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Some functional inequalities in variable exponent spaces with a more generalization of uniform continuity condition

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Abstract

Some functional inequalities in variable exponent Lebesgue spaces are presented. The bi-weighted modular inequality with variable exponent $p(.)$ for the Hardy operator restricted to non-increasing function which is

$$\int_0^\infty (\frac{1}{x}\int_0^x f(t)dt)^{p(x)} v(x)dx \le C\int_0^\infty f(x)^{p(x)}u(x)dx,$$

is studied. We show that the exponent $p(.)$ for which these modular inequalities hold must have constant oscillation. Also we study the boundedness of integral operator $Tf(x) = \int K(x,y)f(x)dy$ on $L^{p(.)}$ when the variable exponent $p(.)$ satisfies some uniform continuity condition that is named β-controller condition and so multiple interesting results which can be seen as a generalization of the same classical results in the constant exponent case, derived.

Keywords: Hardy type inequality; Variable exponent Lebesgue space; Modular type inequality.
2010 MSC: Primary 35A23; Secondary 34A40.

1. Introduction

In any literature on variable exponent functional inequalities, there is a noticeable attention to the boundedness of Hardy type inequalities.

In 2001, Pick and Ruzicka, proved that the uniform continuity condition on p which is

$$|p(x) - p(y)| \le \frac{C}{-\ln|x-y|}, \qquad |x-y| \le \frac{1}{2}; \tag{1.1}$$

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is quite necessary for Hardy-Littelwood maximal operator (H.L.M.O) to be bounded on $L^{p(\cdot)}$; [8]. So this condition appears to be natural in the study of variable exponent $L^{p(\cdot)}$ spaces. The main property to consider the condition (1.1) is that the uniform continuity condition on p is equivalent by $|B|^{p_B^- - p_B^+} \leq C$ for all open ball B, some fixed constant $C > 0$, $p_B^+ = ess\sup_{x\in B}\ p(x)$ and $p_B^- = ess\inf_{x\in B}\ p(x)$.

In 2003, Cruze-uribe, Fiorenza and Neugebauer, obtained that uniform continuity condition (1.1) and moreover

$$|p(x) - p(y)| \leq \frac{C}{\ln(e + |x|)}; \qquad x, y \in \Omega, \quad |y| \geq |x|, \tag{1.2}$$

is sufficient for the boundedness of H.L.M.O on $L^{p(\cdot)}(\Omega)$, where Ω is any open subset of $\mathbb{R}^n$, and is not necessarily bounded; [2].

By overcoming to the boundedness of H.L.M.O, the boundedness of varied operators such as potential type and fractional operators in the space $L^{p(\cdot)}$ is now, one of the interesting topics in related to this generalized type of Lebesgue spaces, for example see [4, 6, 7, 9]. In this paper firstly we study a weighted modular inequalities with variable exponent for the Hardy operator restricted to non-increasing function which is

$$\int_0^\infty (\frac{1}{x}\int_0^x f(t)dt)^{p(x)} v(x)dx \leq C\int_0^\infty f(x)^{p(x)}u(x)dx \tag{1.3}$$

We show that the exponent $p(.)$ for which these modular inequalities hold must have constant oscillation, i.e., $\varphi_{p(.),u}(\delta) = p^+_{B_\delta\cap sptu} - p^-_{B_\delta\cap sptu}$ should be constant where $B_\delta = (0, \delta)$. This result generalized the main result of Boza and Soria in [1]. After wards by introducing a more generalization of uniform continuity on p to $0 < p(x) - p(y) < \alpha(x)$ which α has β-controller condition that introduced in the following we can derive the corresponding norm inequality of (1.3) with $u = v = 1$ which p has no necessarily constant oscillation. Also we generalize the classic general theorems about the boundedness of the integral operator $Tf(x) = \int K(x, y)f(x)dy$ on $L^{p(\cdot)}$ under some appropriate conditions on $p(.)$ and $K(.,.)$ which theorem 3.4 to 3.6 appertain to this. Finally we conclude two special weighted integral inequalities

$$\int_0^\infty (\int_0^y y^\nu (\frac{x}{x + y^\mu})^{\frac{1}{p^-}} f(x)dx)^{p(y)}dy \leq C\int_0^\infty f(x)^{p(x)}dx,$$

where $-\frac{1}{p^+} < \frac{\nu+1}{1-\mu} < 0$ and

$$\int_0^\infty (\int_0^y y^\nu (\frac{x}{y^\mu})^{\lambda-\theta} e^{-\frac{x}{\lambda y^\mu}} f(x)dx)^{p(y)}dy \leq C\int_0^\infty f(x)^{p(x)}dx;$$

where $\theta = \frac{\nu+1}{1-\mu}$ and $\lambda > \frac{1}{p^-}$ and C is independent on f in both of them.

