Int. J. Nonlinear Anal. Appl. 11 (2020) No. 2, 17-28
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2019.17071.1905

A fixed point approach to the stability of additive-quadratic-quartic functional equations

Abasalt Bodaghi[a*], Themistocles Michael Rassias[b], Abbas Zivari-Kazempour[c]

[a] *Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran*
[b] *Department of Mathematics, National Technical University of Athens, Zografou Campus, 157 80, Athens, Greece*
[c] *Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran*

(*Communicated by Madjid Eshaghi Gordji*)

Abstract

In this article, we introduce a class of the generalized mixed additive, quadratic and quartic functional equations and obtain their common solutions. We also investigate the stability of such modified functional equations in the non-Archimedean normed spaces by a fixed point method.

Keywords: additive-quadratic-quartic functional equation, Hyers-Ulam stability, non-Archimedean normed space.
2010 MSC: 39B52, 39B72, 39B82, 46B03.

1. Introduction

The stability of a functional equation originated from a question raised by Ulam in [28] as follows: "when is it true that the solution of an equation differing slightly from a given one must of necessity be close to the solution of the given equation?". The first answer (in the case of Cauchy's functional equation in Banach spaces) to Ulam's question was given by Hyers in [18]. Following his result, a great number of papers on the stability problems of several functional equations have been extensively published as generalizing Ulam's problem and Hyers's theorem in various directions; see for instance [1, 17, 19, 20, 25, 27]. Some results regarding to the stability of various forms of the mixed type additive-quadratic ([5], [31]), additive-cubic ([23], [30]), additive-quartic ([2], [3], [12]), cubic-quartic ([4], [14]), quadratic-quartic ([6], [11], [29]), additive-quadratic-cubic ([13], [21]), and

*Corresponding author
Email addresses: abasalt.bodaghi@gmail.com (Abasalt Bodaghi[a*]), trassias@math.ntua.gr (Themistocles Michael Rassias[b]), zivari6526@gmail.com (Abbas Zivari-Kazempour[c])

Received: January 2019 *Revised:* November 2019

additive-quadratic-sextic [15] functional equations were investigated in some normed spaces and algebras. For some results on the various functional equations in non-Archimedean (2,β)-Banach spaces, refer to [9] and [10].

Mohamadi et al. in [22] introduced the additive-quadratic-quartic functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) &= 2[f(x+y)+f(x-y)f(-x-y)+f(y-x)] \\ &\quad -4[f(-x)+f(y)+f(-y)]-2f(x)+f(y)+f(-2y). \end{aligned} \tag{1.1}$$

They study the Ulam stability problem for (1.1) in complete random normed spaces; for fuzzy stability of (1.1) see [24].

In this article, we consider the class of mixed type additive-quadratic-quartic functional equations as follows:

$$\begin{aligned} &f(ax+by)+f(ax-by)+f(bx+ay)+f(bx-ay) \\ &=2(ab)^2[f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)] \\ &+2[f(ax)+f(bx)+f(ay)+f(by)]-(a+b)(f(y)-f(-y)) \end{aligned} \tag{1.2}$$

for the fixed integer a and any integer b such that $a, b \neq 0, \pm 1$ and $a+b \neq 0$. It is easy to check that the mapping $f(x) = \alpha x^4 + \beta x^2 + \gamma x$ is a common solution of the functional equations given in (1.2). We obtain the general solution and study the Hyers-Ulam stability of the equation (1.2) in the non-Archimedean normed spaces for the fixed integer a and any integer b such that $a, b \neq 0, \pm 1$ and $a+b \neq 0$. In other words, in the proof of our main results, we apply the fixed point method which was used for the first time by Brzdęk et al., in [8]; see also [7] and [26].

2. Solutions of (1.2)

In the sequel, by the mapping $f : X \longrightarrow Y$ satisfies the functional equation (1.2), we mean that f satisfies (1.2) for the fixed integer a and any integer b such that $a, b \neq 0, \pm 1$ and $a+b \neq 0$. Here, we firstly find out the general solutions of (1.2).

Proposition 2.1. *Let X and Y be real vector spaces. Suppose that the mapping $f : X \longrightarrow Y$ satisfies the functional equation* (1.2)*.*

(i) If f is an odd mapping, then it is additive;

(ii) If an even mapping $f : X \longrightarrow Y$ satisfies functional equation (1.2)*, then the mappings $g, h : X \longrightarrow Y$ defined by $g(x) := f(2x) - 16f(x)$ and $h(x) := f(2x) - 4f(x)$ are quadratic and quartic, respectively.*

Proof . (i) By assumption, the equation (1.2) can be rewritten as follows:

$$f(ax+by)+f(ax-by)+f(bx+ay)+f(bx-ay) \tag{2.1}$$

$$\begin{aligned} &=2(ab)^2[f(x+y)+f(x-y)-2f(x)] \\ &+2[f(ax)+f(bx)+f(ay)+f(by)]-2(a+b)f(y) \end{aligned} \tag{2.2}$$

Note that $f(0) = 0$. Replacing (x, y, b) by $(0, x, a)$ in (2.1), we get

$$f(ax) = af(x) \tag{2.3}$$

for all $x \in X$. Putting $a = b$ in (2.1) and using (2.3), we have

$$a(a^3 - 1)[f(x + y) + f(x - y) - 2f(x)] = 0 \tag{2.4}$$

for all $x, y \in X$. Since $a \neq 0, 1$, it follows from (2.4) that

$$f(x + y) + f(x - y) = 2f(x) \tag{2.5}$$

for all $x, y \in X$. Interchanging (x, y) by (y, x) in (2.5), we obtain

$$f(x + y) + f(y - x) = 2f(y) \tag{2.6}$$

for all $x, y \in X$. Now, equalities (2.5), (2.6) and the oddness of f imply that $f(x+y) = f(x)+f(y)$ for all $x, y \in X$. This means that f is an additive mapping.

