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Ulam–Hyers–Rassias stability for stochastic integral equations of Volterra type

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Abstract

In this paper, we study the Ulam–Hyers–Rassias stability for stochastic integral equations of Volterra type by using fixed point theorem and Pachpatte's inequality.

Keywords: Ulam–Hyers stability, Ulam–Hyers–Rassias stability, Fixed point theorem, Pachpatte's inequality.
2010 MSC: 60H20; 34K20; 26D10; 4605.

1. Introduction

Stochastic (or random) integral equations are fundamental for modeling finance, physics and engineering phenomena. Several papers have considered on the problem of existence and uniqueness of solutions of stochastic (or random) integral equations, and the results are established by using various fixed point techniques and the method of successive approximation (see, e.g [1, 3, 4, 6, 11, 24]). Furthermore, asymptotic behavior and stability of solutions of stochastic integral equations are discussed in [5, 23, 27]. In 1940, Ulam [25] posted the open question "Under what conditions, the approximate solution of a given equation can be approximated by its exact solution ?". This question was first answered by D.H. Hyers one year later. Thereafter, T. Aoki [2], D.G. Bourgin [7] and Th.M. Rassias [21] improved the result of D.H. Hyers. For more details and further discussions, we refer the readers to the monographs by Soon-Mo Jung [12] and the papers of Hamid Khodaei [9, 14]. In recent years, accompanied by the development of the Hyers–Ulam stability for ordinary differential equations (see, e.g [13, 20, 22]) and stochastic (or random) integral equations (see, e.g [17, 18]), and

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stochastic (or random) differential equations (see, e.g [10, 16, 26]). In [17], Ngoc proved Ulam-Hyers-Rassias stability results for Volterra type stochastic integral equations by using Gronwall lemma and Banach's fixed point theorem. Ngoc [18] also investigated and established the stability in the sense of Ulam-Hyers and in the sense of Ulam-Hyers-Rassias for the stochastic Ito-Volterra integral equation. Li et al. [16] proved the existence for random impulsive stochastic functional differential equations with finite delays by using Krasnoselskii's fixed point. Authors also showed Hyers-Ulam stability results to the equation under the Lipschitz condition on a bounded and closed interval. Vinodkumar et al. [26] studied the existence and uniqueness of solutions, and Hyers-Ulam-Rassias stability results for random impulsive fractional differential systems by relaxing the linear growth conditions.

To the best of our knowledge, up to now, the number of papers dealing with Ulam–Hyers stability for stochastic integral equations is rather scant as opposed to the amount of publications concerning stochastic integral equations. Based on the motivation stated in the work of Ngoc [17, 18] and Vinodkumar et al. [26] and Li et al. [16]. In this paper, we shall study the Ulam–Hyers–Rassias stability for stochastic integral equations of Volterra type by using fixed point theorem and Pachpatte's inequality.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we shall the Ulam–Hyers–Rassias stability of Equation (2.1) via fixed point theorem and Pachpatte's lemma approach.

2. Preliminaries

Throughout this paper, we denote $\mathcal{F}$ the σ–algebra of subsets of the sample space Ω and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $W(t)$ be a Brownian motion defined on the space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{\mathcal{F}_t, t \in J := [0,T]\}$ be the natural filtration associated to W_t.

Let $X(t,\omega) = \big\{X(t), t \in J, \omega \in \Omega\big\}$ be a stochastic process. For $1 \le p < \infty$, we will use $L_p(\Omega)$ to denote the space of all random variables X with $E(|X|^p) < \infty$. It is a Banach space with norm

$$\|X\|_p = \sqrt[p]{\mathbb{E}|X|^p}.$$

Let $\mathbb{L}^p_{ad}(J,\Omega)$ denote the space of stochastic processes $X(t,\omega)$ satisfying the conditions: $X(t,\omega)$ is adapted to the filtration $\{\mathcal{F}_t\}$ and $\mathbb{E}\big(\int_J |X(t,\omega)|^p dt\big) < \infty$.

Suppose that $F : J \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $G : J \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ and $K : \mathbb{J} \times \mathbb{R}^d \to \mathbb{R}^d$ be measurable, where $\mathbb{J} = \{(t,s) \in T \times T : s \le t\}$, and X_0 be an $\mathcal{F}_t$-measurable $\mathbb{R}^d$–valued random variable such that $\mathbb{E}(|X_0|^p) < \infty$. We consider the following stochastic integral equations of Volterra type of the form

$$X(t) = X_0 + \int_0^t F\big(s, X(s), (\mathcal{Q}X)(s)\big)ds + \int_0^t G\big(s, X(s), (\mathcal{Q}X)(s)\big)dW(s), \tag{2.1}$$

where

$$(\mathcal{Q}X)(t) = \int_0^t K(t,s,X(s))ds, \quad \forall t \in J. \tag{2.2}$$

To investigated the Ulam–Hyers–Rassias stability of Equation (2.1), we will use the following assumptions:

(A1) There exists a constant $L_1 > 0$ such that

$$\max\big\{\big|F(t,X_2,Y_2) - F(t,X_1,Y_1)\big|; \big|G(t,X_2,Y_2) - G(t,X_1,Y_1)\big|\big\} \le L_1\big(\big|X_2 - X_1\big| + \big|Y_2 - Y_1\big|\big),$$

for any $t \in J$ and $X_1, X_2, Y_1, Y_2 \in \mathbb{R}^d$;

(A2) There exists a constant $L_2 > 0$ such that

$$\max\{|F(t,X,Y)|; |G(t,X,Y)|\} \le L_2(1+|X|+|Y|),$$

for any $t \in J$ and $X, Y \in \mathbb{R}^d$;

(A3) There exists a constant $M_1 > 0$ such that

$$|K(t,s,X) - K(t,s,Y)| \le M_1|X-Y|,$$

for any $(t,s) \in \mathbb{J}$ and $X \in \mathbb{R}^d$;

(A4) There exists a constant $M_2 > 0$ such that

$$|K(t,s,X)| \le M_2(1+|X|),$$

for any $(t,s) \in \mathbb{J}$ and $X \in \mathbb{R}^d$;

(A5) The random variable X_0 is $\mathcal{F}$-measurable with $E(X_0^p) < \infty$, where $p \ge 2$.