2. Preliminaries

We refer to [3] for the basic information about variable exponent spaces. But we mention briefly, some of the main properties of variable exponent Lebesgue spaces that are used in the following. Let Ω be an open subset of $\mathbb{R}^N$ with $N \geq 2$, $p \in L^\infty(\Omega)$ and $p^- \geq 1$. The variable exponent Lebesgue space $\mathbf{L}^{p(\cdot)}(\Omega)$ is defined by

$$\mathbf{L}^{p(\cdot)}(\Omega) = \{u : \ u : \Omega \longrightarrow \mathbb{R} \ \ is\ measurable, \int_\Omega |u|^{p(x)}dx < \infty\};$$

which is considered by the norm $|u|_{\mathbf{L}^{p(\cdot)}(\Omega)} = \inf\{\sigma > 0 : \ \int_\Omega |\frac{u}{\sigma}|^{p(x)}dx \leq 1\}$.

We summarize the main properties of $L^{p(\cdot)}(\Omega)$ by the following items:

(i) The space $(\mathbf{L}^{p(x)}(\Omega), |.|_{\mathbf{L}^{p(x)}(\Omega)})$ is a separable, uniform convex Banach space, and its conjugate space is $\mathbf{L}^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $u \in \mathbf{L}^{p(x)}(\Omega)$ and $v \in \mathbf{L}^{q(x)}(\Omega)$, we have

$$|\int_\Omega uvdx| \leq (\frac{1}{p^-} + \frac{1}{q^-})|u|_{\mathbf{L}^{p(x)}(\Omega)}|u|_{\mathbf{L}^{q(x)}(\Omega)};$$

which we named generalized Holder inequality.

(ii) If Ω is bounded, $p_1, p_2 \in C(\overline{\Omega})$ and $1 < p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then there is a continuous embedding $\mathbf{L}^{p_2(x)}(\Omega) \hookrightarrow \mathbf{L}^{p_1(x)}(\Omega)$.

(iii)

$$\min(|u|^{p^-}_{\mathbf{L}^{p(.)}(\Omega)}, |u|^{p^+}_{\mathbf{L}^{p(.)}(\Omega)}) \leq \rho(u) \leq \max(|u|^{p^-}_{\mathbf{L}^{p(.)}(\Omega)}, |u|^{p^+}_{\mathbf{L}^{p(.)}(\Omega)});$$

which $\rho(u) = \int_\Omega |u|^{p(x)}dx$ and $p^+ := ess\sup_{x\in\Omega} p(x)$.

Now we state some new definition which is named β-controller condition.

Definition 2.1. Let α is a measurable function on Ω and $\beta \in L^{p(.)}(\Omega)$ with $\beta^+ < 1$. We say that α admits the β-controller condition on $(\Omega; p(.))$ provided $\beta(x)^{-\alpha(x)} \in L^\infty(\Omega)$.

For example $\frac{1}{\ln x}$ admits the $\frac{1}{x}$- controller condition on $((\frac{3}{2}, +\infty); 2)$.

3. The main results

Let $p : (0, \infty) \to (0, \infty)$ such that $0 < p^- \leq p^+ < \infty$ and a positive weight function u, moreover let $B_\delta = (0, \delta)$ and $\varphi_{p(.),u}(\delta) = p^+_{B_\delta \cap sptu} - p^-_{B_\delta \cap sptu}$, which is called, local oscillation of p; then we have the following theorem.

Theorem 3.1. [1]. If there exists a positive constant C such that the inequality

$$\int_0^\infty (\frac{1}{x}\int_0^x f(t)dt)^{p(x)}u(x)dx \leq C\int_0^\infty f(x)^{p(x)}u(x)dx \tag{3.1}$$

for any positive and non-increasing function f to be held then p has necessarily constant local oscillation.

By the same method similar to [1] we can generalize the above result, as following theorem which two weighted functions are involved.

Theorem 3.2. *Let u, v be two positive weight functions and $p : (0, \infty) \longrightarrow (0, \infty)$ such that $0 < p^- \leq p^+ < \infty$. If there exists a positive constant C such that*

$$\int_0^\infty (\frac{1}{x}\int_0^x f(t)dt)^{p(x)}v(x)dx \leq C\int_0^\infty f(x)^{p(x)}u(x)dx, \tag{3.2}$$

then for any $r, s > 0$ we have

$$\int_r^\infty (\frac{r}{s})^{p(x)}v(x)dx \leq C\int_0^r (\frac{1}{s^{p(x)}})(u(x) + x^{p(x)}v(x))dx,$$

and so p has constant local oscillation.