(ii) We firstly note that $f(0) = 0$. Letting $y = x$ and $b = 2a$ in (1.2) and using the eveness of f, we get

$$f(3ax) + f(ax) = 4a^4[f(2x) - 4f(x)] + 2[f(ax) + f(2ax)] \tag{2.7}$$

for all $x \in X$ (and for the rest of this proof, all the equations are valid for all $x \in X$). Putting $y = x$ and $b = 3a$ in (1.2), we obtain

$$f(4ax) + f(2ax) = 9a^4[f(2x) - 4f(x)] + 2[f(ax) + f(3ax)] \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$f(4ax) + f(2ax) = 17a^4[f(2x) - 4f(x)] + 4[f(ax) + f(2ax)] \tag{2.9}$$

Replacing (b, y) by $(a, 2x)$ in (1.2), we have

$$f(3ax) + f(ax) = 2a^4[f(3x) - f(x) - 2f(2x)] + 4[f(ax) + f(2ax)] \tag{2.10}$$

Substituting (b, y) by $(a, 3x)$ in (1.2), we deduce that

$$f(4ax) + f(2ax) = a^4[f(4x) + f(2x) - 2f(x) - 2f(3x)] + 2[f(3ax) + f(ax)] \tag{2.11}$$

Plugging (2.10) into (2.11), we find

$$f(4ax) + f(2ax) = a^4[f(4x) - 3f(2x) - 4f(x)] + 4[f(ax) + f(2ax)] \tag{2.12}$$

Since $a \neq 0$, one can check that equalities (2.9) and (2.12) necessitate that

$$f(4x) - 20f(2x) + 64f(x) = 0 \tag{2.13}$$

for all $x \in X$. From relation (2.13), the desired results can be obtained. □

Corollary 2.2. *Let X and Y be real vector spaces. Then, a mapping $f : X \longrightarrow Y$ satisfies the functional equation (1.2) for all $x, y \in X$ if and only if there exist a unique additive mapping $\mathcal{A} : X \longrightarrow Y$, a unique symmetric bi-additive mapping $\mathcal{Q} : X \times X \longrightarrow Y$ and a unique symmetric multi-additive mapping $\mathfrak{Q} : X \times X \times X \times X \longrightarrow Y$ such that $f(x) = \mathcal{A}(x) + \mathcal{Q}(x, x) + \mathfrak{Q}(x, x, x, x)$ for all $x \in X$.*

Proof . Suppose that f satisfies (1.2). We decompose f into the odd part and even part by putting

$$f_o(x) = \frac{1}{2}(f(x) - f(-x)), \qquad f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad (x \in X).$$

By Proposition 2.1, $f_o(x) = \mathcal{A}(x)$. If $g, h : X \longrightarrow Y$ are mappings defined as in Proposition 2.1, then the mappings g and h are quadratic and quartic, respectively, and $f_e(x) := \frac{1}{12}(h(x) - g(x))$ for all $x, y \in X$. Thus, there exist a unique symmetric bi-additive mapping $\mathcal{Q} : X \times X \longrightarrow Y$ and a unique symmetric multi-additive mapping $\mathfrak{Q} : X \times X \times X \times X \longrightarrow Y$ such that $g(x) = -12\mathcal{Q}(x,x)$ and $h(x) = 12\mathfrak{Q}(x,x,x,x)$ for all $x \in X$ (see also the proof of [29, Theorem 2.2]). Therefore,

$$f(x) = f_o(x) + f_e(x) = \mathcal{A}(x) + \mathcal{Q}(x,x) + \mathfrak{Q}(x,x,x,x) \qquad (x \in X).$$

The proof of the converse is trivially. □

3. Stability Results for (1.2)

In this section, we prove the generalized Hyers-Ulam-Rassias stability of the mixed type additive, quadratic and quartic functional equation (1.2) in the non-Archimedean normed spaces.

We firstly recall some basic facts concerning non-Archimedean spaces and some preliminary results. By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$ (for more information about the non-Archimedean fields refer to [16]).