Theorem 2.1. *Let $p \ge 2$ and let $g \in L^2_{ad}(J,\Omega)$ be such that*

$$\mathbb{E}\left(\int_0^T |g(t)|^p dt\right) < \infty,$$

then

$$\mathbb{E}\left|\int_0^T g(t)W(s)\right|^p \le \widetilde{C}\mathbb{E}\left(\int_0^T |g(t)|^p dt\right),$$

where $\widetilde{C} := \left(\frac{p(p-2)}{2}\right)^{p/2} T^{(p-2)/2}$.

Theorem 2.2. ([8]) Let $d : \mathbb{X}\times\mathbb{X} \to [0,+\infty)$ be a generalized metric on $\mathbb{X}$ and $(\mathbb{X}, d)$ is a generalized complete metric space. Assume that $T : \mathbb{X} \to \mathbb{X}$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer n such that $d(T^{n+1}x, T^n x) < \infty$ for some $x \in \mathbb{X}$, then the followings are true:

(i) the sequence $\{T^n x\}$ converges to a fixed point x^* of T;

(ii) x^* is the unique fixed point of T in

$$\mathbb{X}^* = \{y \in \mathbb{X} \,|\, d(T^n x, y) < \infty\};$$

(iii) if $y \in \mathbb{X}^*$, then we have

$$d(y, x^*) \le \frac{1}{1-L} d(Ty, y).$$

Lemma 2.3. ([19]) Let $a, b, c \in C(J, \mathbb{R}_+)$ be a real-valued functions satisfying the inequality

$$a(t) \le a_0 + \int_0^t b(s)u(s)ds + \int_0^t b(s)\left(\int_0^s c(r)u(r)dr\right)ds, \forall t \in J$$

holds, where a_0 is positive constant. Then

$$a(t) \le a_0\left(1 + \int_0^t b(s)\exp\left(\int_0^s \big(b(r)+c(r)\big)dr\right)ds\right)$$

for $t \in J$.

Lemma 2.4. ([15]) Let $a, b, c, h \in C(J, \mathbb{R}_+)$ and e be a positive and nondecreasing continuous function defined on J for which in equality

$$a(t) \leq e(t) + \int_0^t b(s) \Big[a(s) + \int_0^s c(r)a(r)dr + \int_0^T h(r)a(r)dr \Big] ds.$$

If

$$r^* = \int_0^T h(r) \exp\Big(\int_0^r \big[b(\tau) + c(\tau) \big] d\tau \Big) dr < 1,$$

then

$$a(t) \leq \frac{e(t)}{1 - r^*} \exp\Big(\int_0^t \big[b(s) + c(s) \big] ds \Big)$$

for any $t \in J$.

Definition 2.5. Equation (2.1) is Ulam–Hyers stable with respect to ϵ if there exists M_ϵ such that for each solution $X(t) \in \mathbb{L}^p_{ad}(J, \Omega)$ of the following inequality

$$\Big\| X(t) - X_0 + \int_0^t F\big(s, X(s), (\mathfrak{Q}X)(s)\big) ds + \int_0^t G\big(s, X(s), (\mathfrak{Q}X)(s)\big) dW(s) \Big\|_p \leq \epsilon, \tag{2.3}$$

for any $t \in J$, there exists a solution $U(t) \in \mathbb{L}^p_{ad}(J, \Omega)$ of Equation (2.1) such that

$$\big\| X(t) - U(t) \big\|_p \leq M_\epsilon \epsilon, \tag{2.4}$$

where M_ϵ is a constant that does not depend on $X(t)$.

Definition 2.6. Equation (2.1) is Ulam–Hyers–Rassias stable with respect to $\varphi(t) \in C(J, \mathbb{R}_+)$ if there exists M_φ such that for each solution $X(t) \in \mathbb{L}^p_{ad}(J, \Omega)$ of the following inequality

$$\Big\| X(t) - X_0 + \int_0^t F\big(s, X(s), (\mathfrak{Q}X)(s)\big) ds + \int_0^t G\big(s, X(s), (\mathfrak{Q}X)(s)\big) dW(s) \Big\|_p \leq \varphi(t), \tag{2.5}$$

for any $t \in J$, there exists a solution $U(t) \in \mathbb{L}^p_{ad}(J, \Omega)$ of Equation (2.1) such that

$$\big\| X(t) - U(t) \big\|_p \leq M_\varphi \varphi(t), \forall t \in J, \tag{2.6}$$

where M_φ is a constant that does not depend on $X(t)$.

Remark 2.7. We see that Definition 2.5 $\Longrightarrow$ Definition 2.6.

3. Main results

3.1. Fixed point approach to the Ulam–Hyers–Rassias stability of Equation (2.1)

In this part, we present the Ulam–Hyers stability and Ulam–Hyers–Rassias stability for Equation (2.1) by using the fixed point approach.

Theorem 3.1. *Assume that (A1)-(A5) are satisfied. If*

$$\Big(2^{2p-2}(T^{p-1}+\widetilde{C})T^pL_1^p+2^{2p-1}(T^{p-1}+\widetilde{C})T^{p+1}L_1^pM_1^p\Big)^{1/p}<1$$

then

(i) Equation (2.1) has a unique solution in $L^p_{ad}(J,\Omega)$.