Proof . For $r, s > o$ let $f_{r,s}(x) = \frac{1}{s}\chi_{(0,r)}(x)$. Then (3.2) is equivalent by

$$\int_0^r (\frac{x}{s})^{p(x)} v(x)dx + \int_r^{+\infty} (\frac{r}{s})^{p(x)} v(x)dx \leq C \int_0^r (\frac{1}{s})^{p(x)} u(x)dx. \tag{3.3}$$

If p has no constant local oscillation then there exists $\delta_1 > 0$ such that for

$$\alpha = esssup\{p(x); x \in (0, \delta_1) \cap (sptu \cup sptv)\}$$

and

$$p^+_{u,v} = esssup\{p(x); x \in (0, +\infty) \cap (sptu \cup sptv)\}$$

we have $\alpha < p^+_{u,v}$ or there exists $\delta_2 > 0$ such that for

$$\beta = essinf\{p(x); x \in (0, \delta_2) \cap (sptu \cup sptv)\}$$

and

$$p^-_{u,v} = essinf\{p(x); x \in (0, +\infty) \cap (sptu \cup sptv)\}$$

we have $\beta > p^-_{u,v}$.

When $\alpha < p^+_{u,v}$ then for $\varepsilon > 0$, $|\{x \geq \delta_1; x \in sptu \cup sptv, p(x) \geq \alpha + \varepsilon\}| > 0$. Now by letting $r = \delta_1$ and $s < \min\{1, \delta_1\}$ in (3.3) we obtain

$$(\frac{\delta_1}{s})^{\alpha+\varepsilon} \int_{x \geq \delta_1, p(x) > \alpha+\varepsilon} v(x)dx \leq C(\frac{1}{s})^{\alpha} \int_0^{\delta_1} (u(x) + x^{p(x)} v(x))dx$$

which by tending $s \to 0$ get to contradiction. By similar arguments when $\beta > p^-_{u,v}$ for $\varepsilon > 0$ we have $|\{x \geq \delta_2; x \in sptu \cup sptv, p(x) \leq \beta - \varepsilon\}| > 0$. Now by letting $r = \delta_2$ and $s > \max\{1, \delta_2\}$ in (3.3) we deduce

$$(\frac{\delta_2}{s})^{\beta-\varepsilon} \int_{x \geq \delta_2, p(x) < \beta-\varepsilon} v(x)dx \leq C(\frac{1}{s})^{\beta} \int_0^{\delta_2} (u(x) + x^{p(x)} v(x))dx$$

which by tending $s \to \infty$ get to contradiction. $\square$

When $p > 1$ is a constant value, the modular inequality $\int_\Omega f(x)^p dx \leq C \int_\Omega g(x)^p dx$ and the norm inequality $\|f\|_{L^p(\Omega)} \leq C\|g\|_{L^p(\Omega)}$ are equivalent for any domain Ω, whereas these are not true for general function $p(.) > 1$. When $p : \Omega \to [1, \infty)$ is not constant function, the modular inequality shows the corresponding norm inequality but the inverse is not true in general case. So from this point of view, we can consider the next theorem that present the norm inequality corresponding to (3.1) for some p with not necessarily constant local oscillation.

Theorem 3.3. *Suppose that $p : (0, +\infty) \to [1, \infty)$, α is measurable function with β-controller condition on $((0, +\infty); p(.))$ where*

$$p(y) - p^-(B_x) < \alpha(y); \qquad y \leq (0, x], \tag{3.4}$$

and

$$\alpha(y) \leq \frac{c}{|\ln y|}, \quad 0 < y < 1; \tag{3.5}$$

where c is a positive constant and $p^-(B_x) = p^-_{B_x} = essinf\{p(y); y \in B_x = (0, x)\}$. Then the Hardy operator $Sf(x) = \frac{1}{x}\int_0^x f(y)dy$ is bounded in $L^{p(\cdot)}(0, +\infty)$, i.e.; there exist $C > 0$ such that

$$\|Sf\|_{L^{p(\cdot)}(0,\infty)} \leq C\|f\|_{L^{p(\cdot)}(0,\infty)}; \qquad f \in L^{p(\cdot)}(0, +\infty).$$