Let $\mathcal{X}$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : \mathcal{X} \longrightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\|, \quad (x \in \mathcal{X}, r \in \mathbb{K})$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x+y\| \leq \max\{\|x\|, \|y\|\} \qquad (x, y \in \mathcal{X}).$$

Then $(\mathcal{X}, \|\cdot\|)$ is called a *non-Archimedean normed space*. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\|; m \leq j \leq n-1\} \quad (n \geq m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $\mathcal{X}$. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

Here and subsequently, for the given mapping $f : X \longrightarrow Y$, we define the difference operators $\Delta_{a,b}f : X \times X \longrightarrow Y$ via

$$\begin{aligned}\Delta_{a,b}f(x,y) &= f(ax+by) + f(ax-by) + f(bx+ay) + f(bx-ay)\\ &- 2(ab)^2[f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)]\\ &- 2[f(ax) + f(bx) + f(ay) + f(by)] + (a+b)(f(y) - f(-y))\end{aligned}$$

for the fixed integer a and any integer b such that $a, b \neq 0, \pm 1$ and $a + b \neq 0$.

Throughout, for two sets A and B, the set of all mappings from A to B is denoted by B^A. We also denote the nonnegative real numbers by $\mathbb{R}_+$. The proof of the stability of the functional equation (1.2) in the non-Archimedean normed spaces is based on a fixed point result that can be derived from [8, Theorem 1]. To present it, we introduce the following three hypotheses:

(H1) E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j \in \mathbb{N}$, $g_1, \ldots, g_j : E \longrightarrow E$ and $L_1, \ldots, L_j : E \longrightarrow \mathbb{R}_+$,

(H2) $\mathcal{T} : Y^E \longrightarrow Y^E$ is an operator satisfying the inequality

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq \max_{i\in\{1,\ldots,j\}} L_i(x) \|\lambda(g_i(x)) - \mu(g_i(x))\|, \quad \lambda, \mu \in Y^E, x \in E,$$

(H3) $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ is an operator defined through

$$\Lambda\delta(x) := \max_{i\in\{1,\ldots,j\}} L_i(x)\delta(g_i(x)) \qquad \delta \in \mathbb{R}_+^E, x \in E.$$

Here, we bring the following theorem which is a fundamental result in fixed point theory [8]. This result plays a key tool to reach our aim in this paper.

Theorem 3.1. *Let hypotheses* (H1)-(H3) *hold and the function* $\epsilon : E \longrightarrow \mathbb{R}_+$ *and the mapping* $\varphi : E \longrightarrow Y$ *fulfill the following two conditions:*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \epsilon(x), \quad \lim_{l\to\infty} \Lambda^l \epsilon(x) = 0 \qquad (x \in E).$$

Then, for every $x \in E$, $\lim_{l\to\infty} \mathcal{T}^l\varphi(x) =: \psi(x)$ *and the function* $\psi \in Y^E$, *defined in this way, is a fixed point of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \leq \sup_{l\in\mathbb{N}_0} \Lambda^l \epsilon(x) \qquad (x \in E).$$

From now on, let X be a linear space and let Y be a complete non-Archimedean normed space over a non-Archimedean field $\mathbb{K}$ of the characteristic different from 2, unless otherwise explicitly stated. Furthermore, we assume that $|2| < 1$.

Theorem 3.2. *Let* $s \in \{-1, 1\}$ *be fixed. Suppose that* $\phi : X \times X \longrightarrow \mathbb{R}_+$ *is a mapping satisfying the equality*

$$\lim_{l\to\infty} \frac{1}{|2|^{sl}} \phi\left(2^{sl}x, 2^{sl}y\right) = 0 \tag{3.1}$$

for all $x, y \in X$. *Assume also* $f : X \longrightarrow Y$ *is an odd mapping satisfying the inequality*

$$\|\Delta_{a,b} f(x, y)\| \leq \phi(x, y) \tag{3.2}$$

for all $x, y \in X$. *Then, there exists a unique additive mapping* $\mathcal{A} : X \longrightarrow Y$ *such that*

$$\|f(x) - \mathcal{A}(x)\| \leq \sup_{l\in\mathbb{N}} \frac{1}{|2|^{\frac{s+1}{2}+sl}} \Phi\left(2^{\frac{s-1}{2}+sl}x\right) \tag{3.3}$$

for all $x \in X$, *where*

$$\Phi(x) = \frac{1}{|2a(1-a^3)|} \max\left\{\frac{1}{|2|}\phi(0, 2x), |2|\phi(0, x), \phi(x, x)\right\}. \tag{3.4}$$

Proof . We firstly note that $f(0) = 0$. Replacing (x, y, b) by $(0, x, a)$ in (3.2) and applying the oddness of f, we have

$$\|f(ax) - af(x)\| \leq \frac{1}{|4|}\phi(0, x) \tag{3.5}$$

for all $x \in X$. Putting $y = x$ and $b = a$ in (3.2), we get

$$\|2f(2ax) - 2a^4(f(2x) - 2f(x)) - 8f(ax) + 4af(x)\| \leq \phi(x, x) \tag{3.6}$$

for all $x \in X$. The inequality (3.6) can be modified as

$$\|2(f(2ax) - af(2x)) + 2a(1 - a^3)(f(2x) - 2f(x)) - 8(f(ax) - af(x))\| \leq \phi(x, x) \tag{3.7}$$

for all $x \in X$. It follows from (3.5) and (3.7) that

$$\|f(2x) - 2f(x)\| \leq \Phi(x) \tag{3.8}$$

for all $x \in X$, where $\Phi(x)$ is defined in (3.4). In other words, the inequality (3.8) is equivalent to

$$\|f(x) - \frac{1}{2^s}f(2^s x)\| \leq \frac{1}{|2|^{\frac{s+1}{2}}}\Phi\left(2^{\frac{s-1}{2}}x\right) \tag{3.9}$$

for all $x \in X$. For each $x \in X$, set

$$\mathcal{T}\xi(x) := \frac{1}{2^s}\xi(2^s x) \text{ and } \epsilon(x) := \frac{1}{|2|^{\frac{s+1}{2}}}\Phi\left(2^{\frac{s-1}{2}}x\right) \qquad (\xi \in Y^X).$$