(ii) Equation (2.1) has Ulam-Hyers stable in $L^p_{ad}(J,\Omega)$.

Proof . Consider a space $L^p_{ad}(J,\Omega)$ consisting of all stochastic processes $X(t)$, which it is $\mathcal{F}$-adapted and continuous such that

$$\big\|X(t)\big\|_{p,\infty}=\Big(\sup_{t\in J}\mathbb{E}\big(|X(t)|^p\big)\Big)^{1/p}. \tag{3.1}$$

Now, we consider the operator $\mathbf{G}:L^p_{ad}(J,\Omega)\to L^p_{ad}(J,\Omega)$ defined by

$$\big(\mathbf{G}X\big)(t)=X_0+\int_0^t F\big(s,X(s),(\mathcal{Q}X)(s)\big)ds+\int_0^t G\big(s,X(s),(\mathcal{Q}X)(s)\big)dW(s), \tag{3.2}$$

where

$$(\mathcal{Q}X)(t)=\int_0^t K(t,s,X(s))ds,\quad \forall t\in J, X\in L^p_{ad}(J,\Omega).$$

For $X\in L^p_{ad}(J,\Omega)$ and $t\in J$, and by the Hölder inequality, we have

$$\int_0^t |X(s)|ds\le T^{(p-1)/p}\Big(\int_0^t |X(s)|^pds\Big)^{1/p} \tag{3.3}$$

$$\int_0^t |(\mathcal{Q}X)(s)|ds\le T^{(p-1)/p}\Big(\int_0^t |(\mathcal{Q}X)(s)|^pds\Big)^{1/p} \tag{3.4}$$

Using the elementary inequality $(a+b)^p\le 2^{p-1}(a^p+b^p)$ and assumtion (A4), we have the following estimate

$$\begin{aligned}\big|(\mathcal{Q}X)(t)\big|^p&\le\Big(\int_0^t\big|K(t,s,X(s))\big|ds\Big)^p\le\Big(\int_0^t M_2\big|1+X(s)\big|ds\Big)^p\\&\le 2^{p-1}T^{p-1}\Big\{M_2^pT^p+M_2^p\Big(\int_0^t\big|X(s)\big|ds\Big)^p\Big\}\end{aligned} \tag{3.5}$$

Form (3.3), (3.4) and (3.5), we obtain for $t\in J$

$$\begin{aligned}\int_0^t\big|(\mathcal{Q}X)(s)\big|^pds&\le 2^{p-1}T^{p-1}\Big\{M_2^pT^p+M_2^p\Big(\int_0^s\big|X(r)\big|dr\Big)^p\Big\}\\&\le 2^{p-1}T^{p-1}\int_0^t\Big\{M_2^pT^p+M_2^pT^{p-1}\Big(\int_0^s\big|X(r)\big|^pdr\Big)\Big\}ds\\&\le 2^{p-1}T^{p-1}\Big\{M_2^pT^{p+1}+M_2^pT^{p-1}\int_0^t\Big(\int_0^s\big|X(r)\big|^pdr\Big)ds\Big\}\end{aligned} \tag{3.6}$$

Using the inequality $(a+b+c)^p \le 3^{p-1}(a^p+b^p+c^p)$ and the Hölder inequalities (3.3), (3.4), and assumtions (A1)-(A4), we have the following estimates

$$\begin{aligned}
\Big|\int_0^t F\big(s,X(s),(\mathfrak{Q}X)(s)\big)ds\Big|^p &\le \bigg(\int_0^t L_2\Big(1+|X(s)|+\big|(\mathfrak{Q}X)(s)\big|\Big)ds\bigg)^p\\
&\le 3^{p-1}L_2^p\bigg\{T^p+\bigg(\int_0^t \big|X(s)\big|ds\bigg)^p+\bigg(\int_0^t \big|(\mathfrak{Q}X)(s)\big|ds\bigg)^p\bigg\}\\
&\le 3^{p-1}L_2^p\bigg\{T^p+T^{p-1}\bigg(\int_0^t \big|X(s)\big|^p ds\bigg)+T^{p-1}\bigg(\int_0^t \big|(\mathfrak{Q}X)(s)\big|^p ds\bigg)\bigg\} \qquad (3.7)
\end{aligned}$$

Inserting (3.6) into (3.7), we have

$$\Big|\int_0^t F\big(s,X(s),(\mathfrak{Q}X)(s)\big)ds\Big|^p \le C_1 + C_2\int_0^t \big|X(s)\big|^p ds + C_3\int_0^t\int_0^s \big|X(r)\big|^p dr ds, \qquad (3.8)$$

where $C_1 := 3^{p-1}L_2^pT^p + 6^{p-1}L_2^pM_2^pT^{3p-1}$, $C_2 := 3^{p-1}L_2^pT^{p-1}$ and $C_3 := 6^{p-1}L_2^pT^{3p-p}M_2^p$.

Using Theorem 2.1 and assumption (A2), we have

$$\begin{aligned}
\mathbb{E}\Big|\int_0^t G\big(s,X(s),(\mathfrak{Q}X)(s)\big)dW(s)\Big|^p &\le \widetilde{C}\mathbb{E}\bigg(\int_0^t \big|G\big(s,X(s),(\mathfrak{Q}X)(s)\big)\big|^p ds\bigg)\\
&\le \widetilde{C}L_2^p\mathbb{E}\bigg(\int_0^t \big(1+|X(s)|+\big|(\mathfrak{Q}X)(s)\big|\big)^p ds\bigg)\\
&\le 3^{p-1}\widetilde{C}L_2^p\mathbb{E}\bigg(\int_0^t \Big(1+\big|X(s)\big|^p+\big|(\mathfrak{Q}X)(s)\big|^p\Big)ds\bigg)\\
&\le 3^{p-1}\widetilde{C}L_2^p\bigg\{T^p+\mathbb{E}\int_0^t \big|X(s)\big|^p ds+\mathbb{E}\int_0^t \big|(\mathfrak{Q}X)(s)\big|^p ds\bigg\}. \qquad (3.9)
\end{aligned}$$