Proof . Fix $f \in L^{p(.)}(0,\infty)$ with $\|f\| = \|f\|_{L^{p(.)}(0,\infty)} = 1$. Let $\tilde{p}(t) := \frac{p(t)}{p^-}$ then for any $x \in (0,+\infty)$, $\tilde{p}^-(B_x) \geq 1$. By applying Jensen inequality we obtain

$$\begin{aligned} I &:= \int_0^\infty (\frac{1}{x}\int_0^x f(y)dy)^{p(x)}dx \leq \int_0^\infty (\frac{1}{x}\int_0^x f(y)^{\tilde{p}^-(B_x)}dy)^{\frac{p(x)}{\tilde{p}^-(B_x)}}dx \\ &= \int_0^\infty (\frac{1}{x}(\int_{B_x^+} + \int_{B_x^-})f(y)^{\tilde{p}^-(B_x)}dy)^{\frac{p(x)}{\tilde{p}^-(B_x)}}dx \\ &\leq C[\int_0^\infty (\frac{1}{x}\int_{B_x^+} f(y)^{\tilde{p}^-(B_x)}dy)^{\frac{p(x)}{\tilde{p}^-(B_x)}}dx + \int_0^\infty (\frac{1}{x}\int_{B_x^-} f(y)^{\tilde{p}^-(B_x)}dy)^{\frac{p(x)}{\tilde{p}^-(B_x)}}dx] \\ &:= I_1 + I_2 \end{aligned}$$

where $B_x^+ := B_x \cap \{y; f(y) \geq \beta(y)\}$ and $B_x^- := B_x \cap \{y; f(y) < \beta(y)\}$. Since $\|f\| = 1$, when $x < 1$, by applying Holder inequality we obtain

$$\int_0^x f(y)^{\tilde{p}(y)}dy < x^{\frac{p^- - 1}{p^-}}(\int_0^x f(y)^{p(y)}dy)^{\frac{1}{p^-}} \leq 1;$$

and for $x > 1$ we have

$$\frac{1}{x}\int_0^x f(y)^{\tilde{p}(y)}dy \leq (\frac{1}{x}\int_0^x f(y)^{p(y)}dy)^{\frac{1}{p^-}} \leq 1.$$

Hence

$$\begin{aligned} I_1 &\leq \int_0^\infty (\frac{1}{x}\int_{B_x^+} f(y)^{\tilde{p}(y)}\beta(y)^{\tilde{p}^-(B_x)-\tilde{p}(y)}dy)^{\frac{p(x)}{\tilde{p}^-(B_x)}}dx \leq C\int_0^\infty (\frac{1}{x}\int_{B_x^+} f(y)^{\tilde{p}(y)}dy)^{\frac{p(x)}{\tilde{p}^-(B_x)}}dx \\ &= C[\int_0^1 (\frac{1}{x}\int_{B_x^+} f(y)^{\tilde{p}(y)}dy)^{\frac{p(x)}{\tilde{p}^-(B_x)}}dx + \int_1^\infty (\frac{1}{x}\int_{B_x^+} f(y)^{\tilde{p}(y)}dy)^{\frac{p(x)}{\tilde{p}^-(B_x)}}dx] \\ &= C\ [\int_0^1 (\frac{1}{x})^{\frac{p(x)}{\tilde{p}^-(B_x)}-p^-}(\frac{1}{x}\int_0^x f(y)^{\tilde{p}(y)}dy)^{p^-}dx + \int_1^\infty (\frac{1}{x}\int_0^x f(y)^{\tilde{p}(y)}dy)^{p^-}dx]; \end{aligned}$$

By using the growth conditions on p, we observe

$$(\frac{1}{x})^{\frac{p(x)}{\tilde{p}^-(B_x)}-p^-} \leq (\frac{1}{x})^{(p(x)-p^-(B_x))} \leq (\frac{1}{x})^{\frac{C}{|\ln x|}} \leq C.$$

So $I_1 \leq C\int_0^\infty (\frac{1}{x}\int_0^x f(y)^{\tilde{p}(y)}dy)^{p^-}dx$. On the other hand,

$$I_2 \leq \int_0^\infty (\frac{1}{x}\int_{B_x^-} \beta(y)^{\tilde{p}^-(B_x)}dy)^{\frac{p(x)}{\tilde{p}^-(B_x)}}dx \leq C\int_0^\infty \beta(x)^{p(x)}dx.$$