The inequality (3.9) can be rewritten as follows:

$$\|f(x) - \mathcal{T}f(x)\| \leq \epsilon(x) \qquad (x \in X). \tag{3.10}$$

Define $\Lambda\eta(x) := \frac{1}{|2|^s}\eta(2^s x)$ for all $\eta \in \mathbb{R}_+^X, x \in X$. It is easy to see that Λ has the form described in (H3) with $E = X$, $g_1(x) := 2^s x$ for all $x \in X$ and $L_1(x) = \frac{1}{|2|^s}$. Moreover, for each $\lambda, \mu \in Y^X$ and $x \in X$, we get

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| = \left\|\frac{1}{2^s}\lambda(2^s x) - \frac{1}{2^s}\mu(2^s x)\right\| \leq L_1(x)\,\|\lambda(g_1(x)) - \mu(g_1(x))\|\,.$$

The above relation shows that the hypothesis (H2) holds. By induction on l, one can check that for any $l \in \mathbb{N}$ and $x \in X$ that

$$\Lambda^l\epsilon(x) := \frac{1}{|2|^{sl}}\epsilon\left(2^{sl}x\right) = \frac{1}{|2|^{\frac{s+1}{2}+sl}}\Phi\left(2^{\frac{s-1}{2}+sl}x\right) \tag{3.11}$$

for all $x \in X$. Relations (3.1) and (3.11) show that all assumptions of Theorem 3.1 are satisfied. Hence, there exists a unique mapping $\mathcal{A} : X \longrightarrow Y$ such that $\mathcal{A}(x) = \lim_{l\to\infty}(\mathcal{T}^l f)(x)$ for all $x \in X$, and also (3.3) holds. Clearly,

$$\|\Delta_{a,b}(\mathcal{T}^l f)(x, y)\| \leq \frac{1}{|2|^{sl}}\phi\left(2^{sl}x, 2^{sl}y\right) \tag{3.12}$$

for all $x, y \in X$ and $l \in \mathbb{N}$. Letting $l \to \infty$ in (3.12) and applying (3.1), we arrive at $\Delta_{a,b}\mathcal{A}(x, y) = 0$ for all $x, y \in X$. This means that the odd mapping $\mathcal{A}$ satisfies (1.2) and so by Proposition 2.1 it is additive. Therefore, the proof is now completed. □

In the oncoming corollary which is a direct consequence of Theorem 3.2, the norm of $\Delta_{a,b}f(x, y)$ is bounded by a positive real number δ.

Corollary 3.3. *Let δ be a positive real number. Suppose that $f : X \longrightarrow Y$ is an odd mapping satisfying the inequality*

$$\|\Delta_{a,b} f(x, y)\| \leq \delta$$

for all $x, y \in X$. Then, there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$ such that

$$\|f(x) - \mathcal{A}(x)\| \leq \frac{\delta}{|2a(1 - a^3)|} \qquad (x \in X).$$

Proof . It is enough to put $s = -1$ and $\phi(x, y) = \delta$ in Theorem 3.2. □

We have the next result which is analogous to Theorem 3.2 for functional equation (1.2) in the even case.

Theorem 3.4. *Let $s \in \{-1, 1\}$ be fixed. Suppose that $\phi : X \times X \longrightarrow \mathbb{R}_+$ is a mapping satisfying the equality*

$$\lim_{l\to\infty} \frac{1}{|2|^{2sl}} \phi\left(2^{sl}x, 2^{sl}y\right) = 0 \tag{3.13}$$

for all $x, y \in X$. Assume also $f : X \longrightarrow Y$ is an even mapping satisfying $f(0) = 0$ (in the case $s = -1$) and the inequality

$$\|\Delta_{a,b} f(x, y)\| \leq \phi(x, y) \tag{3.14}$$

for all $x, y \in X$. Then, there exists a unique quadratic mapping $\mathcal{Q} : X \longrightarrow Y$ such that

$$\|f(2x) - 16f(x) - \mathcal{Q}(x)\| \leq \sup_{l\in\mathbb{N}} \frac{1}{|2|^{s+1+2sl}} \Psi\left(2^{\frac{s-1}{2}+sl}x\right) \tag{3.15}$$

for all $x \in X$, where

$$\Psi(x) = \frac{1}{|2a^4|} \max\{(1 + |2|)\phi(x, x), |2|\phi(x, 2x), \phi(x, 3x)\} \tag{3.16}$$