Inserting (3.6) into (3.9), we obtain

$$\mathbb{E}\Big|\int_0^t G\big(s,X(s),(\mathfrak{Q}X)(s)\big)dW(s)\Big|^p \le C_4 + C_5\int_0^t \mathbb{E}\big|X(s)\big|^p ds + C_6\int_0^t\int_0^s \mathbb{E}|X(r)|^p dr ds \qquad (3.10)$$

where $C_4 := 3^{p-1}\widetilde{C}L_2^p\big(T^p+2^{p-1}M_2^pT^{2p}\big)$, $C_5 := 3^{p-1}\widetilde{C}L_2^p$ and $C_6 := 6^{p-1}\widetilde{C}L_2^pM_2^pT^{2p-2}$.

Using the inequalities $(a+b+c)^p \le 3^{p-1}(a^p+b^p+c^p)$ and (3.8), (3.10), we obtain for $t \in J$

$$\begin{aligned}
\mathbb{E}\big|\big(\mathbf{G}X\big)(t)\big|^p &\le 3^{p-1}\bigg\{\mathbb{E}\big|X_0\big|^p+\mathbb{E}\Big|\int_0^t F\big(s,X(s),(\mathfrak{Q}X)(s)\big)ds\Big|^p\\
&\qquad +\mathbb{E}\Big|\int_0^t G\big(s,X(s),(\mathfrak{Q}X)(s)\big)dW(s)\Big|^p\bigg\}\\
&\le 3^{p-1}\big(C_1+C_4+\mathbb{E}|X_0|^0\big)+3^{p-1}\big(C_2+C_5\big)\int_0^t \mathbb{E}\big|X(s)\big|^p ds\\
&\qquad +3^{p-1}\big(C_3+C_6\big)\int_0^t\int_0^s \mathbb{E}|X(r)|^p dr ds \qquad (3.11)
\end{aligned}$$

Form (3.1) and inequality (3.10), we infer that $\big\|\big(\mathbf{G}X\big)(t)\big\|_{p,\infty} < \infty$. Hence, $\mathbf{G}(L^p_{ad}(J,\Omega)) \subset L^p_{ad}(J,\Omega)$.

For $X, Y \in L^p_{ad}(J,\Omega)$ and using assumptions (A1), and (A3), we have for $t \in J$

$$\begin{aligned}
&\bigg(\int_0^t \big|F\big(s,X(s),(\mathfrak{Q}X)(s)\big) - F\big(s,Y(s),(\mathfrak{Q}Y)(s)\big)\big|ds\bigg)^p \\
&\quad\leq T^{p-1}\int_0^t \big|F\big(s,X(s),(\mathfrak{Q}X)(s)\big) - F\big(s,Y(s),(\mathfrak{Q}Y)(s)\big)\big|^p ds \\
&\quad\leq T^{p-1}L_1^p\int_0^t \Big(\big|X(s)-Y(s)\big| + \big|(\mathfrak{Q}X)(s)-(\mathfrak{Q}Y)(s)\big|\Big)^p ds \\
&\quad\leq 2^{p-1}T^{p-1}L_1^p\int_0^t \Big(\big|X(s)-Y(s)\big|^p + \big|(\mathfrak{Q}X)(s)-(\mathfrak{Q}Y)(s)\big|^p\Big) ds \\
&\quad\leq 2^{p-1}T^{p-1}L_1^p\int_0^t \big|X(s)-Y(s)\big|^p ds + 2^{p-1}T^{2p-2}L_1^pM_1^p\int_0^t\int_0^s \big|X(r)-Y(r)\big|^p drds
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\mathbb{E}\bigg|\int_0^t \Big(G\big(s,X(s),(\mathfrak{Q}X)(s)\big) - G\big(s,Y(s),(\mathfrak{Q}Y)(s)\big)\Big)dW(s)\bigg|^p \\
&\leq \widetilde{C}\mathbb{E}\int_0^t \Big|G\big(s,X(s),(\mathfrak{Q}X)(s)\big) - G\big(s,Y(s),(\mathfrak{Q}Y)(s)\big)\Big|^p ds \\
&\leq 2^{p-1}\widetilde{C}L_1^p\mathbb{E}\bigg(\int_0^t \Big(\big|X(s)-Y(s)\big| + \big|(\mathfrak{Q}X)(s)-(\mathfrak{Q}Y)(s)\big|\Big)^p ds\bigg) \\
&\leq 2^{p-1}\widetilde{C}L_1^p\int_0^t \mathbb{E}\big|X(s)-Y(s)\big|^p ds + \widetilde{C}2^{p-1}T^{p-1}L_1^pM_1^p\int_0^t\int_0^s \mathbb{E}\big|X(r)-Y(r)\big|^p drds
\end{aligned} \tag{3.13}$$