Thus

$$I \leq C\int_0^\infty (\frac{1}{x}\int_0^x f(y)^{\tilde{p}(y)}dy)^{p^-}dx + C\int_0^\infty \beta(x)^{p(x)}dx;$$

which is bounded by using the boundedness of Hardy operator in $L^{p^-}(0,\infty)$ and considering $\beta \in L^{p(.)}(0,\infty)$. Hence $\|Sf\| \leq C$ for any $f \in L^{p(x)}(0,\infty)$ with $\|f\| = 1$. Now suppose $f \in L^{p(.)}(0,\infty)$ with $\|f\| \neq 1$ then define $g = \frac{f}{\|f\|}$ so $g \in L^{p(.)}(0,\infty)$ and $\|g\| = 1$. Hence by the first part of the proof we have $\|Sf\| = \|\|f\|Sg\| = \|f\|\|Sg\| \leq C\|f\|$. $\square$

When p is constant , and K is a measurable function on $\Omega_1 \times \Omega_2 \subset \mathbb{R}^N \times \mathbb{R}^N$, with $\int_{\Omega_1} |K(x,y)|dx \leq C$ for a.e $y \in \Omega_2$ and $\int_{\Omega_2} |K(x,y)|dy \leq C$ for a.e $x \in \Omega_1$; then the integral operator $Tf(x) = \int K(x,y)f(y)dy$ is converge absolutely for a.e $x \in \Omega_2$ and $\|Tf\|_{L^p(\Omega_1)} \leq C\|f\|_{L^p(\Omega_2)}$; [5]. Now we present similar result for non constant p.

Theorem 3.4. *Let Ω_1,Ω_2 be two measurable subsets of $\mathbb{R}^N$. $p:\mathbb{R}^N \to [1,\infty)$ and α is a measurable function with β- controller condition on $(\Omega_1; p(.))$ where*

$$0 < p(x) - p(y) < \alpha(x)$$

for any $x \in \Omega_1$, $y \in \Omega_2$. K is a measurable function on $\Omega_1 \times \Omega_2$ and there exist $C_1 > 0$ such that $\int |K(x,y)|dy \leq C_1$ for a.e. $x \in \Omega_1$ and $\int |K(x,y)|dx \leq C_1$ for a.e. $y \in \Omega_2$. If $f \in L^{p(.)}(\Omega_1)$, the integral operator

$$Tf(y) = \int_{\Omega_1} K(x,y)f(x)dx$$

converges absolutely for a.e. $y \in \Omega_2$ and $\|Tf\|_{L^{p(.)}(\Omega_2)} \leq C\|f\|_{L^{p(.)}(\Omega_1)}$.

Proof . By applying Holder inequality for fixed $y \in \Omega_2$ we have,

$$I := \int_{\Omega_2} (\int_{\Omega_1} |K(x,y)f(x)|dx)^{p(y)}dy \leq \int_{\Omega_2} (\int_{\Omega_1} |K(x,y)|dx)^{\frac{p(y)}{p'(y)}} (\int_{\Omega_1} |K(x,y)f(x)^{p(y)}|dx)dy.$$

Let $\Omega_1^+ := \Omega_1 \cap \{x; f(x) \geq \beta(x)\}$ and $\Omega_1^- := \Omega_1 \cap \{x; f(x) < \beta(x)\}$. So

$$I \leq C\int_{\Omega_2} (\int_{\Omega_1} |K(x,y)f(x)^{p(y)}|dx)dy = C\int_{\Omega_2} (\int_{\Omega_1^+} + \int_{\Omega_1^-})(|K(x,y)f(x)^{p(y)}|dx)dy =: I_1 + I_2.$$

Thus

$$\begin{aligned} I_1 &= \textstyle\int_{\Omega_2}\int_{\Omega_1^+} |K(x,y)f(x)^{p(x)}||f(x)|^{p(y)-p(x)}dxdy \leq C\int_{\Omega_1^+} |f(x)|^{p(x)}(\int_{\Omega_2} |K(x,y)|dy)dx \\ &\leq C\textstyle\int_{\Omega_1} |f(x)|^{p(x)}dx. \end{aligned}$$

And

$$I_2 < \int_{\Omega_2}\int_{\Omega_1^-} |K(x,y)\beta(x)^{p(y)}|dxdy \leq C\int_{\Omega_1^-} |\beta(x)|^{p(x)}(\int_{\Omega_1} |K(x,y)|dy)dx \leq C.$$

Hence $\int_{\Omega_2}(\int_{\Omega_1} |K(x,y)f(x)|dx)^{p(y)}dy \leq C\int_{\Omega_1} |f(x)|^{p(x)}dx$. $\square$

In the case p is a constant value and K is a Lebesgue measurable function on $(0,\infty) \times (0,\infty)$, such that $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$, for all $\lambda > 0$; $Tf(x) = \int_0^\infty K(x,y)f(y)dy$ is bounded operator on $L^p(0,\infty)$ provided that $\int_0^\infty K(x,y)x^{-\frac{1}{p}}dx < \infty$; [5]. In the following theorem by additional assumptions we present a counterpart result in variable exponent case.