Proof . For the case $s = 1$, since $|2| < 1$ is assumed, $\phi(0, 0) = 0$ by (3.13) and so $f(0) = 0$ by (3.14). Interchanging (y, b) by $(x, 2a)$ in (3.14) and using the evenness of f, we obtain

$$\|2(f(3ax) + f(ax)) - 8a^4(f(2x) - 4f(x)) - 4(f(ax) + f(2ax))\| \leq \phi(x, x) \tag{3.17}$$

for all $x \in X$. Putting $y = x$ and $b = 3a$ in (3.14), we find

$$\|2(f(4ax) + f(2ax)) - 18a^4(f(2x) - 4f(x)) - 4(f(3ax) + f(ax))\| \leq \phi(x, x) \tag{3.18}$$

for all $x \in X$. Plugging (3.17) into (3.18), we have

$$\|2(f(4ax) + f(2ax)) - 34a^4(f(2x) - 4f(x)) - 8(f(ax) + f(2ax))\| \leq (1 + |2|)\phi(x, x) \tag{3.19}$$

for all $x \in X$. Once more, replacing (y, b) by $(2x, a)$ in (3.14), we get

$$\|2(f(3ax) + f(ax)) - 2a^4(f(3x) - f(x) - 2f(2x)) - 4(f(ax) + f(2ax))\| \leq \phi(x, 2x) \tag{3.20}$$

for all $x \in X$. Setting $y = 3x$ and $b = a$ in (3.14), we arrive at

$$\|2(f(4ax) + f(2ax)) - 2a^4(f(4x) + f(2x) - 2f(x) - 2f(3x)) - 4(f(3ax) + f(ax))\| \\ \leq \phi(x, 3x) \tag{3.21}$$

for all $x \in X$. It follows from (3.20) and (3.21) that

$$\|2(f(4ax) + f(2ax)) - a^4(2f(4x) - 6f(2x) - 8f(x)) - 8(f(ax) + f(2ax))\| \\ \leq \max\{|2|\phi(x, 2x), \phi(x, 3x)\} \tag{3.22}$$

for all $x \in X$. Now, inequalities (3.19) and (3.22) imply that

$$\|f(4x) - 20f(2x) + 64f(x))\| \leq \Psi(x) \tag{3.23}$$

for all $x \in X$, where $\Psi(x)$ is defined in (3.16). Similar to the proof of Theorem 3.2, for each $x \in X$ we consider

$$\mathcal{T}\xi(x) := \frac{1}{2^{2s}}\xi(2^s x) \text{ and } \epsilon(x) := \frac{1}{|2|^{s+1}}\Psi\left(2^{\frac{s-1}{2}}x\right) \qquad (\xi \in Y^X).$$

Then, inequality (3.23) will be rewritten as

$$\|g(x) - \mathcal{T}g(x)\| \leq \epsilon(x) \qquad (x \in X), \tag{3.24}$$

where $g(x) := f(2x) - 16f(x)$. Define $\Lambda\eta(x) := \frac{1}{|2|^{2s}}\eta\,(2^s x)$ for all $\eta \in \mathbb{R}_+^X, x \in X$. Hence, the mapping Λ satisfies in (H3) with $E = X$, $g_1(x) = 2^s x$ for all $x \in X$ and $L_1(x) = \frac{1}{|2|^{2s}}$. Furthermore, for each $\lambda, \mu \in Y^X$ and $x \in X$, we have

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| = \left\|\frac{1}{2^{2s}}\lambda(2^s x) - \frac{1}{2^{2s}}\mu(2^s x)\right\| \leq \frac{1}{|2|^{2s}}\left\|\lambda(2^s x) - \mu(2^s x)\right\|.$$

Thus, the hypothesis (H2) holds. In addition, for any $l \in \mathbb{N}$ and $x \in X$ we have

$$\Lambda^l\epsilon(x) := \frac{1}{|2|^{2sl}}\epsilon\left(2^{sl}x\right) = \frac{1}{|2|^{s+1+2sl}}\Psi\left(2^{\frac{s-1}{2}+sl}x\right) \tag{3.25}$$

for all $x \in X$. It follow from (3.13) that $\lim_{l\to\infty}\Lambda^l\epsilon(x) = 0$ and hence all assumptions of Theorem 3.1 are satisfied. Therefore, there exists a unique mapping $\mathcal{Q} : X \longrightarrow Y$ defined through $\mathcal{Q}(x) = \lim_{l\to\infty}(\mathcal{T}^l f)(x)$ such that

$$\|g(x) - \mathcal{Q}(x)\| \leq \sup_{l\in\mathbb{N}}\frac{1}{|2|^{s+1+2sl}}\Psi\left(2^{\frac{s-1}{2}+sl}x\right) \qquad (x \in X).$$

In other words, the relation (3.15) holds. One can show that

$$\|\Delta_{a,b}(\mathcal{T}^l f)(x, y)\| \leq \frac{1}{|2|^{2sl}}\phi\left(2^{sl}x, 2^{sl}y\right) \tag{3.26}$$

for all $x, y \in X$ and $l \in \mathbb{N}$. Taking $l \to \infty$ in (3.26) and using (3.13), we find $\Delta_{a,b}\mathcal{Q}(x, y) = 0$ for all $x, y \in X$. Hence, the mapping $\mathcal{Q}$ satisfies (1.2). Therefore, it is quadratic by Proposition 2.1. This finishes the proof. □

We have the next result which is analogous to Theorem 3.4 for the functional equation (1.2) in another even case.