Let $X, Y \in L^p_{ad}(J,\Omega)$ and the estimations (3.12), (3.13), we have for $t \in J$

$$\begin{aligned}
\mathbb{E}\big|\big(\mathbf{G}X\big)(t) - \big(\mathbf{G}Y\big)(t)\big|^p &= \mathbb{E}\bigg|\int_0^t F\big(s,X(s),(\mathfrak{Q}X)(s)\big)ds + \int_0^t G\big(s,X(s),(\mathfrak{Q}X)(s)\big)dW(s) \\
&\quad - \int_0^t F\big(s,Y(s),(\mathfrak{Q}Y)(s)\big)ds - \int_0^t G\big(s,Y(s),(\mathfrak{Q}Y)(s)\big)dW(s)\bigg|^p \\
&\leq 2^{p-1}\bigg\{\mathbb{E}\bigg|\int_0^t \big[F\big(s,X(s),(\mathfrak{Q}X)(s)\big) - F\big(s,Y(s),(\mathfrak{Q}Y)(s)\big)\big]ds\bigg|^p \\
&\quad + \mathbb{E}\bigg|\int_0^t \big[G\big(s,X(s),(\mathfrak{Q}X)(s)\big) - G\big(s,Y(s),(\mathfrak{Q}Y)(s)\big)\big]dW(s)\bigg|^p\bigg\} \\
&\leq 2^{2p-2}(T^{p-1}+\widetilde{C})L_1^p\int_0^t \mathbb{E}\big|X(s)-Y(s)\big|^p ds \\
&\quad + 2^{2p-2}(T^{p-1}+\widetilde{C})T^{p-1}L_1^pM_1^p\int_0^t\int_0^s \mathbb{E}\big|X(r)-Y(r)\big|^p drds.
\end{aligned} \tag{3.14}$$

By taking the supremum of both sides of (3.14) for $t \in J$, we have

$$\big\|\big(\mathbf{G}X\big)(t) - \big(\mathbf{G}Y\big)(t)\big\|_{p,\infty} \leq C_7(T)\big\|X(t)-Y(t)\big\|_{p,\infty}$$

where $C_7(T) := \Big(2^{2p-2}(T^{p-1}+\widetilde{C})T^pL_1^p + 2^{2p-1}(T^{p-1}+\widetilde{C})T^{p+1}L_1^pM_1^p\Big)^{1/p}$. By choosing a suitable $T_1 \in (0,T)$ sufficient small such that $C_7(T) < 1$, hence $\mathbf{G}$ is contraction mapping. By Banach fixed point theorem, then there exists a unique $U \in L^p_{ad}(J,\Omega)$ such that $\mathbf{G}(U) = U$.

Next, we assume that $X(t)$ be a solution of Inequation (2.3). For $t \in J$, we have

$$\big\|X(t) - \mathbf{G}\big(\mathfrak{Q}X\big)(t)\big\|_p \le \epsilon$$

which implies that

$$\big\|X(t) - \mathbf{G}\big(\mathfrak{Q}X\big)(t)\big\|_{p,\infty} \le \epsilon.$$

By Theorem 2.2 and for $t \in J$, we obtain

$$\big\|X(t) - U(t)\big\|_{p,\infty} \le \frac{\epsilon}{1 - C_7}.$$

On the other hand, we get

$$\big\|X(t) - U(t)\big\|_p \le \big\|X(t) - U(t)\big\|_{p,\infty}$$

for $t \in J$. Hence we infer that

$$\big\|X(t) - U(t)\big\|_p \le \frac{\epsilon}{1 - C_7}.$$

That is, Equation (2.1) has Ulam-Hyers stability. The proof is completed. □

Theorem 3.2. *Assume that the assumptions (A1)-(A5) are satisfied and there exists a constant $N_\varphi > 0$ such that*

$$\int_0^t \varphi^p(s)ds \le N_\varphi\, \varphi^p(t),$$

where $\varphi : J \to \mathbb{R}_+$ is continuous function. If

$$\Big(2^{p-1}(1+\widetilde{C})T^{p-1}L_1^p N_\varphi + 4^{p-1}(1+\widetilde{C})T^{2p-2}L_1^p M_1^p N_\varphi^2\Big)^{1/p} < 1$$

then

(i) Equation (2.1) has a unique solution in $L^p_{ad}(J,\Omega)$.

(ii) Equation (2.1) has Ulam–Hyers–Rassias stability with respect to $\varphi(t)$ in $L^p_{ad}(J,\Omega)$.

Proof .

Firstly, we choose continuous function $\Psi : J \to \mathbb{R}_+$ such that

$$\int_0^t \Psi^p(s)ds \le N_\varphi\, \Psi^p(t), \quad \forall t \in J.$$

Suppose that there exists two constants $\alpha_\varphi, \beta_\varphi > 0$ satisfies the following inequality

$$\alpha_\varphi \Psi(t) \le \varphi(t) \le \beta_\varphi \Psi(t), \quad \forall t \in J. \tag{3.15}$$

For $X(t), Y(t) \in L^p_{ad}(J,\Omega)$, we define

$$d_\Psi\big(X(t), Y(t)\big) = \sup_{t\in J} \frac{1}{\Psi(t)} \big\|X(t) - Y(t)\big\|_p. \tag{3.16}$$

It is easy to see that the space $\big(L^p_{ad}(J,\Omega), d_\Psi\big)$ is a complete generalized metric space.

Consider the operator $\mathbf{H} : L^p_{ad}(J,\Omega) \to L^p_{ad}(J,\Omega)$ defined by

$$\big(\mathbf{H}X\big)(t) = X_0 + \int_0^t F\big(s, X(s), (\mathfrak{Q}X)(s)\big)ds + \int_0^t G\big(s, X(s), (\mathfrak{Q}X)(s)\big)dW(s), \tag{3.17}$$

where

$$(\mathfrak{Q}X)(t) = \int_0^t K(t,s,X(s))ds, \quad \forall t \in J, X \in L^p_{ad}(J,\Omega).$$