Theorem 3.5. *Let K be a measurable function on $(0,\infty)\times(0,\infty)$ such that $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$ for all $\lambda > 0$ and for any $y > 0$, $K(.,y) \in L^{q^+}(0,\infty)$ (q is conjugate exponent of p).*

Suppose $\{w_i\}_{i=1}^n$ is a finite family of measurable sets which is a finite cover for $[0,\infty)$, i.e.; $[0,\infty) \subset \cup_{i=1}^n w_i$, $p:\mathbb{R} \to [1,\infty)$ is a measurable function such that

$$p(y) - p_i \leq \frac{c}{\ln(1+r_y)} \qquad \textit{for all} \quad y \in w_i, \quad 1 \leq i \leq n; \tag{3.6}$$

where $p_i = \inf_{y\in w_i} p(y)$ and $r_y = \sup\{x; K(x,y) \leq 1\} < \infty$.

Define $Tf(y) = \int_0^\infty K(x,y)f(x)dx$, then T is bounded operator from $L^{p(.)}(0,\infty) \cap_{i=1}^n L^{p_i}(0,\infty)$ in to $L^{p(.)}(0,\infty)$ provided that

$$\int_0^\infty K(t,1)(t^{-\frac{1}{p^-}}\chi_{(0,1)}(t) + t^{-\frac{1}{p^+}}\chi_{(1,\infty)}(t))dt < \infty.$$

Proof . It is enough to prove the boundedness of modular operator $\rho_{L^{p(.)}(o,\infty)}(Tf(.))$.

$$\rho_{L^{p(.)}(o,\infty)}(Tf(.)) = \int_0^\infty (\int_0^\infty |K(x,y)f(x)|dx)^{p(y)}dy$$
$$\leq \sum_{i=1}^n \int_{w_k} (\int_0^\infty |K(x,y)f(x)|dx)^{p(y)-p_i+p_i}dy. \tag{3.7}$$

Let us to show the uniform estimate below,

$$I_{i,y} := (\int_0^\infty |K(x,y)f(x)|dx)^{p(y)-p_i} \leq C; \qquad \text{for any } y \in w_i,\ 1 \leq i \leq n. \tag{3.8}$$

We have,

$$\int_0^\infty |K(x,y)f(x)|dx \leq C\|K(.,y)\|_{q(.)}\|f\|_{p(.)} \leq C(\int_0^\infty |K(x,y)|^{q(x)}dx)^\theta$$
$$\leq C(r_y + \|K(.,y)\|^{q^+}_{L^{q^+}})^\theta \leq C(r_y+1)^\theta$$

where $\frac{1}{q^+} \leq \theta \leq \frac{1}{q^-}$. Thus

$$I_{i,y} \leq C(r_y+1)^{\theta(p(y)-p_i)} \leq C(r_y+1)^{\frac{c\theta}{\ln(1+r_y)}} \leq C.$$

Using the estimate (3.8) in (3.7) we obtain

$$\rho_{L^{p(.)}(o,\infty)}(Tf(.)) \leq C\sum_{i=1}^n \int_{w_i} (\int_0^\infty |K(x,y)f(x)|dx)^{p_i}dy.$$

For any i, p_i is constant, so we may apply the Minkowski inequality for integrals after an appropriate change of variable. Indeed,

$$J := \sum_{i=1}^n \int_{w_i}(\int_0^\infty |K(x,y)f(x)|dx)^{p_i}dy = \sum_{i=1}^n \int_{w_i}(\int_0^\infty |K(t,1)f(ty)|dt)^{p_i}dy$$
$$\leq \sum_{i=1}^n(\int_0^\infty(\int_{w_i} K(t,1)^{p_i}f(ty)^{p_i}dy)^{\frac{1}{p_i}}dt)^{p_i} = \sum_{i=1}^n(\int_0^\infty K(t,1)t^{\frac{-1}{p_i}}(\int_{tw_i} f(x)^{p_i}dx)^{\frac{1}{p_i}}dt)^{p_i}$$
$$\leq (\int_0^\infty(K(t,1)(t^{-\frac{1}{p^-}}\chi_{(0,1)}(t) + t^{-\frac{1}{p^+}}\chi_{(1,\infty)}(t))dt)^\theta(\sum_{i=1}^n \int_0^\infty f(x)^{p_i}dx) \leq C;$$

which $p^- \leq \theta \leq p^+$. □

Theorem 3.6. *Let K be a measurable function on $[0,\infty)\times[0,\infty)$ which $K(y^\mu z, y) = y^\nu K(z,1)$ for fixed $\mu > 1, \nu > -1$. Suppose that for any x, y which $0 < x < y$; $0 < p(x) - p(y) < \alpha(x)$ is held where α is a measurable function with β- controller on $([0,\infty); p(.))$.*

