 2.2 of [8], that the mappings of the form $\theta^\phi : \mathfrak{A} \to \mathfrak{A}^*$ which are defined, for every $a \in \mathfrak{A}$, by $a \mapsto \phi \cdot a$ are all compact operators. By $\mathfrak{L}(\mathfrak{A}, \mathfrak{A}^*) \cong (\mathfrak{A} \hat{\otimes} \mathfrak{A})^*$, for each $\phi \in \mathfrak{A}^*$ and every $a, b \in \mathfrak{A}$, we have

$$
\begin{aligned}
\langle \Delta^*(\phi)(a), b\rangle &= \langle \Delta^*(\phi), a \otimes b\rangle \\
&= \langle \phi, ab\rangle \\
&= \langle \phi \cdot a, b\rangle \\
&= \langle \theta^\phi(a), b\rangle.
\end{aligned}
$$

Thus, for each $\phi \in \mathfrak{A}^*$, we have $\Delta^*(\phi) = \theta^\phi$, that is, $\Delta^*(\mathfrak{A}^*) \subset \mathfrak{K}(\mathfrak{A}, \mathfrak{A}^*)$. Since $\mathfrak{A}^{**}$ has the approximation property, $\mathfrak{A}^*$ has too. So by Proposition 5.3 of [3], $\mathfrak{K}(\mathfrak{A}, \mathfrak{A}^*) \cong \mathfrak{A}^* \check{\otimes} \mathfrak{A}^*$. Also, since $\mathfrak{A}$ is reflexive, $\mathfrak{A}$ has Radon-Nikodim property, see D3 of [3]. The approximation and Radon-Nikodim properties of $\mathfrak{A}$ imply that $\mathfrak{A} \hat{\otimes} \mathfrak{A} \cong (\mathfrak{A}^* \check{\otimes} \mathfrak{A}^*)^*$ [3].

Now let $(d_\alpha)_\alpha \subseteq \mathfrak{A} \hat{\otimes} \mathfrak{A}$ be an approximate diagonal for $\mathfrak{A}$. So $(d_\alpha)_\alpha$ has a w^*- accumalation point. Without loss of generality, assume that

$$d = w^* - \lim_\alpha d_\alpha.$$

For each $a \in \mathfrak{A}$ and $T \in \mathfrak{A}^* \check{\otimes} \mathfrak{A}^*$, we have

$$
\begin{aligned}
\langle a \cdot d, T \rangle &= \langle d, T \cdot a \rangle \\
&= \lim_\alpha \langle d_\alpha, T \cdot a \rangle \\
&= \lim_\alpha \langle a \cdot d_\alpha, T \rangle \\
&= \lim_\alpha \langle d_\alpha \cdot a, T \rangle \\
&= \lim_\alpha \langle d_\alpha, a \cdot T \rangle \\
&= \langle d \cdot a, T \rangle.
\end{aligned}
$$

Thus, $a \cdot d = d \cdot a$ for each $a \in \mathfrak{A}$.

We claim that $\Delta(d_\alpha) \to \Delta(d)$ with respect to the weak topology on $\mathfrak{A}$. To see, let $\psi \in \mathfrak{A}^*$, we have

$$
\begin{aligned}
\lim_\alpha \langle \psi, \Delta(d_\alpha) \rangle &= \lim_\alpha \langle \Delta^*(\psi), d_\alpha \rangle \\
&= \langle \Delta^*(\psi), d \rangle \qquad (\Delta^*(\psi) \in \mathfrak{A}^* \check{\otimes} \mathfrak{A}^*) \\
&= \langle \psi, \Delta(d) \rangle.
\end{aligned}
$$

This proves the claim.

Now let $\psi \in \mathfrak{A}^*$ be an arbitrary element. For each $a \in \mathfrak{A}$, we have

$$
\begin{aligned}
\langle \psi, a\Delta(d) \rangle &= \langle \psi \cdot a, \Delta(d) \rangle \\
&= \lim_\alpha \langle \psi \cdot a, \Delta(d_\alpha) \rangle \\
&= \lim_\alpha \langle \psi, a\Delta(d_\alpha) \rangle \\
&= \langle \psi, a \rangle.
\end{aligned}
$$

By the Hahn-Banach theorem, for every $a \in \mathfrak{A}$, we have $a\Delta(d) = a$. Therefore, d is a diagonal for $\mathfrak{A}$ and consequently $\mathfrak{A}$ is contractible. The proof is now complete by Theorem 4.1.5 of [11]. $\square$

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