Now, we need to check the operator $\mathbf{H}$ is strictly contractive on $L^p_{ad}(J,\Omega)$. For any $X, Y \in L^p_{ad}(J,\Omega)$ and let $C_{X,Y} \in [0,\infty)$ be a arbitrary constant with $d_\Psi\big(X,Y\big) \le C_{X,Y}$, that is, by (3.16), we have

$$\big\|X(t) - Y(t)\big\|_p \le C_{X,Y}\Psi(t), \quad \forall t \in J.$$

For any $X, Y \in L^p_{ad}(J,\Omega)$ and performing similar calculations as in Theorem 3.1, we have the following estimate

$$\begin{aligned}
\mathbb{E}\big|\big(\mathbf{H}X\big)(t) - \big(\mathbf{H}Y\big)(t)\big|^p &\le 2^{2p-2}(T^{p-1} + \widetilde{C})L_1^p \int_0^t \big\|X(r) - Y(r)\big\|_p^p ds \\
&+ 2^{2p-2}(T^{p-1} + \widetilde{C})T^{p-1}L_1^p M_1^p \int_0^t \int_0^s \big\|X(r) - Y(r)\big\|_p^p drds \\
&\le 2^{2p-2}(T^{p-1} + \widetilde{C})L_1^p \int_0^t C^p_{X,Y}\Psi^p(s)ds \\
&+ 2^{2p-2}(T^{p-1} + \widetilde{C})T^{p-1}L_1^p M_1^p \int_0^t \int_0^s C^p_{X,Y}\Psi^p(r)drds \\
&\le 2^{2p-2}(T^{p-1} + \widetilde{C})L_1^p C^p_{X,Y} N_\varphi \Psi^p(t) \\
&+ 2^{2p-2}(T^{p-1} + \widetilde{C})T^{p-1}L_1^p M_1^p \int_0^t C^p_{X,Y} N_\varphi \Psi^p(s)ds \\
&\le 2^{2p-2}(T^{p-1} + \widetilde{C})L_1^p C^p_{X,Y} N_\varphi \Psi^p(t) \\
&+ 2^{2p-2}(T^{p-1} + \widetilde{C})T^{p-1}L_1^p M_1^p C^p_{X,Y} N_\varphi^2 \Psi^p(t)
\end{aligned}$$

which implies that

$$\big\|\big(\mathbf{H}X\big)(t) - \big(\mathbf{H}Y\big)(t)\big\|_p \le C_8 C_{X,Y}\Psi(t), \quad \forall t \in J,$$

where $C_8(T) := \Big(2^{2p-2}(T^{p-1} + \widetilde{C})L_1^p N_\varphi + 2^{2p-2}(T^{p-1} + \widetilde{C})T^{p-1}L_1^p M_1^p N_\varphi^2\Big)^{1/p}$. It implies that

$$d_\Psi\big(\mathbf{H}X\big)(t), \mathbf{H}Y\big)(t)\big) \le C_8 C_{X,Y},$$

for any $t \in J$. Therefore, we conclude that

$$d_\Psi\big(\big(\mathbf{H}X\big)(t), \mathbf{H}Y\big)(t)\big) \le C_8 d_\Psi\big(X(t), Y(t)\big), \tag{3.18}$$

where $X, Y \in L^p_{ad}(J,\Omega)$. By choosing a suitable $T_2 \in (0,T)$ sufficient small such that $C_8(T) < 1$, hence $\mathbf{H}$ is strictly contractive on $\big(L^p_{ad}(J,\Omega), d_\Psi\big)$. Applying Banach fixed point theorem, then there exists a unique $U \in L^p_{ad}(J,\Omega)$ such that $\mathbf{H}(U) = U$.

Let $X(t)$ be a solution of (2.5) and by assumption (3.15) and for any $t \in J$, we have

$$\big\|\big(\mathbf{H}X\big)(t) - \big(\mathbf{H}Y\big)(t)\big\|_p \le \varphi(t) \le \beta_\varphi \Psi(t). \tag{3.19}$$

For any $X(t), U(t) \in L^p_{ad}(J,\Omega)$ and $t \in J$, we have the following estimate

$$\begin{aligned} d_\Psi\big(X(t), U(t)\big) &\le d_\Psi\big(X(t), \big(\mathbf{H}X\big)(t)\big) + d_\Psi\big(\big(\mathbf{H}X\big)(t), U(t)\big) \\ &= d_\Psi\big(X(t), \big(\mathbf{H}X\big)(t)\big) + d_\Psi\big(\big(\mathbf{H}X\big)(t), \mathbf{H}(U)(t)\big) \end{aligned} \tag{3.20}$$

By the definition of the metric d_Ψ and (3.18), (3.19), (3.20), we obtain

$$d_\Psi\big(X(t), U(t)\big) \le \beta_\varphi + C_8 d_\Psi\big(X(t), Y(t)\big) \tag{3.21}$$

for any $t \in J$, which implies that

$$d_\Psi\big(X(t), U(t)\big) \le \frac{\beta_\varphi}{1 - C_8}, \quad \forall t \in J. \tag{3.22}$$

Therefore,

$$\big\|X(t) - Y(t)\big\|_p \le \frac{\beta_\varphi}{1 - C_8}\Psi(t) \le M_\varphi \varphi(t),$$

where $M_\varphi = \dfrac{\beta_\varphi}{(1 - C_8)\alpha_\varphi}$, that is, Equation (2.1) has the Ulam–Hyers–Rassias stability. This completes the proof the theorem. □

3.2. Pachpatte's lemma approach to the Ulam–Hyers–Rassias stability of Equation (2.1)

In this part, we present the Ulam–Hyers stability and Ulam–Hyers–Rassias stability for Equation (2.1) by using Pachpatte's inequality.

Theorem 3.3. *Let $\varphi : J \to \mathbb{R}_+$ be a continuous function and $\varphi^p(t)$ is non-decreasing on J. Assume that the assumptions (A1)-(A5) are satisfied. Then*

(i) Equation (2.1) has a unique solution in $L^p_{ad}(J,\Omega)$.

(ii) Equation (2.1) has Ulam–Hyers–Rassias stability with respect to $\varphi(t)$ in $L^p_{ad}(J,\Omega)$.