Theorem 3.5. *Let $s \in \{-1, 1\}$ be fixed. Suppose that $\phi : X \times X \longrightarrow \mathbb{R}_+$ is a mapping satisfying the equality*

$$\lim_{l\to\infty} \frac{1}{|2|^{4sl}} \phi\left(2^{sl}x, 2^{sl}y\right) = 0$$

for all $x, y \in X$. Assume also $f : X \longrightarrow Y$ is an even mapping satisfying $f(0) = 0$ (in the case $s = -1$) and the inequality (3.14)*. Then, there exists a unique quartic mapping $\mathfrak{Q} : X \longrightarrow Y$ such that*

$$\|f(2x) - 4f(x) - \mathfrak{Q}(x)\| \leq \sup_{l\in\mathbb{N}} \frac{1}{|2|^{2(s+1)+4sl}} \Psi\left(2^{\frac{s-1}{2}+sl}x\right)$$

for all $x \in X$, where $\Psi(x)$ is defined in (3.16)*.*

Proof . From the proof of Theorem 3.4, we have

$$\|f(4x) - 20f(2x) + 64f(x))\| \leq \Psi(x) \tag{3.27}$$

for all $x \in X$, where $\Psi(x)$ is defined in (3.16). For each $x \in X$, consider

$$\mathcal{T}\xi(x) := \frac{1}{2^{4s}}\xi(2^s x) \text{ and } \epsilon(x) := \frac{1}{|2|^{2(s+1)}}\Psi\left(2^{\frac{s-1}{2}}x\right).$$

We modify relation (3.27) as follows:

$$\|h(x) - \mathcal{T}h(x)\| \leq \epsilon(x) \qquad (x \in X), \tag{3.28}$$

where $h(x) := f(2x) - 4f(x)$. Define $\Lambda\eta(x) := \frac{1}{|2|^{4s}}\eta(2^s x)$ for all $\eta \in \mathbb{R}_+^X, x \in X$. Thus, the mapping Λ is as in (H3) with $E = X$, $g_1(x) = 2^s x$ for all $x \in X$ and $L_1(x) = \frac{1}{|2|^{4s}}$. The rest of the proof is similar to the proof of Theorem 3.4. □

The next corollary is a direct result of Theorems 3.4 and 3.5.

Corollary 3.6. *Let δ be a positive real number. Suppose that $f : X \longrightarrow Y$ is an even mapping satisfying the inequality*

$$\|\Delta_{a,b}f(x, y)\| \leq \delta$$

for all $x, y \in X$. Then, there exists a unique quadratic mapping $\mathcal{Q} : X \longrightarrow Y$ and a unique quartic mapping $\mathfrak{Q} : X \longrightarrow Y$ such that

$$\|f(2x) - 16f(x) - \mathcal{Q}(x)\| \leq \frac{|2|(1 + |2|)}{|a|^4}\delta$$

and

$$\|f(2x) - 4f(x) - \mathfrak{Q}(x)\| \leq \frac{|2|^3(1 + |2|)}{|a|^4}\delta, \qquad (x \in X).$$

In the upcoming result, we prove the stability for the functional equation (1.2) when f is an arbitrary mapping.

Theorem 3.7. *Let $s \in \{-1, 1\}$ be fixed. Suppose that $\phi : X \times X \longrightarrow \mathbb{R}_+$ is a mapping satisfying the equalities*

$$\lim_{l\to\infty} \frac{1}{|2|^{4l}}\phi\left(2^l x, 2^l y\right) = 0 \text{ and } \lim_{l\to\infty} |2|^l \phi\left(\frac{x}{2^l}, \frac{y}{2^l}\right) = 0 \tag{3.29}$$

for all $x, y \in X$. Assume also $f : X \longrightarrow Y$ is a mapping satisfying the inequality

$$\|\Delta_{a,b} f(x, y)\| \leq \phi(x, y)$$

for all $x, y \in X$. Then, there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$, a unique quadratic mapping $\mathcal{Q} : X \longrightarrow Y$ and a unique quartic mapping $\mathfrak{Q} : X \longrightarrow Y$ such that

$$\begin{aligned}&\|f(x) - \mathcal{A}(x) - \mathcal{Q}(x) - \mathfrak{Q}(x)\| \\ &\leq \sup_{l\in\mathbb{N}} \left\{ \frac{1}{|2|^{\frac{s+1}{2}+sl}} \widetilde{\Phi}\left(2^{\frac{s-1}{2}+sl}x\right), \frac{1}{|12|}\frac{1}{|2|^{2(s+1)+4sl}} \widetilde{\Psi}\left(2^{\frac{s-1}{2}+sl}x\right), \frac{1}{|12|}\frac{1}{|2|^{s+1+2sl}} \widetilde{\Psi}\left(2^{\frac{s-1}{2}+sl}x\right) \right\}\end{aligned}$$