Consider the integral operator

$$Tf(y) = \int_0^y |K(x,y)f(x)|dx.$$

Then Tf is defined a.e. and T is bounded operator in $L^{p(.)}(0,\infty)$ provided that

$$\int_0^\infty (\frac{(z^{\frac{\nu+1}{1-\mu}}K(z,1))^{p^+} + (z^{\frac{\nu+1}{1-\mu}}K(z,1))^{p^-}}{z})dz < \infty. \tag{3.9}$$

Proof . By applying Jensen inequality we have

$$\begin{aligned} I &:= \int_0^\infty (\int_0^y |K(x,y)f(x)|dx)^{p(y)}dy \\ &= \int_0^\infty (\frac{1}{y}\int_0^y |yK(x,y)f(x)|dx)^{p(y)}dy \le \int_0^\infty \frac{1}{y}\int_0^y |yK(x,y)f(x)|^{p(y)}dxdy. \end{aligned}$$

Let $\Omega_y^+ := [0,y] \cap \{x; f(x) \ge \beta(x)\}$ and $\Omega_y^- := [0,y] \cap \{x; f(x) < \beta(x)\}$. So

$$I := \int_0^\infty \frac{1}{y}(\int_{\Omega_y^+} + \int_{\Omega_y^-})|yK(x,y)f(x)|^{p(y)}dxdy =: I_1 + I_2.$$

for the first equality we have,

$$I_1 \le C\int_0^\infty \frac{1}{y}\int_{\Omega_y^+} |yK(x,y)|^{p(y)}|f(x)|^{p(x)}dxdy.$$

Use the change of variable $x := y^\mu z$ thus

$$\begin{aligned} I_1 &\le C\int_0^\infty \tfrac{1}{y}\int_0^{y^{1-\mu}} |yK(y^\mu z,y)|^{p(y)}|f(y^\mu z)|^{p(y^\mu z)}y^\mu dzdy \\ &\le C\int_0^\infty \int_0^{z^{\frac{1}{1-\mu}}} |y^{\nu+1}K(z,1)|^{p(y)}|f(y^\mu z)|^{p(y^\mu z)}y^{\mu-1}dydz \\ &\le C\int_0^\infty \int_0^{z^{\frac{1}{1-\mu}}} |(\tfrac{t}{z})^{\frac{\nu+1}{\mu}}K(z,1)|^{p(\sqrt[\mu]{\frac{t}{z}})}|f(t)|^{p(t)}\tfrac{1}{\mu z}dtdz. \end{aligned}$$

Since $\frac{\nu+1}{\mu} > 0$ and $t < z^{\frac{1}{1-\mu}}$ the last inequality leads to

$$\le C\int_0^\infty (\frac{(z^{\frac{\nu+1}{1-\mu}}K(z,1))^{p^+} + (z^{\frac{\nu+1}{1-\mu}}K(z,1))^{p^-}}{z})dz\int_0^\infty f(t)^{p(t)}dt \le C\int_0^\infty f(t)^{p(t)}dt.$$

For the second integral, I_2 we have

$$I_2 \le \int_0^\infty \tfrac{1}{y}\int_{\Omega_y^-} |yK(x,y)f(x)|^{p(y)}dxdy \le \int_0^\infty \tfrac{1}{y}\int_{\Omega_y^-} |yK(x,y)|^{p(y)}|\beta(x)|^{p(y)}dxdy$$

$$\le C\int_0^\infty \tfrac{1}{y}\int_{\Omega_y^-} |y^{\nu+1}K(\tfrac{x}{y^\nu},1)|^{p(y)}|\beta(x)|^{p(x)}dxdy$$

Now, by replacement $\frac{x}{y^\mu} =: z$, change the order of integral and so let $y^\mu z = t$ we have ;

$$I_2 \le C\int_0^\infty \int_0^{z^{\frac{1}{1-\mu}}} |(\frac{t}{z})^{\frac{\nu+1}{\mu}}K(z,1)|^{p(\sqrt[\mu]{\frac{t}{z}})}|\beta(t)|^{p(t)}\frac{1}{\mu z}dtdz.$$

Thus by hypothesis on β and (3.9), and the same discussion as in the first integral we obtained the desired result; $I_2 < \infty$. $\Box$

Remark 3.7. If $K(z,1) \le 1$, it suffice to take

$$\int_0^\infty (\frac{(z^{p^+\frac{\nu+1}{1-\mu}} + z^{p^-\frac{\nu+1}{1-\mu}})K(z,1)^{p^-}}{z})dz < \infty. \tag{3.10}$$

instead of (3.9) in Theorem 3.6.