Proof . Let $U(t)$ be the solution of Equation (2.1). For any $t \in J$, we have

$$U(t) = X_0 + \int_0^t F\big(s, U(s), (\mathfrak{Q}U)(s)\big)ds + \int_0^t G\big(s, U(s), (\mathfrak{Q}U)(s)\big)dW(s), \tag{3.23}$$

where

$$(\mathfrak{Q}U)(t) = \int_0^t K(t, s, U(s))ds, \quad \forall t \in J. \tag{3.24}$$

Using the inequality $(a + b + c)^p \le 3^{p-1}(a^p + b^p + c^p)$ and assumptions (A1)-(A5), we obtain

$$\begin{aligned} \mathbb{E}\big|U(t)\big|^p \le 3^{p-1}\Big\{\mathbb{E}\big|X_0\big|^p &+ \mathbb{E}\Big|\int_0^t F\big(s, U(s), (\mathfrak{Q}U)(s)\big)ds\Big|^p \\ &+ \mathbb{E}\Big|\int_0^t G\big(s, U(s), (\mathfrak{Q}U)(s)\big)dW(s)\Big|^p\Big\}. \end{aligned} \tag{3.25}$$

Performing similar calculations as in Theorem 3.1, we have the following estimates

$$\begin{aligned}\mathbb{E}\big|U(t)\big|^p &\leq 3^{p-1}\big(C_1+C_4+\mathbb{E}|X_0|^0\big)+3^{p-1}\big(C_2+C_5\big)\int_0^t \mathbb{E}\big|X(s)\big|^p ds\\ &+3^{p-1}\big(C_3+C_6\big)\int_0^t\int_0^s \mathbb{E}|X(r)|^p drds\\ &\leq 3^{p-1}\big(C_1+C_4+\mathbb{E}|X_0|^0\big)+\int_0^t 3^{p-1}\big(C_2+C_5\big)\mathbb{E}\big|X(s)\big|^p ds\\ &+\int_0^t 3^{p-1}\big(C_2+C_5\big)\int_0^s \frac{C_3+C_6}{C_2+C_5}\mathbb{E}|X(r)|^p drds. \end{aligned}\tag{3.26}$$

Applying Gronwall-Bellman lemma 2.3, we obtain

$$\mathbb{E}\big|U(t)\big|^p \leq C_9\bigg(1+\exp\bigg(\frac{TC_{10}}{C_{10}+C_{11}}\bigg)\bigg)<+\infty,\tag{3.27}$$

for any $t\in J$, where $C_9 := 3^{p-1}\big(C_1+C_4+\mathbb{E}|X_0|^0\big)$, $C_{10} := 3^{p-1}\big(C_2+C_5\big)$ and $C_{11} := \dfrac{C_3+C_6}{C_2+C_5}$.

Therefore, we infer that $U(t)\in L^p_{ad}(J,\Omega)$.

Next, let $X(t)$ be a solution of (2.5) and let $U(t)$ be a solution of (2.1). For any $t\in J$ and by the inequalities $(a+b+c)^p\leq 3^{p-2}(a^p+b^p+c^p)$, we have

$$\begin{aligned}\big|X(t)-U(t)\big|^p \leq 3^{p-1}\bigg\{&\bigg|X(t)-X_0-\int_0^t F\big(s,U(s),(\mathfrak{Q}U)(s)\big)ds-\int_0^t G\big(s,U(s),(\mathfrak{Q}U)(s)\big)dW(s)\bigg|^p\\ &+\bigg|\int_0^t \big|F\big(s,X(s),(\mathfrak{Q}X)(s)-F\big(s,U(s),(\mathfrak{Q}U)(s)\big|ds\bigg|^p\\ &+\bigg|\int_0^t \big|G\big(s,X(s),(\mathfrak{Q}X)(s)\big)-G\big(s,U(s),(\mathfrak{Q}U)(s)\big)\big|dW(s)\bigg|^p\bigg\}\end{aligned}\tag{3.28}$$

From Inequalities (2.5) and the assumption $\varphi^p(t)$ is non-decreasing function on J, we get

$$\bigg|X(t)-X_0-\int_0^t F\big(s,U(s),(\mathfrak{Q}U)(s)\big)ds-\int_0^t G\big(s,U(s),(\mathfrak{Q}U)(s)\big)dW(s)\bigg|^p\leq\varphi^p(t),\tag{3.29}$$

for any $t\in J$.

On the other hand, performing similar calculations as in Theorem 3.1, we obtain the following estimate

$$\begin{aligned}&\bigg|\int_0^t \big|F\big(s,X(s),(\mathfrak{Q}X)(s)-F\big(s,U(s),(\mathfrak{Q}U)(s)\big|ds\bigg|^p\\ &\quad\leq 2^{p-1}L_1^pT^{p-1}\int_0^t\big|X(s)-U(s)\big|^p ds+2^{p-1}L_1^pT^{2p-2}M_1^p\int_0^t\int_0^s\big|X(r)-U(r)\big|^p drds\end{aligned}\tag{3.30}$$

and

$$\begin{aligned}&\mathbb{E}\bigg|\int_0^t \big|G\big(s,X(s),(\mathfrak{Q}X)(s)\big)-G\big(s,U(s),(\mathfrak{Q}U)(s)\big)\big|dW(s)\bigg|^p\\ &\qquad\leq 2^{2p-2}(T^{p-1}+\widetilde{C})L_1^p\int_0^t\mathbb{E}\big|X(s)-Y(s)\big|^p ds\\ &\qquad+2^{2p-2}(T^{p-1}+\widetilde{C})T^{p-1}L_1^pM_1^p\int_0^t\int_0^s\mathbb{E}\big|X(r)-Y(r)\big|^p drds,\end{aligned}\tag{3.31}$$

for any $t \in J$.