for all $x \in X$, where

$$\widetilde{\Phi}(x) = \frac{1}{|2a(1 - a^3)|}\max\left\{\frac{1}{|2|}\varphi(0, 2x), |2|\varphi(0, x), \varphi(x, x)\right\} \tag{3.30}$$

and

$$\widetilde{\Psi}(x) = \frac{1}{|2a^4|}\max\left\{(1 + |2|)\varphi(x, x), |2|\varphi(x, 2x), \varphi(x, 3x)\right\} \tag{3.31}$$

in which

$$\varphi(x, y) = \frac{1}{2}[\phi(x, y) + \phi(-x, -y)]. \tag{3.32}$$

Proof . To achieve our aim, we decompose f into the even part and odd part by setting

$$f_o(x) = \frac{1}{2}(f(x) - f(-x)), \qquad f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad (x \in X).$$

We have

$$\|\Delta_{a,b} f_o(x, y)\| \leq \varphi(x, y), \quad \||\Delta_{a,b} f_e(x, y)\| \leq \varphi(x, y)$$

for all $x, y \in X$, where $\varphi(x, y)$ is given in (3.32). Since $|2| < 1$, the equalities in (3.29) imply that

$$\lim_{l\to\infty} \frac{1}{|2|^{slt}}\varphi\left(2^{sl}x, 2^{sl}y\right) = 0 \qquad (t \in \{1, 2, 4\}).$$

It follows from Theorem 3.2 that there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$ such that

$$\|f_o(x) - \mathcal{A}(x)\| \leq \sup_{l\in\mathbb{N}} \frac{1}{|2|^{\frac{s+1}{2}+sl}} \widetilde{\Phi}\left(2^{\frac{s-1}{2}+sl}x\right) \tag{3.33}$$

for all $x \in X$, where $\widetilde{\Phi}(x)$ is defined in (3.30). Furthermore, Theorems 3.4 and 3.5 imply that there exists a unique quadratic mapping $\mathcal{Q}_0 : X \longrightarrow Y$ and a unique quartic mapping $\mathfrak{Q}_0 : X \longrightarrow Y$ such that

$$\|f_e(2x) - 16f_e(x) - \mathcal{Q}_0(x)\| \leq \sup_{l\in\mathbb{N}} \frac{1}{|2|^{s+1+2sl}} \widetilde{\Psi}\left(2^{\frac{s-1}{2}+sl}x\right) \tag{3.34}$$

and

$$\|f_e(2x) - 4f_e(x) - \mathfrak{Q}_0(x)\| \leq \sup_{l\in\mathbb{N}} \frac{1}{|2|^{2(s+1)+4sl}} \widetilde{\Psi}\left(2^{\frac{s-1}{2}+sl}x\right) \tag{3.35}$$

for all $x \in X$, where $\widetilde{\Psi}(x)$ is defined in (3.31). By the inequalities (3.34) and (3.35), we have

$$\|f_e(x) - \mathcal{Q}(x) - \mathfrak{Q}(x)\| \leq \frac{1}{|12|}\sup_{l\in\mathbb{N}} \left\{\frac{1}{|2|^{2(s+1)+4sl}}\widetilde{\Psi}\left(2^{\frac{s-1}{2}+sl}x\right), \frac{1}{|2|^{s+1+2sl}}\widetilde{\Psi}\left(2^{s-1+sl}x\right)\right\} \tag{3.36}$$

for all $x \in X$, where $\mathcal{Q}(x) = -\frac{1}{12}\mathcal{Q}_0(x)$ and $\mathfrak{Q}(x) = \frac{1}{12}\mathfrak{Q}_0(x)$. Plugging relation (3.33) into (3.36), we obtain the desired result. $\square$

Corollary 3.8. *Let δ be a positive real number. Suppose that $f : X \longrightarrow Y$ is a mapping satisfying the inequality*

$$\|\Delta_{a,b}f(x, y)\| \leq \delta$$

for all $x, y \in X$. Then, there exists a unique additive mapping $\mathcal{A} : X \longrightarrow Y$, a unique quadratic mapping $\mathcal{Q} : X \longrightarrow Y$ and a unique quartic mapping $\mathfrak{Q} : X \longrightarrow Y$ such that such that

$$\|f(x) - \mathcal{A}(x) - \mathcal{Q}(x) - \mathfrak{Q}(x)\| \leq \max\left\{\frac{\delta}{|2a(1-a^3)|}, \frac{1+|2|}{|6||a|^4}\delta\right\}.$$

References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan., 2 (1950), 64–66.

[2] A. Bodaghi, Approximate mixed type additive and quartic functional equation, Bol. Soc. Paran. Mat., 35 (1) (2017), 43–56.

[3] A. Bodaghi, Stability of a mixed type additive and quartic function equation, Filomat, 28 (8) (2014), 1629-1640.

[4] A. Bodaghi, D. Kang and J. M. Rassias, The mixed cubic-quartic functional equation, An. Stiint. Univ. Al. I. Cuza Iasi Mat. (N.S.), 63 (2017), No. 1, 215–227.

[5] A. Bodaghi and S. O. Kim, Stability of a functional equation deriving from quadratic and additive functions in non-Archimedean normed spaces, Abst. Appl. Anal., 2013, Art. ID 198018, 10 pages, doi:10.1155/2013/198018.