Corollary 3.8. *Let p as in Theorem 3.6 and consider the integral operator*

$$Tf(y) = \int_0^y y^\nu \left(\frac{x}{x+y^\mu}\right)^{\frac{1}{p^-}} f(x)dx.$$

for fixed μ and ν which $-\frac{1}{p^+} < \frac{\nu+1}{1-\mu} < 0$. Then T is bounded operator on $L^{p(\cdot)}[0,\infty)$, i.e.;

$$\int_0^\infty \left(\int_0^y y^\nu \left(\frac{x}{x+y^\mu}\right)^{\frac{1}{p^-}} f(x)dx\right)^{p(y)} dy \le C \int_0^\infty f(x)^{p(x)}dx,$$

for some positive constant C.

Proof . let $K(x,y) = y^\nu(\frac{x}{x+y^\mu})^{\frac{1}{p^-}}$ so $K(y^\mu z, y) = y^\nu K(z,1)$ which $K(z,1) = (\frac{z}{1+z})^{\frac{1}{p^-}}$. By Remark 3.7 it is suffice to show that (3.10) is held.

$$\int_0^\infty \left(\frac{(z^{p^+\frac{\nu+1}{1-\mu}} + z^{p^-\frac{\nu+1}{1-\mu}})K(z,1)^{p^-}}{z}\right)dz = \int_0^\infty \left(\frac{z^{p^+\frac{\nu+1}{1-\mu}} + z^{p^-\frac{\nu+1}{1-\mu}}}{z+1}\right)dz =: I_m + I_k.$$

Where $p^+\frac{\nu+1}{1-\mu} =: m$ and $p^-\frac{\nu+1}{1-\mu} =: k$ hence $-1 < m, k < 0$. By calculation we deduce

$$I_m = \int_0^\infty \frac{z^m}{z+1}dz = \left(\int_0^1 + \int_1^\infty\right)\frac{z^m}{z+1}dz = \int_1^\infty \frac{z^{-1-m}}{z+1}dz + \int_1^\infty \frac{z^m}{z+1}dz = J_{-1-m} + J_m.$$

Since $-1 < m < 0$ so $-1 + \frac{1}{n+1} < m$ for some integer $n \ge 2$ and hence

$$J_m = \int_1^\infty \frac{z^m}{z+1}dz < \int_1^\infty \frac{z^{-1+\frac{1}{n+1}}}{z+1}dz = (n+1)\int_1^\infty \frac{1}{r^n+1} < \frac{n+1}{n-1} \le 3.$$

Note that the last equality deduced from the change of variable $z = r^{n+1}$.

The similar argument can be done for J_{-1-m} and so for I_k. Thus we get the desired result. □

Corollary 3.9. *By assuming the condition on p as in Theorem 3.6, there exist $C > 0$ such that*

$$\int_0^\infty \left(\int_0^y y^\nu \left(\frac{x}{y^\mu}\right)^{\lambda-\theta} e^{-\frac{x}{\lambda y^\mu}} f(x)dx\right)^{p(y)} dy \le C \int_0^\infty f(x)^{p(x)}dx;$$

where $\theta = \frac{\nu+1}{1-\mu}$ and $\lambda > \frac{1}{p^-}$ is a positive constant.

Proof . By considering $K(x,y) := y^\nu(\frac{x}{y^\mu})^{\lambda-\theta}e^{-\frac{x}{\lambda y^\mu}}$, it is obvious that the homogeneity property of K is satisfied. Moreover

$$I = \int_0^\infty \frac{z^{p^+\theta}K(z,1)^{p^+}}{z} = \left(\int_0^1 + \int_1^\infty\right)\left(z^{p^+\lambda-1}e^{-\frac{p^+}{\lambda}z}dz\right) =: I_1 + I_\infty.$$

Since $\lambda > \frac{1}{p^-}$, $I_1 < \int_0^1 e^{-\frac{p^+}{\lambda}z} < \infty$ and $\lambda > 0$ by Limiting comparison test with $\int_1^\infty \frac{1}{z^2}dz$ we obtain $I_\infty < \infty$. So I is finite and hence, the same result is hold for the second part of (3.9). Now the result follows from Theorem 3.6. □

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