Combining the estimaties (3.31),(3.30) and (3.29) with (3.28), we get

$$\mathbb{E}\big|X(t)-U(t)\big|^p \le C_{12}\varphi^p(t) + C_{13}\int_0^t \mathbb{E}\big|X(s)-Y(s)\big|^p ds + C_{14}\int_0^t\int_0^s \mathbb{E}\big|X(r)-Y(r)\big|^p dr ds, \quad \forall t \in J, \tag{3.32}$$

where $C_{12} := 3^{p-1}$, $C_{13} := 2^{2p-2}3^{p-1}(T^{p-1}+\widetilde{C})L_1^p$ and $C_{14} := 2^{2p-2}3^{p-1}(T^{p-1}+\widetilde{C})T^{p-1}L_1^pM_1^p$.

Applying Pachpatte's Inequality 2.4, we obtain

$$\mathbb{E}\big|X(t)-U(t)\big|^p \le C_{12}\varphi^p(t)\exp\bigg(\int_0^t \big(C_{13}+C_{14}\big)ds\bigg) \le C_{12}\varphi^p(t)\exp\big(TC_{13}+TC_{14}\big), \tag{3.33}$$

which implies that

$$\big\|X(t)-U(t)\big\|_p \le M_\varphi \varphi(t), \quad \forall t \in J,$$

where $M_\varphi = \big(C_{12}\exp\big(TC_{13}+TC_{14}\big)\big)^{1/p}$. This completes the proof the theorem. □

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References

[1] G. Adomian, *Random volterra integral equations*, Math. Comput. Model. 22(8) (1995) 101–102.
[2] T. Aoki, *On the stability of the linear transformation in banach spaces*, J. Math. Soc. Japan 2(1-2) (1950) 64–66.
[3] K. Balachandran and J.K. Kim, *Existence of solutions of nonlinear stochastic volterra fredholm integral equations of mixed type*, Int. J. Math. Math. Sci. 2010 (2010) Article ID 603819.
[4] K. Balachandran, K. Sumathy, and J.K. Kim, *Existence and stability of solutions of general stochastic integral equations*, Nonlinear Funct. Anal. Appl. 12(2) (2007) 219–235.
[5] J. Banas, D. Szynal and S. Wedrychowicz, *On existence,asymptotic behaviour and stability of solutions of stochastic integral equations*, Stochastic Anal. Appl. 9(4) (1991) 363–385.
[6] A.T. Bharucha-Reid and M.J. Christensen, *Approximate solution of random integral equations: General methods*, Math. Comput. Simul. 26(4) (1984) 321–328.
[7] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. 57(4) (1951) 223–237.
[8] J.B. Diaz and B. Margolis, *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. 74(2) (1968) 305–309.
[9] M. Eshaghi Gordji and H. Khodaei, *On the generalized Hyers-Ulam-Rassias stability of quadratic functional equations*, Abstr. Appl. Anal. 2009 (2009) Article ID: 923476.
[10] Y. Guo, X.-B. Shu, Y. Li and F. Xu, *The existence and Hyers-Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order* $1 < \beta < 2$, Bound. Value Probl. 2019(1) (2019) 59.
[11] S.T. Hardiman and C.P. Tsokos, *On the Uryson type of stochastic integral equations*, Math. Proc. Cambridge Philos. Soc. 76(1) (1974) 297305.
[12] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, 2011.
[13] S.-M. Jung and J. Brzdek, *Hyers-Ulam stability of the delay equation*, Abstr. Appl. Anal. 2010 (2010) Article ID: 372176.

[14] H. Khodaei, *Selections of generalized convex set-valued functions satisfying some inclusions*, J. Math. Anal. Appl. 474(2) (2019) 1104–1115.
[15] K.D. Kucche and P.U. Shikhare, *Ulam stabilities via pachpattes inequality for volterra fredholm delay integrodifferential equations in banach spaces*, Note di Mate. 38(1) (2018) 67–82.
[16] S. Li, L. Shu, X.-B. Shu and F. Xu, *Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays*, Stochast. 2018 (2018) 1–16.
[17] N.P.N. Ngoc, *Ulam-Hyers- Rassias stability of a nonlinear stochastic integral equation of volterra type*, Differ. Equ. Appl. 9(2) (2017) 183–193.
[18] N.P.N. Ngoc and N.V. Vinh, *Ulam-Hyers-Rassias stability of a nonlinear stochastic itovolterra integral equation*, Differ. Equ. Appl. 10(4) (2018) 397–411.
[19] B.G. Pachpatte, *A note on gronwall-bellman inequality*, J. Math. Anal. Appl. 44(3) (1973) 758–762.
[20] D. Popa and I. Raa, *On the Hyers-Ulam stability of the linear differential equation*, J. Math. Anal. Appl. 381(2) (2011) 530–537.
[21] T.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72(2) (1978) 297–300.
[22] I.A. Rus, *Ulam stability of ordinary differential equations*, Studia Univ. Babes Bolyai, Math. LIV(4) (2009) 1–9.
[23] R. Sakthivel and J. Luo, *Asymptotic stability of nonlinear impulsive stochastic differential equations*, Stat. Probab. Lett. 79(9) (2009) 1219–1223.
[24] C.P. Tsokos, *Stochastic approximation of a random integral equation*, Math. Nachr. 51(16) (1971) 101–110.
[25] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York-London, 1960.
[26] A. Vinodkumar, K. Malar, M. Gowrisankar and P. Mohankumar, *Existence, uniqueness and stability of random impulsive fractional differential equations*, Acta Math. Sci. 36(2) (2016) 428–442.
[27] H.F. Yashima and A. Gianesini, *Asymptotic behavior of the solution of an integral stochastic equation in* r^n, Acta Math. Vietn. 31(2) (2006) 115–129.