[6] A. Bodaghi and P. Narasimman, Stability of the general form of quadratic-quartic functional equations in non-Archimedean $\mathcal{L}$-fuzzy normed spaces, Tbilisi Math. J., 11 (1) (2018), 15–29.

[7] J. Brzdęk, Stability of additivity and fixed point methods, Fixed Point Theory Appl., 2013, 2013:285, 9 pp.

[8] J. Brzdęk and K. Ciepliński, A fixed point approach to the stability of functional equations in non-Archimedean metric spaces, Nonlinear Anal., 74 (2011), 6861–6867.

[9] Iz. El-Fassi, A new type of approximation for the radical quintic functional equation in non-Archimedean (2,β)-Banach spaces, J. Math. Anal. Appl., 457 (2018), 322–335.

[10] Iz. El-Fassi, H. Khodaie and Th. M. Rassias, Approximate Solution of a p-th root functional equation in non-Archimedean (2,β)-Banach spaces, Acta Math. Sci., 39B(2) (2019), 369–381.

[11] M. Eshaghi Gordji, H. Khodaei and Hark-Mahn Kim, Approximate quartic and quadratic mappings in quasi-Banach spaces, Int. J. Math. Math., Sci., 2011, Art. ID 734567, 18 pages, doi:10.1155/2011/734567.

[12] M. Eshaghi Gordji, Stability of a functional equation deriving from quartic and additive functions, Bull. Korean Math. Soc., 47, No. 3 (2010), 491–502.

[13] M. Eshaghi Gordji, S. Kaboli Gharetapeh, J. M. Rassias and S. Zolfaghari, Solution and stability of a mixed type additive, quadratic, and cubic functional equation, Abstr. Appl. Anal., 2009, Art. ID 826130, doi:10.1155/2009/826130

[14] M. Eshaghi Gordji, M. B. Savadkouhi, J. M. Rassias, S. Zolfaghari, Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces, Abstr. Appl. Anal., 2009, Art. ID 417473, 14 pages.

[15] S. Falihi, B. Shojaee, A. Bodaghi and A. Zivari-Kazempour, Approximation on the mixed type additive-quadratic-sextic functional equation, U.P.B. Sci. Bull., Series A, 81 (2019), no. 3, 13–22.

[16] K. Hensel, Uber eine neue Begrndung der Theorie der algebraischen Zahlen, Jahresber, Deutsche Mathematiker-Vereinigung, 6 (1897), 83–88.

[17] V. Ho and L. S. Dong, Ulam-Hyers-Rassias stability for stochastic integral equations of Volterra type, Int. J. Nonlinear Anal. Appl., 10 (2019) No. 2, 213-225.

[18] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA., 27 (1941), 222–224.

[19] Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer, 2009.

[20] K. Kucche and P. Shikhare, Ulam stabilities for nonlinear Volterra-Fredholm delay integrodifferential equations, Int. J. Nonlinear Anal. Appl., 9 (2018) No. 2, 145-159.

[21] M. Maghsoudi, A. Bodaghi, A. Niazi Motlagh and M. Karami, Almost additive-quadratic-cubic mappings in modular spaces, Revista de la Union Mat. Arg., 60 (2019), No. 2, 359–379.

[22] M. Mohamadi,Y. J. Cho, C. Park, P. Vetro, and R. Saadati, Random stability of an additive-quadratic-quartic functional equation, J. Inequ. Appl., 2010, Article ID 754210, 18 pp. doi:10.1155/2010/754210.

[23] P. Narasimman and A. Bodaghi, Solution and stability of a mixed type functional equation, Filomat, 31 (5) (2017), 1229–1239.

[24] C. Park, Fuzzy stability of an additive-quadratic-quartic functional equation, J. Inequ. Appl., 2010, Article ID 253040, 22 pp. doi:10.1155/2010/253040.

[25] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72 (2) (1978) 297–300.

[26] S. Salimi and A. Bodaghi, A fixed point application for the stability and hyperstability of multi-Jensen-quadratic mappings, J. Fixed Point Theory Appl., (2020), 22:9, https://doi.org/10.1007/s11784-019-0738-3

[27] F. Skof, Propriet- locali e approssimazione di operatori, Rend. Sem. Mat. Fis. Milano, 53 (1983), 113–129.

[28] S. M. Ulam, Problems in Modern Mathematics, Chapter VI, Science Ed., Wiley, New York, 1940.

[29] T. Z. Xu, J. M. Rassias, and W. X. Xu, A generalized mixed quadratic-quartic functional equation, Bull. Malay. Math. Sci. Soc., 35 (2012), No. 3, 633–649.

[30] T. Z. Xu, M. J. Rassias and W. X. Xu, Stability of a general mixed additive-cubic functional equation in non-Archimedean fuzzy normed spaces, J. Math. Physics, 51 (2010), Art. 093508, 19 pages.

[31] G. Zamani Eskandani, H.Vaezi and Y. N. Dehghan, Stability of a mixed additive and quadratic functional equation in non-Archimedean Banach modules, Taiwanese J. Math., 14 (2010), No. 4, 1309–1324.